INTEGRAL TRANSFORM PRINCIPLE

Signal $f(x)$
to be analyzed

Analyzing
functions $\psi_k(x)$

Analysis

$$\hat{f}(k) = \int f(x)\, \psi_k(x)\, dx$$

Synthesis

$$f(x) = \frac{1}{c} \int \hat{f}(k)\, \psi_k(x)\, d\mu(k)$$

Energy conservation (Parseval)

$$\int |f(x)|^2 dx = \frac{1}{c} \int |\hat{f}(k)|^2\, dk$$

DIFFERENT FOURIER TRANSFORMS

Used for Signal Processing

$$\hat{f}(k) = \int_{-\infty}^{+\infty} f(x)\, e^{-2i\pi kx}\, dx$$

$$f(x) = \int_{-\infty}^{+\infty} \hat{f}(k)\, e^{+2i\pi kx}\, dk$$

$$e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi k^2}$$

$$\int |f(x)|^2\, dx = \int |\hat{f}(k)|^2\, dk$$

$$\widehat{f(x) * g(x)} = \hat{f}(k) \cdot \hat{g}(k)$$

$$\widehat{\frac{\partial^n f(x)}{\partial x^n}} = (2i\pi k)^n f(k)$$

Used for Harmonic Analysis

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(k)\, e^{ikx}\, dk$$

$$e^{-\frac{x^2}{2}} \xrightarrow{\mathcal{F}} e^{-\frac{k^2}{2}}$$

$$\int |f(x)|^2\, dx = \int |\hat{f}(k)|^2\, dk$$

$$\widehat{f(x) * g(x)} = \frac{1}{2\pi} \hat{f}(k) \cdot \hat{g}(k)$$

$$\widehat{\frac{\partial^n f(x)}{\partial x^n}} = (ik)^n \hat{f}(k)$$

Used for Group Theory

$$\hat{f}(k) = \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(k)\, e^{ikx}\, dk$$

$$e^{-\pi x^2} \longrightarrow e^{-\frac{k^2}{4\pi}}$$

$$\int |f(x)|^2\, dx = \frac{1}{2\pi} \int |\hat{f}(k)|^2\, dk$$

$$\widehat{f(x) * g(x)} = \hat{f}(k) \cdot \hat{g}(k)$$

$$\widehat{\frac{\partial^n f(x)}{\partial x^n}} = (ik)^n \hat{f}(k)$$

PROPERTIES

Similarity (Dilatation)

$$\widehat{f(ax)} = \frac{1}{a}\hat{f}\left(\frac{k}{a}\right)$$

Modulation (Homogeneity in k)

$$\widehat{e^{i\pi x}} = \delta\left(k-\frac{1}{2}\right)$$

Translation (Homogeneity in x)

$$\widehat{\delta\left(x-\frac{1}{2}\right)} = e^{-i\pi k}$$

$$\widehat{f(x-x_0)} = e^{-ix_0k}\hat{f}(k)$$

Linearity (Addition)

$$\widehat{f(x)+g(x)} = \hat{f}(k)+\hat{g}(k)$$

Average

$$\int_{-\infty}^{+\infty} f(x)\,dx = \hat{f}(0)$$

Reality

$$\hat{f}(k) = \hat{f}^{*}(-k)$$

if $f(x)$ real

Convolution

$$f * g(x) = \int_{-\infty}^{+\infty} f(x')\, g(x - x')\, dx'$$

$$\widehat{f * g} = \frac{1}{2\pi} \hat{f} \cdot \hat{g}$$

Derivation

$$\widehat{\frac{\partial^m}{\partial x^m} f(x)} = (ik)^m \hat{f}(k)$$

Moments

$$\widehat{x^m f(x)} = i^m \frac{\partial^m}{\partial k^m} \hat{f}(k)$$

$$\Longrightarrow \hat{M} = \widehat{\int x^m f(x)\, dx} = i^m \frac{\partial^m}{\partial k^m} \hat{f}(0)$$

Parity conservation

f even (pair) $\quad f(-x) = f(x) \Longleftrightarrow \hat{f}$ even $\quad \hat{f}(-k) = \hat{f}(k)$

f odd $\quad f(-x) = -f(x) \Longleftrightarrow \hat{f}$ odd $\quad \hat{f}(-k) = -\hat{f}(k)$

Scalar product conservation

$$\langle f | g \rangle = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx = \int_{-\infty}^{+\infty} \hat{f}(k)\, \hat{g}^*(k)\, dk$$

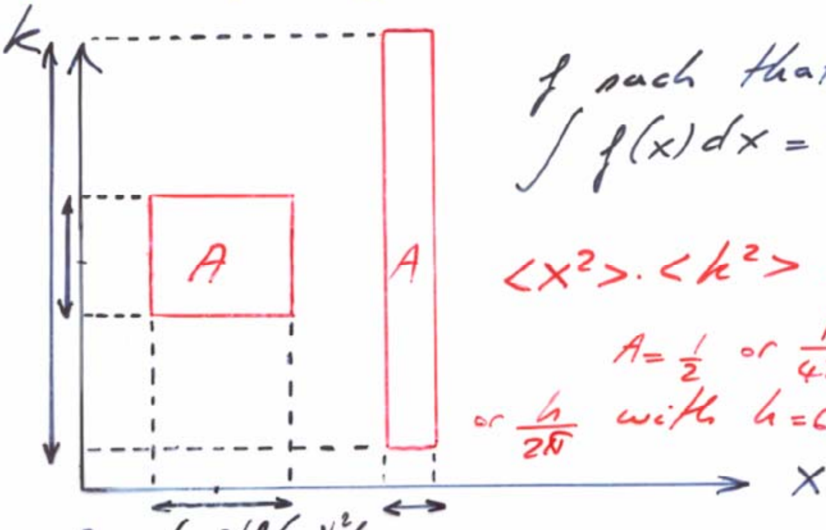
Energy conservation (Parseval)

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |\hat{f}(k)|^2 dk$$

HEISENBERG'S UNCERTAINTY PRINCIPLE

A signal cannot be concentrated in both time and frequency, or position and wavenumber.

This is not a limitation of our perception of reality. This is just by definition of the frequency or wavenumber.



f such that $\int f(x)\,dx = 1$

$$\langle x^2\rangle . \langle k^2\rangle \geq A$$

$A = \frac{1}{2}$ or $\frac{1}{4\pi}$ or $\frac{1}{16\pi^2}$ or $\frac{h}{2\pi}$ with $h = 6.6\ 10^{-27}\ gcm^2/s$

$\langle k^2\rangle = \int k^2 |\hat{f}(k)|^2 dk$

$\widehat{f(\lambda x)} = \frac{1}{\lambda} \hat{f}\left(\frac{k}{\lambda}\right)$

Space dilation $\Longleftrightarrow$ Wavenumber contraction

$\langle x^2\rangle = \int x^2 |f(x)|^2 dx$

Time-frequency plane,

or position-wavenumber plane,

or position-impulsion ($p = \frac{hk}{2\pi}$, h Planck's constant),

or phase space,

or information plane.