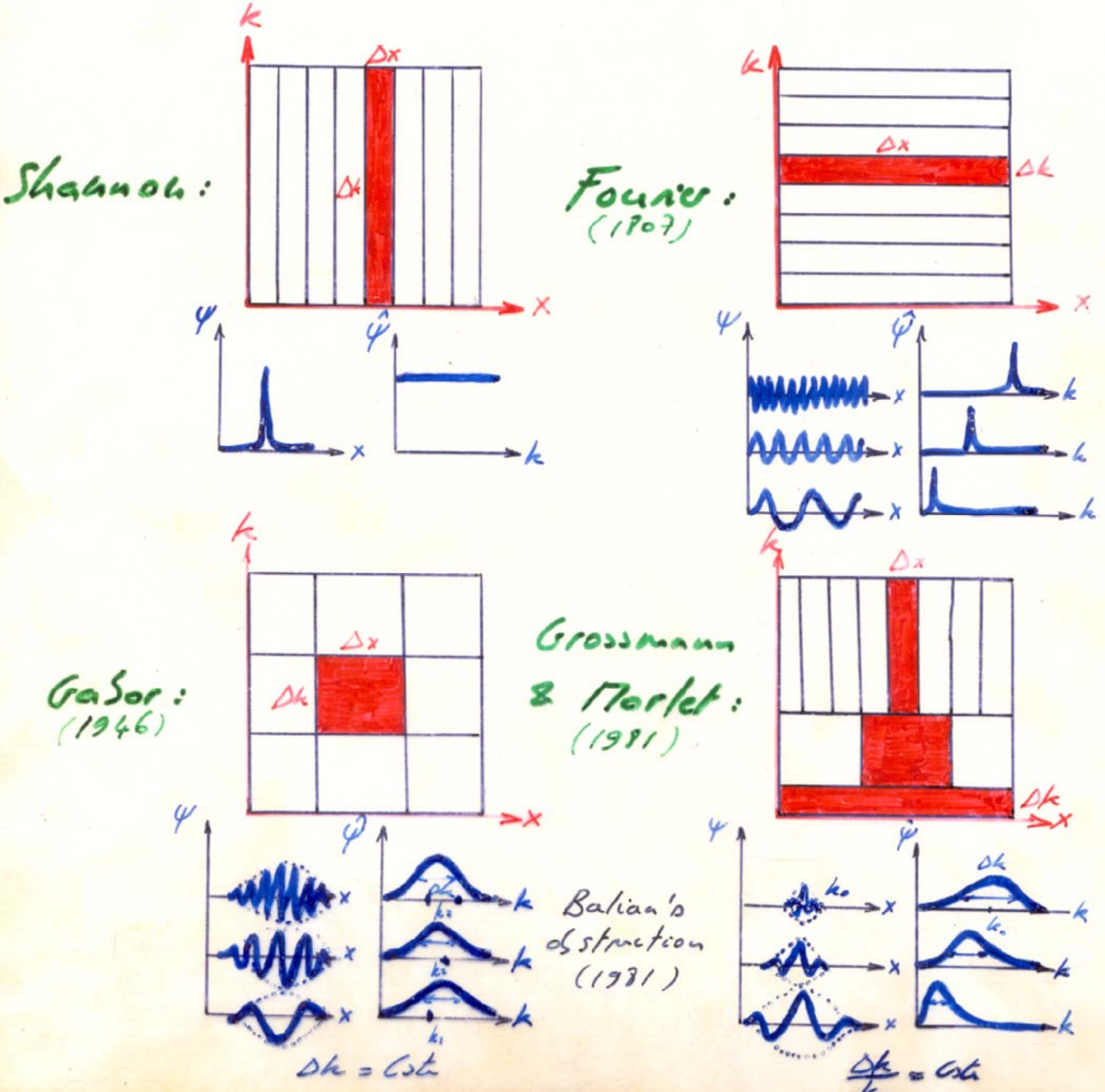
INTEGRAL TRANSFORM PRINCIPLE

Analysis $\quad T_f(k) = \int f(\vec{x})\, \psi_k(\vec{x})\, d\vec{x}$

Synthesis $\quad f(\vec{x}) = \frac{1}{c} \int T_f(k)\, \psi_x(\vec{k})\, d\mu(k)$

$$\Delta x \cdot \Delta k \geqslant Cst$$



CONTINUOUS WAVELET TRANSFORM

Choice of the 'mother wavelet'

Admissibility condition: $\psi \in \{L^1 \cap L^2\}$

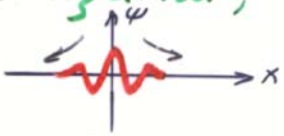
$$C = \int |\hat{\psi}(k)|^2 \frac{dk}{|k|} < \infty \Rightarrow \hat{\psi}(0) = 0 \quad \text{i.e.} \int \psi(x)\,dx = 0$$

in order to have a finite energy reproducing kernel

And, if possible:

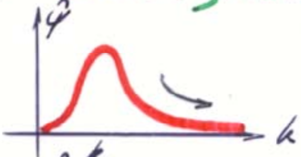
Good space (/time) localization,

$$|\psi(x)| < \frac{1}{1+|x|^n}$$



Good scale (/frequency) localization,

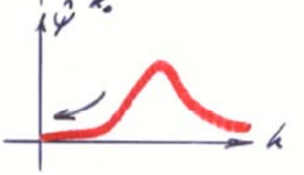
$$|\hat{\psi}(k)| < \frac{1}{1+|k-k_0|^n}$$



Zero high order moments,

$$\int \psi(x)\, x^m\, dx = 0$$

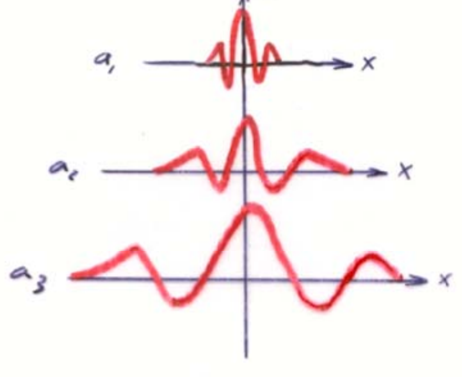
$$= \left(\frac{d}{dk}\right)^m \hat{\psi}(k)\Big|_{k=0}$$

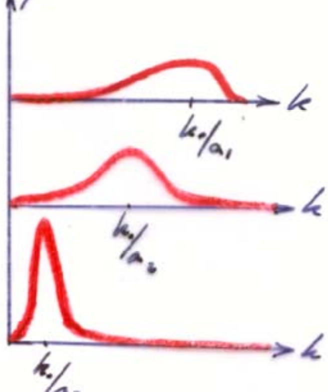


Generation of the 'wavelet family'

Affine Group $\begin{cases} \text{by translation (parameter } s) \\ \text{and dilation (parameter } a): \end{cases}$

$$\psi_{s,a}(x) = \frac{1}{\sqrt{a}}\, \psi\left(\frac{x-s}{a}\right)$$





ANALYSIS/SYNTHESIS

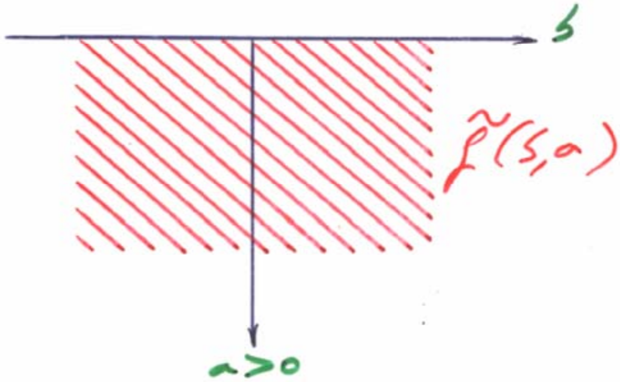
Analysis: $\bar{\psi}$ complex conjugate

$$\tilde{f}(b,a) = \int \bar{\psi}_{a,b}(x)\, f(x)\, dx$$

$$= \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} \bar{\psi}\left(\frac{x-b}{a}\right) f(x)\, dx$$

$$= a^{1/2} \int_{-\infty}^{+\infty} \underbrace{\hat{\bar{\psi}}(ak)}_{\text{Filter with } \frac{\Delta k}{|k|} = \text{Cst}} e^{ibk} \hat{f}(k)\, dk$$

The wavelet coefficients are defined on the open half-plane $(b,a) \in \mathbb{R} \times \mathbb{R}^+$



Synthesis:

$$f(x) = \frac{1}{C} \iint \psi_{a,b}(x)\, \tilde{f}(b,a)\, \frac{da\, db}{a^2}$$

$$= \frac{1}{C} \int_{0^+}^{+\infty} \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} \psi\left(\frac{x-b}{a}\right) \tilde{f}(b,a)\, \frac{da\, db}{a^2}$$

avec $C = 2\pi \int_{-\infty}^{+\infty} |\hat{\psi}(k)|^2 \frac{dk}{|k|}$

Energy conservation (Parseval):

$$\int |f(x)|^2 dx = \frac{1}{C} \iint |\tilde{f}(b,a)|^2 \underbrace{\frac{da\, db}{a^2}}_{\text{Haar Measure}}$$

WAVELET TRANSFORM OF HARMONIC SIGNALS

Progressive wavelet $\psi(x) \in \mathcal{H}$ Hardy space

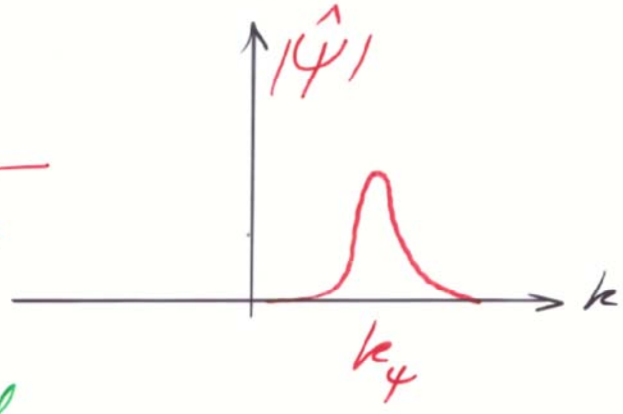
$$\Longleftrightarrow \psi(x) \in \mathbb{C} \;/\; \hat{\psi}(k \leq 0) = 0,$$

i.e. $\mathcal{R}(\psi) \xleftrightarrow{H} \mathcal{I}(\psi)$

H Hilbert transform,

with maximum of $\psi(x)$ at

$$k_\psi = \frac{\int_{-\infty}^{+\infty} k^2 \psi^2(k)\, dk}{\int_{-\infty}^{+\infty} \psi^2(k)\, dk}$$



Harmonic signal

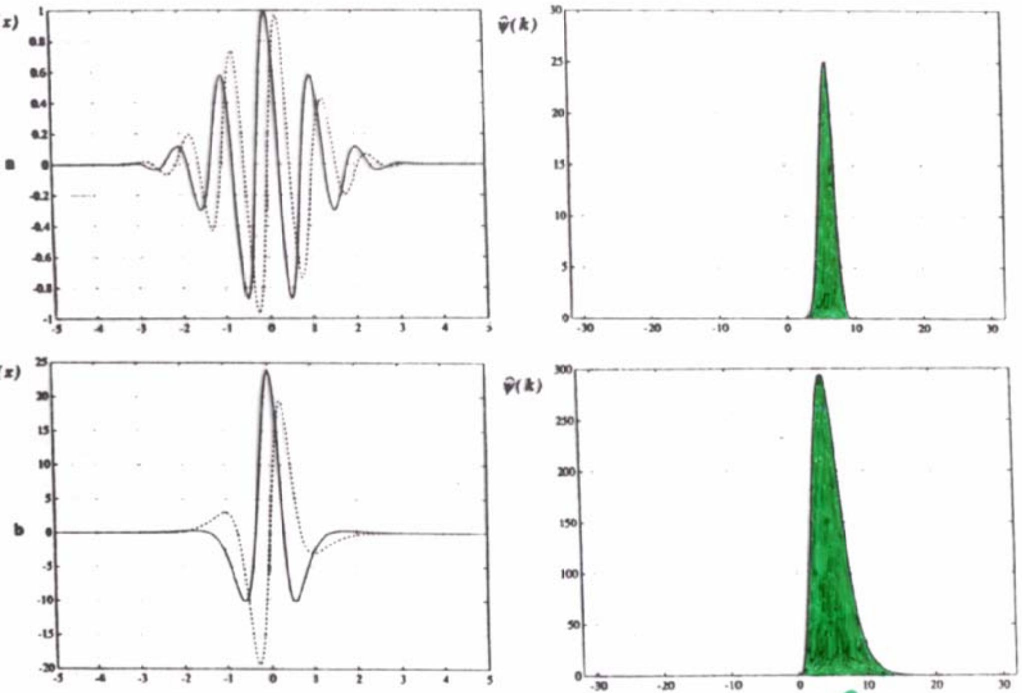
$$f(x) = \cos k_0 x = \frac{e^{ikx} + e^{-ikx}}{2}$$

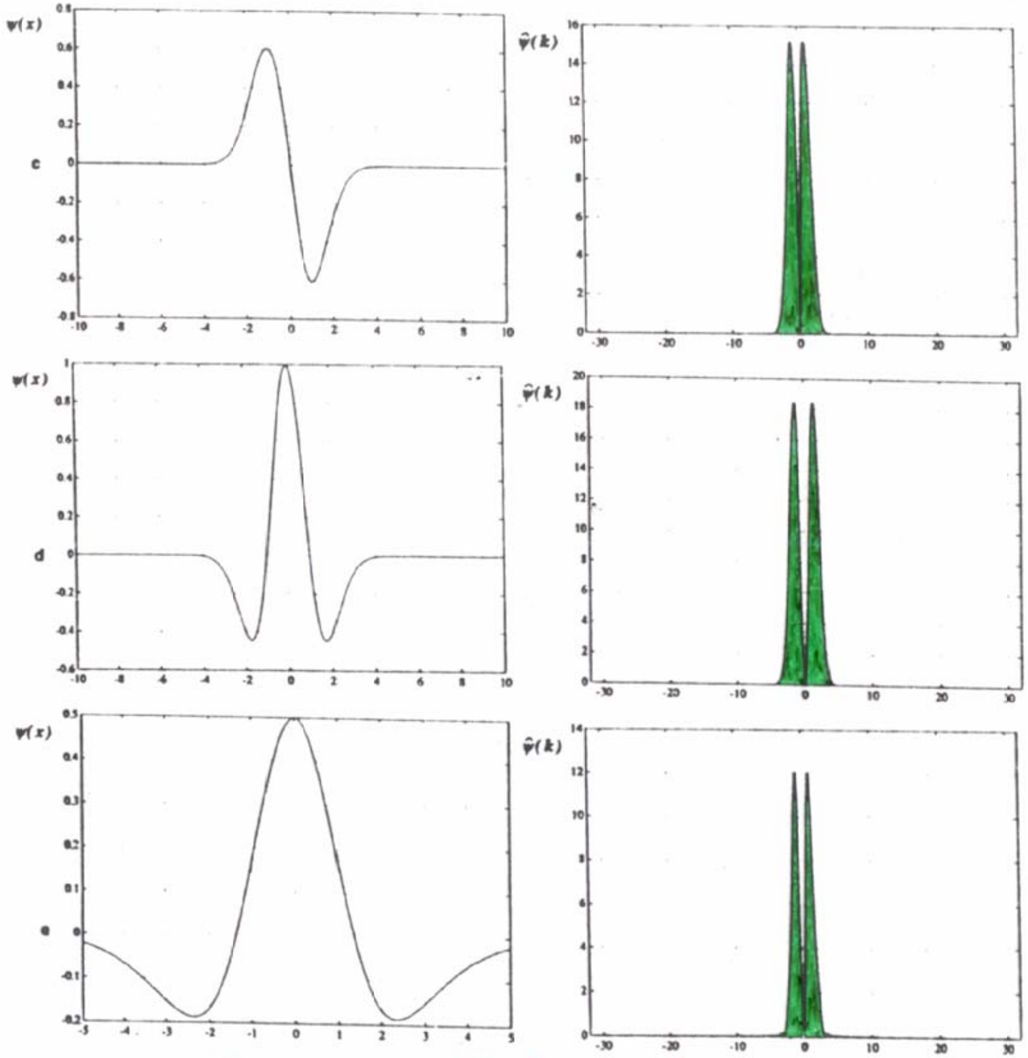
$$\tilde{f}(s, a) = \sqrt{a}\, e^{i s k_0}\, \bar{\hat{\psi}}(a \nu_0)$$

- Modulus behaves as $\bar{\hat{\psi}}(a\nu_0)$ which is maximal for $\hat{\psi}(\nu_\psi) \Rightarrow a = \frac{k_\psi}{k_0}$
- Phase varies linearly with s and therefore unfolds the signal phase in space $\Rightarrow \frac{\partial \varphi}{\partial s} = \frac{k_0}{k_\psi}$

Derivatives of a Gaussian

Morlet





ψ $\hat{\psi}$

Paul

Difference of two Gaussians

ψ $\hat{\psi}$

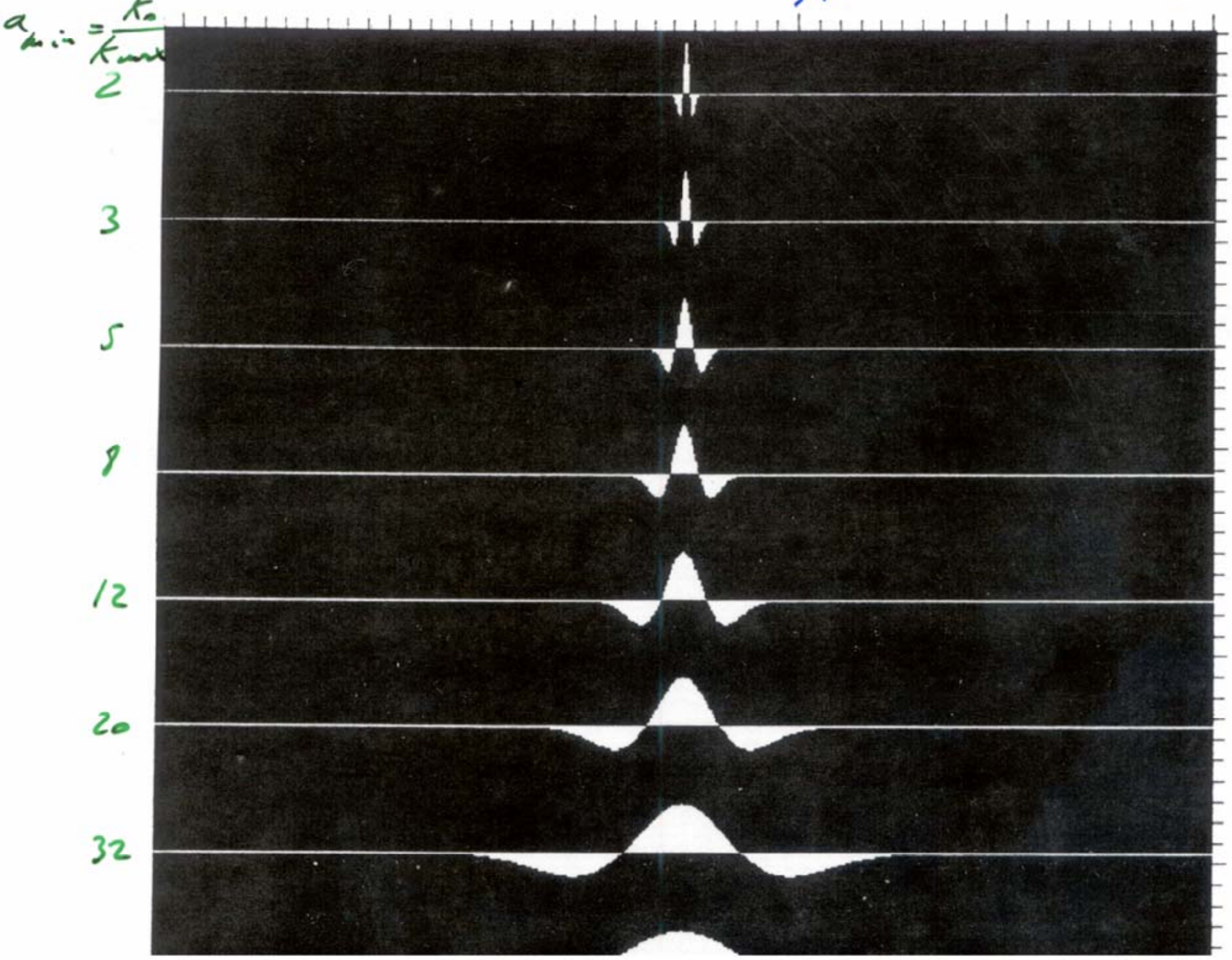
Complex-valued wavelets

Real-valued wavelets

WAVELET FAMILY IN L2-NORM
WITH REAL-VALUED WAVELETS

Scale in pixel units

L signal duration or length
K_{max} Nyquist cut-off frequency



WAVELET FAMILY IN L2-NORM
WITH COMPLEX-VALUED WAVELETS

scale in pixel units

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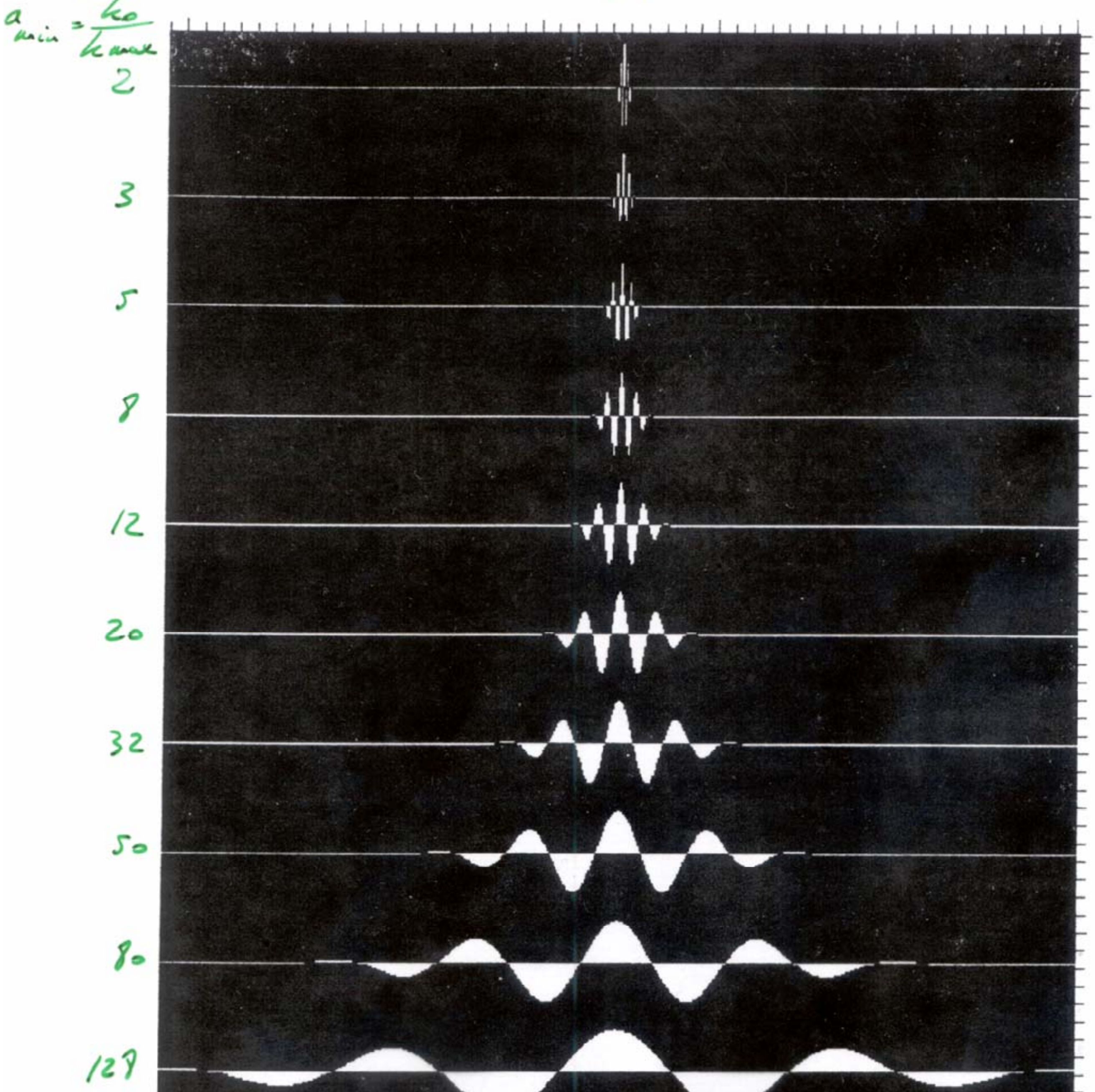
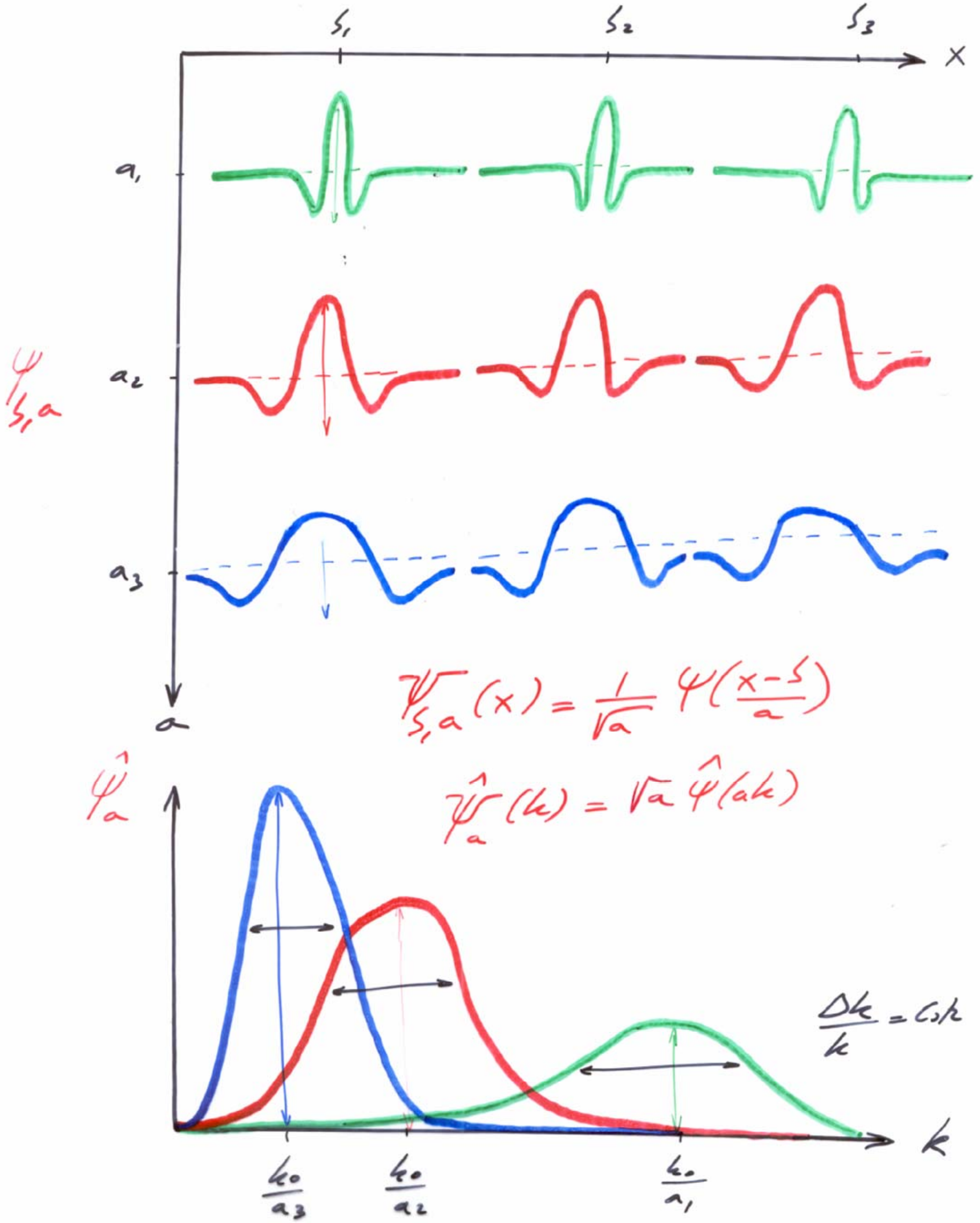


Morlet Wavelet

$$\psi(x) = e^{-ik_0 x} e^{-\frac{x^2}{2}}$$

$$\psi \in \mathbb{C}$$

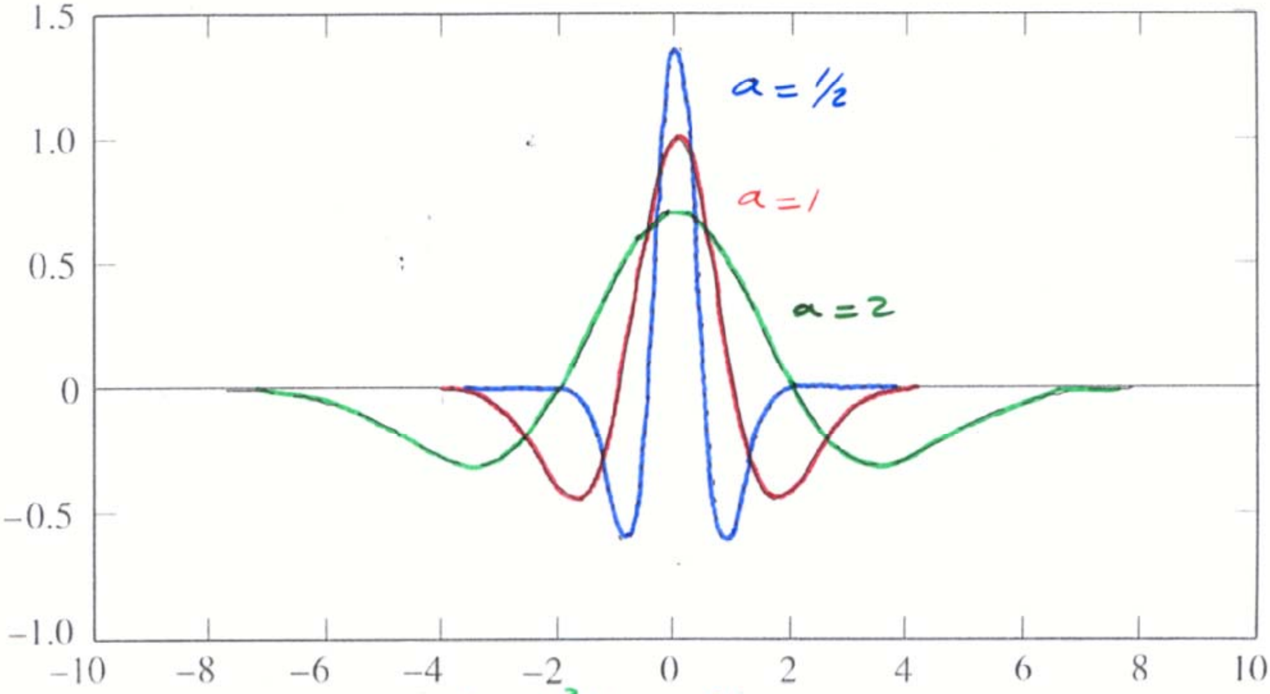
WAVELET FAMILY IN L^2-NORM



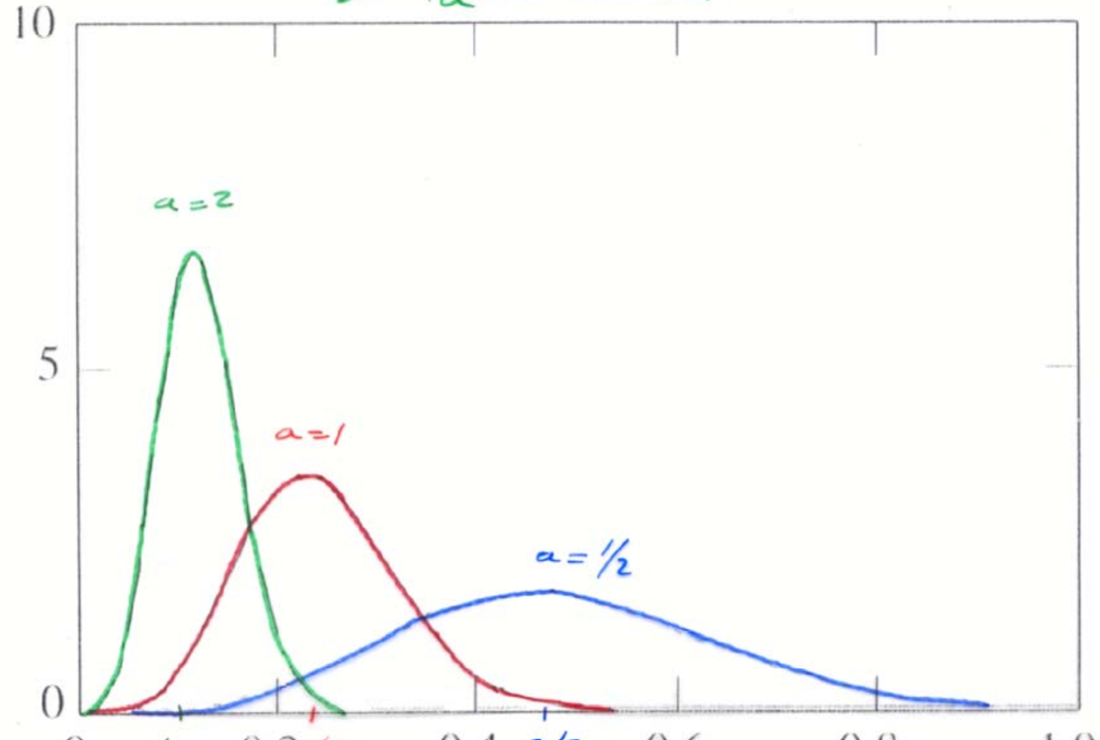
$$\psi_{\varsigma,a}(x) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{x-\varsigma}{a}\right)$$

$$\hat{\psi}_a(k) = \sqrt{a}\,\hat{\psi}(ak)$$

WAVELET FAMILY WITH L2-NORM

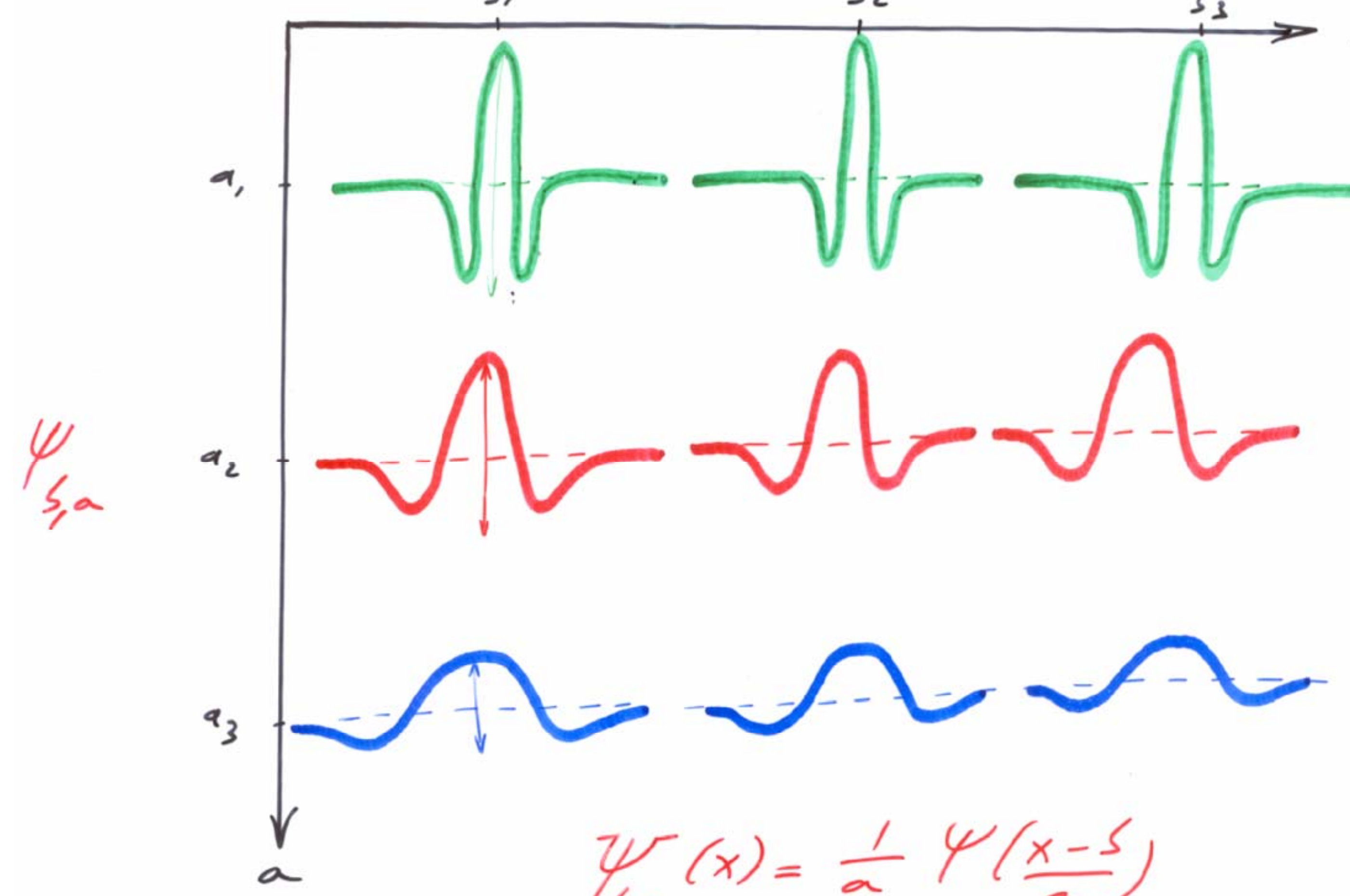


$$\int \psi_a^2(t)\, dt = 1$$



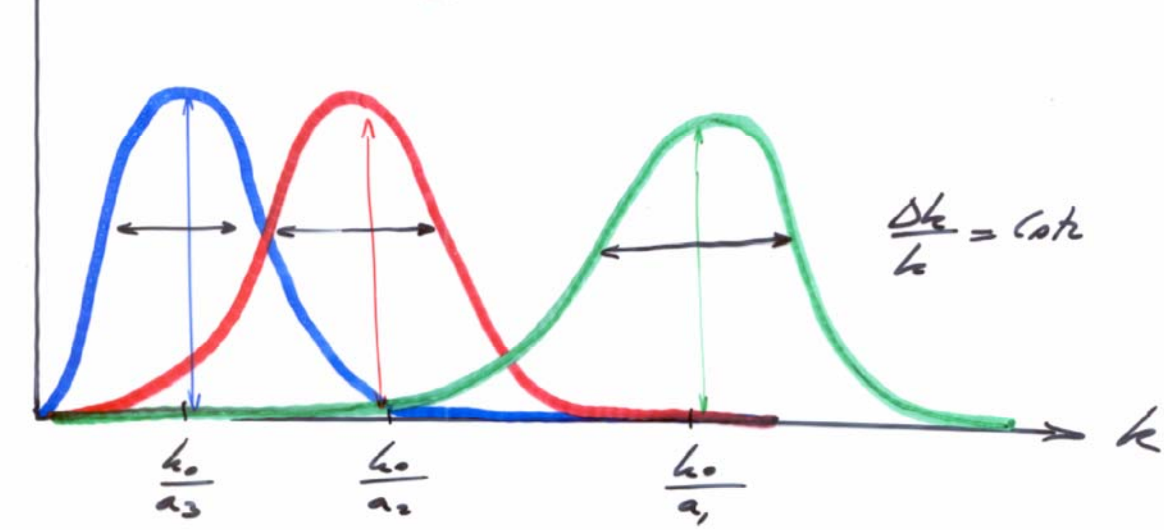
$$\Rightarrow \tilde{f}_{L^2} = \sqrt{a}\, \tilde{f}_{L^1}$$

WAVELET FAMILY IN L^1-NORM

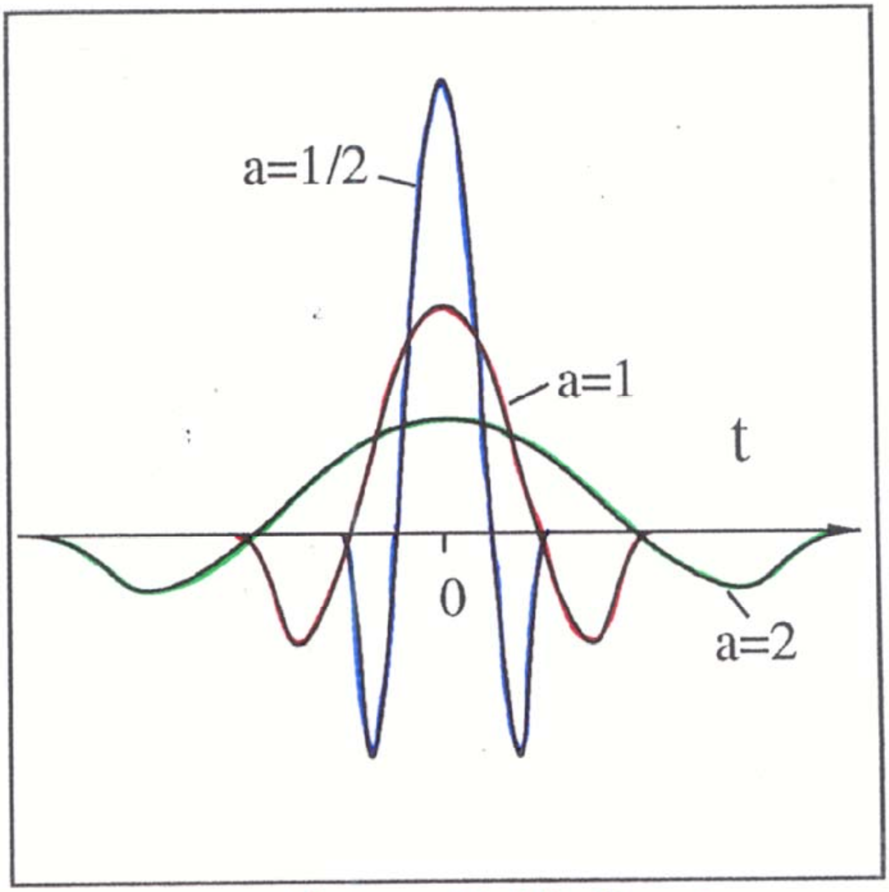


$$\psi_{s,a}(x) = \frac{1}{a}\,\psi\left(\frac{x-s}{a}\right)$$

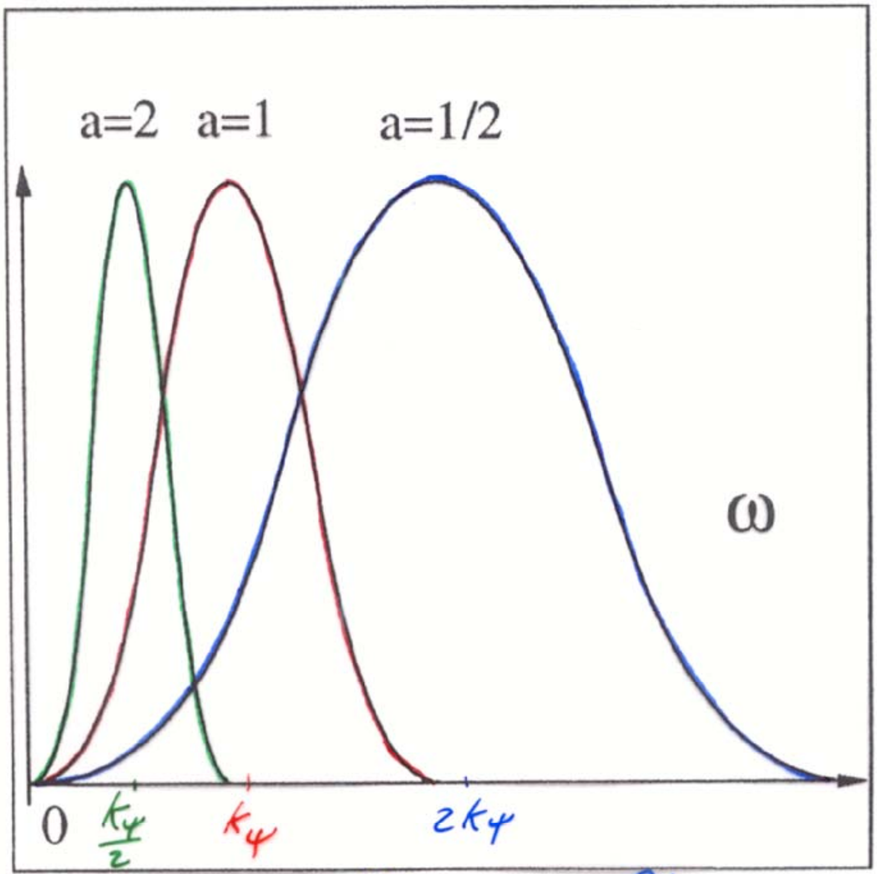
$$\hat{\psi}_a(k) = \hat{\psi}(ak)$$



WAVELET FAMILY WITH L1-NORM



$\int |\psi_a(t)|\, dt = 1$



$\Longrightarrow \tilde{f}_{L^1} = \frac{1}{\sqrt{a}} \tilde{f}_{L^2}$

Real part

Parameters:

Sampling frequency: 44.1 kHz

Number of voice per octave = 1

Highest voice: index = 1 frequency = 6000 Hz

Lowest voice: index = 7 frequency = 93.7 Hz

Time window = 18 ms

Signal de parole (1983)

Jean Morlet

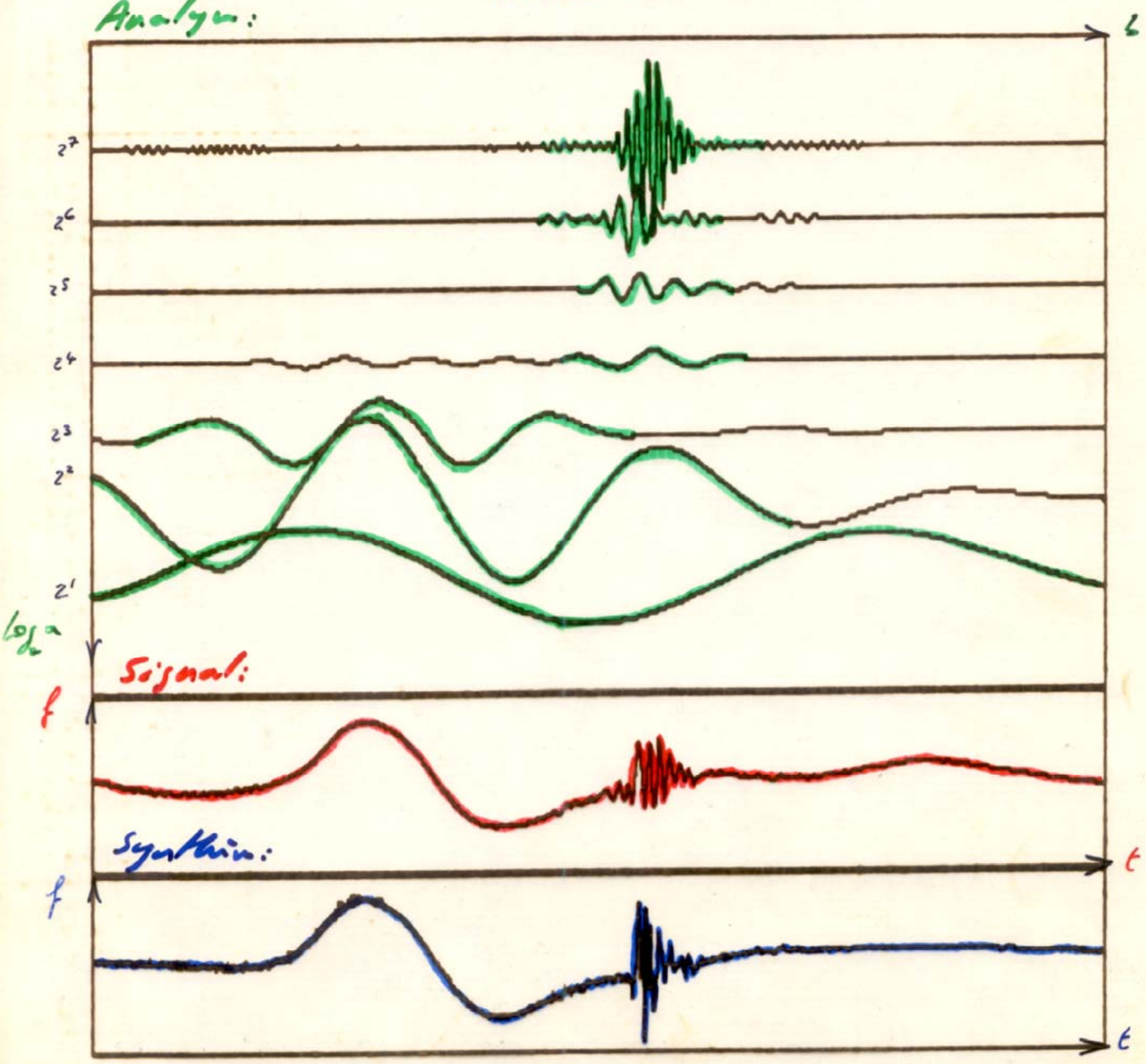


Figure 5

TWO WAVELET FORMULA

We can use different wavelets for analysis ψ_A and synthesis ψ_S.

Admissibility condition:

$$C_\psi = \int_0^\infty \bar{\hat{\psi}}_A(k)\,\hat{\psi}_S(k)\,\frac{dk}{k} = \int_0^\infty \bar{\hat{\psi}}_A(-k)\,\hat{\psi}_S(-k)\,\frac{dk}{k} < \infty$$

Then isometry:

$$f(x) = \frac{1}{C_\psi}\int_{0^+}^{\infty}\frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty}\tilde{f}(b,a)\,\psi_S(b,a)\,\frac{da\,db}{a^2}$$

with $\tilde{f}(b,a) = \frac{1}{\sqrt{a}}\int f(x)\,\bar{\psi}_A(b,a)\,dx$

Morlet's reconstruction formula:

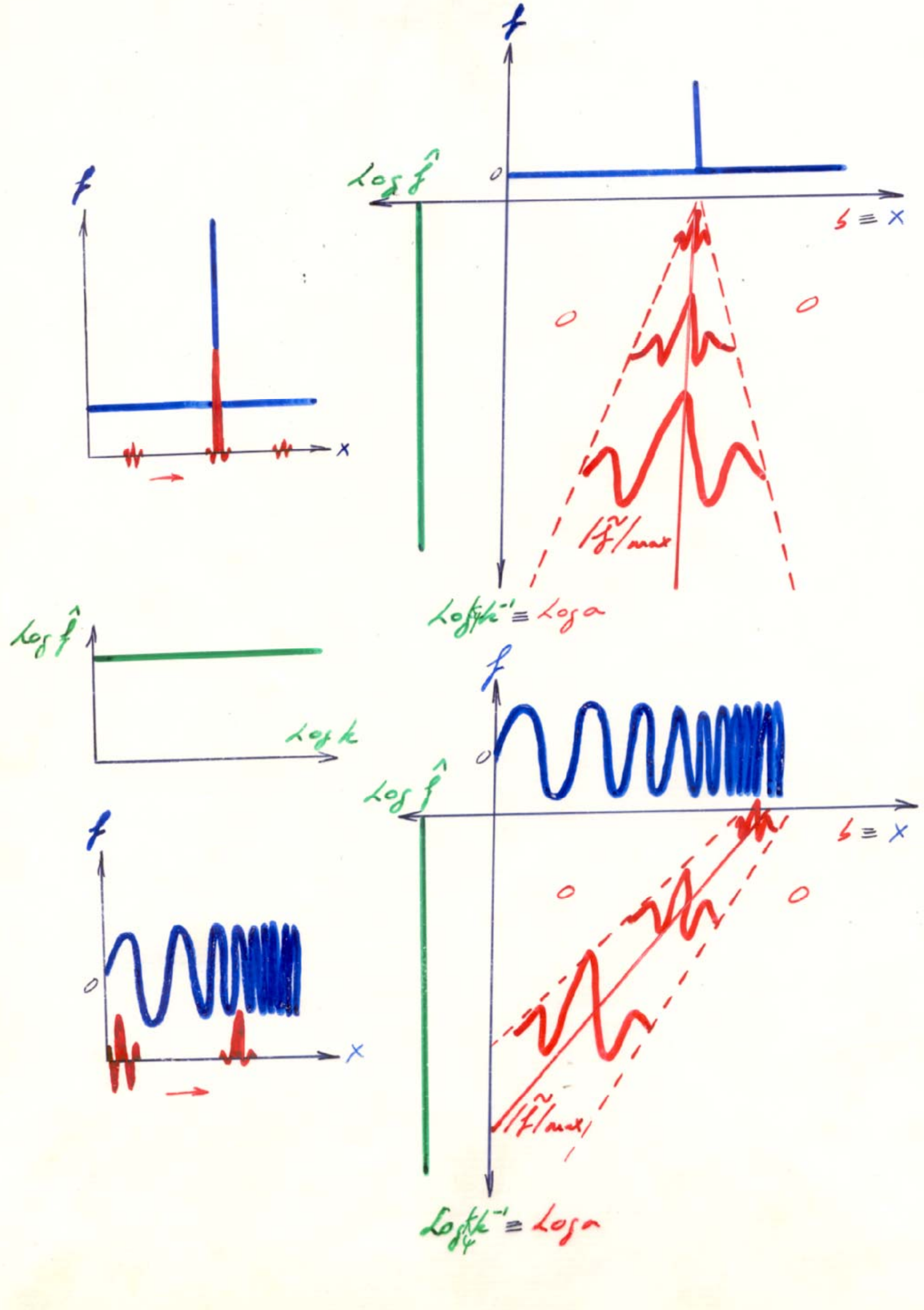
$$\psi_S(x) = \delta(x)$$

then $f(x) = \frac{1}{C_\psi}\int_{0^+}^{+\infty}\frac{1}{\sqrt{a}}\tilde{f}(b,a)\frac{da}{a}$

with $C_\psi = \int_0^\infty \bar{\hat{\psi}}_A(k)\frac{dk}{k} = \int_0^\infty \bar{\hat{\psi}}_A(-k)\frac{dk}{k} < \infty$

We need only one integration to reconstruct.

TeO : EXEMPLES



WAVELET TRANSFORM OF HARMONIC SIGNALS

Progressive wavelet $\psi(x) \in \mathcal{H}$ Hardy space

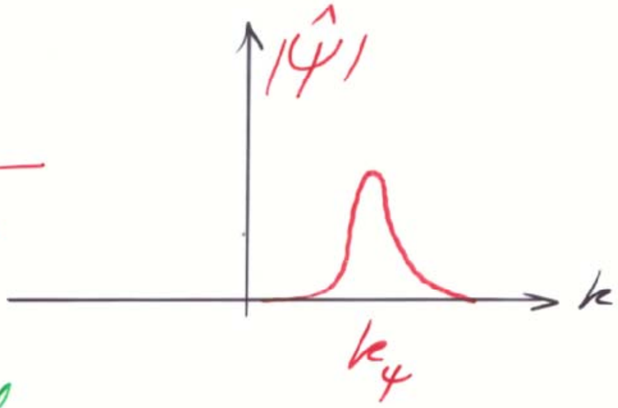
$\Longleftrightarrow \psi(x) \in \mathbb{C} \,/\, \hat{\psi}(k \leq 0) = 0$,

i.e. $\mathcal{R}(\psi) \xleftrightarrow{H} \mathcal{I}(\psi)$

H Hilbert transform,

with maximum of $\psi(x)$ at

$$k_\psi = \frac{\int_{-\infty}^{+\infty} k^2 \psi^2(k)\, dk}{\int_{-\infty}^{+\infty} \psi^2(k)\, dk}$$

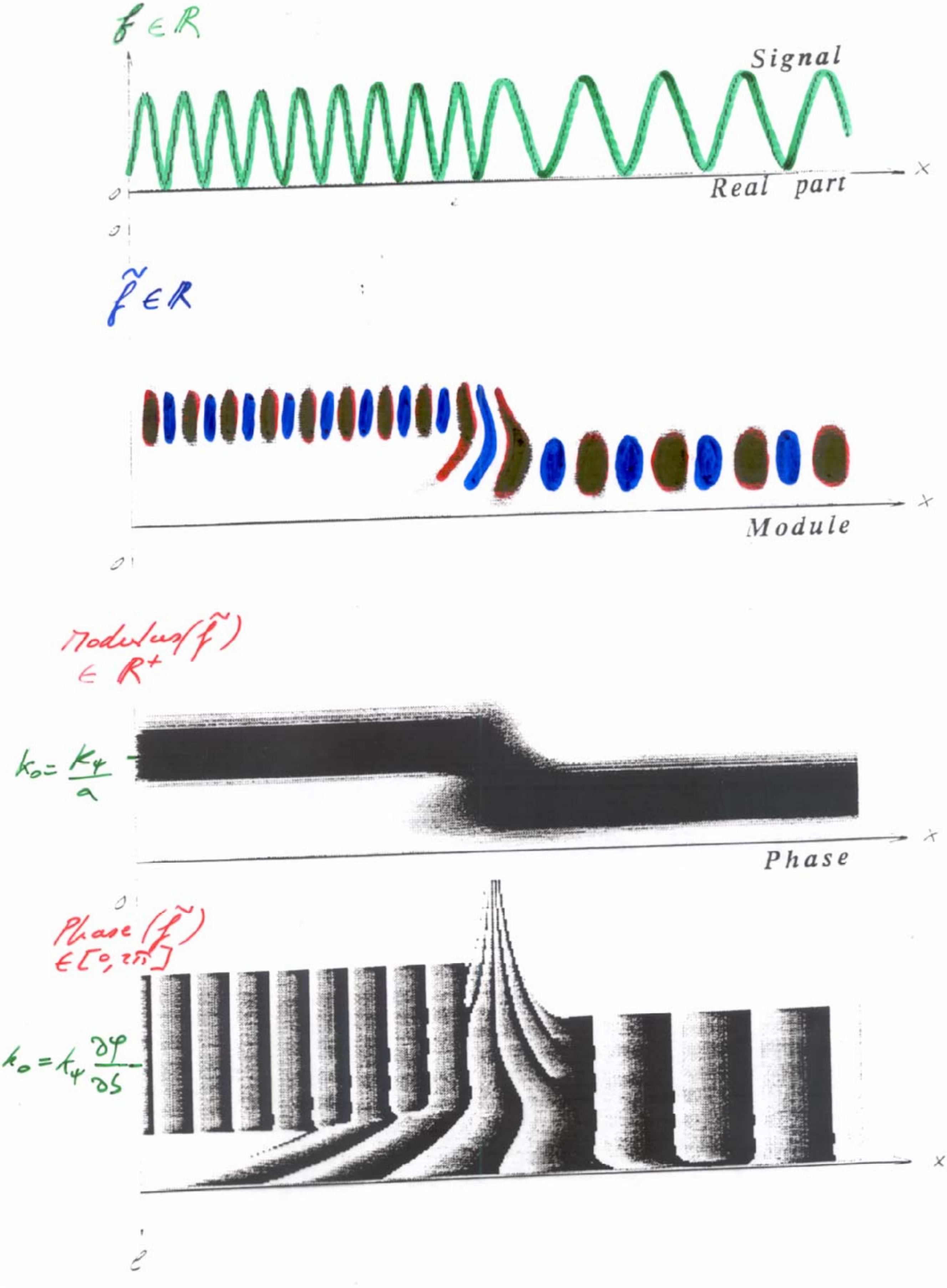


Harmonic signal

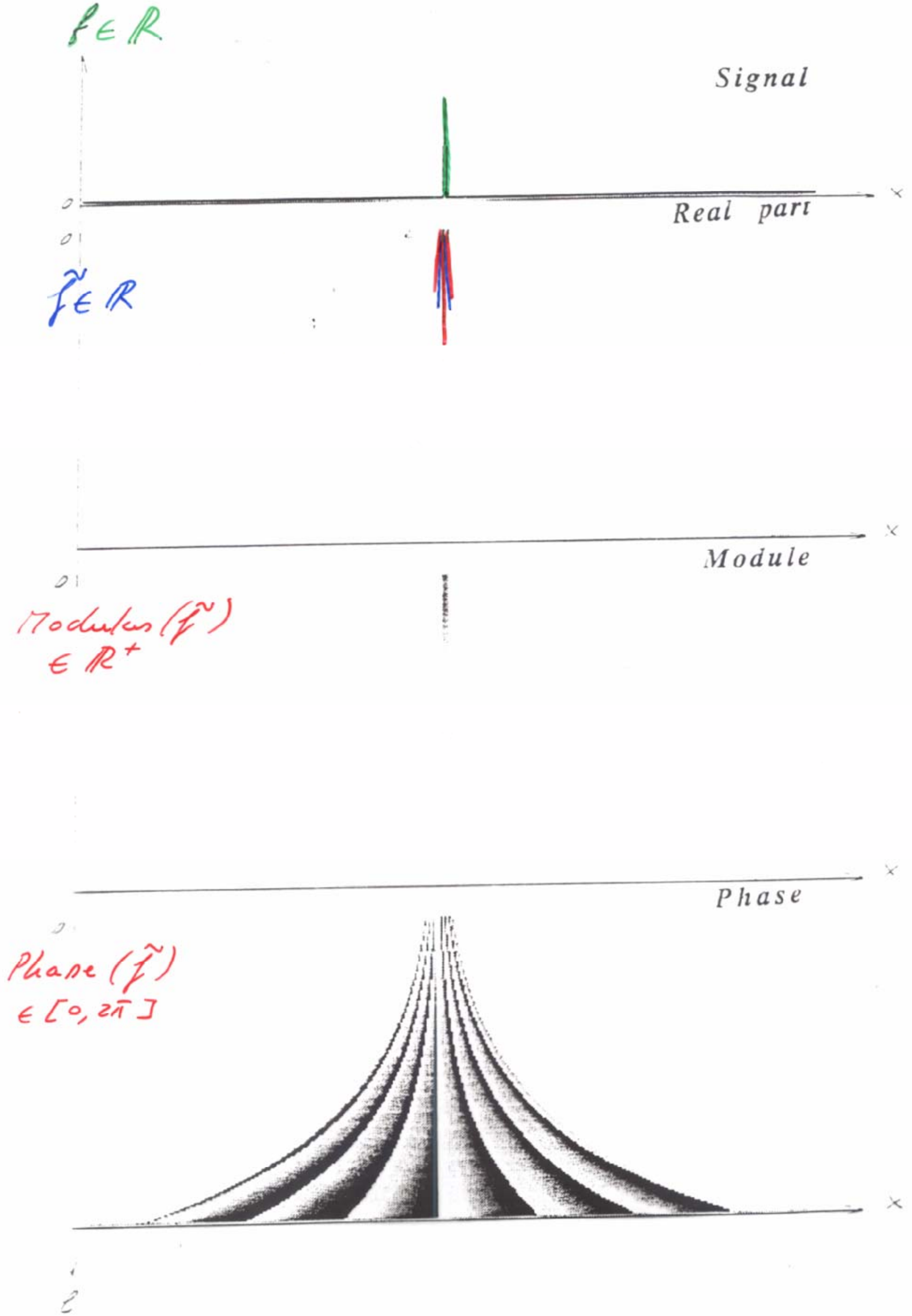
$$f(x) = \cos k_0 x = \frac{e^{ikx} + e^{-ikx}}{2}$$

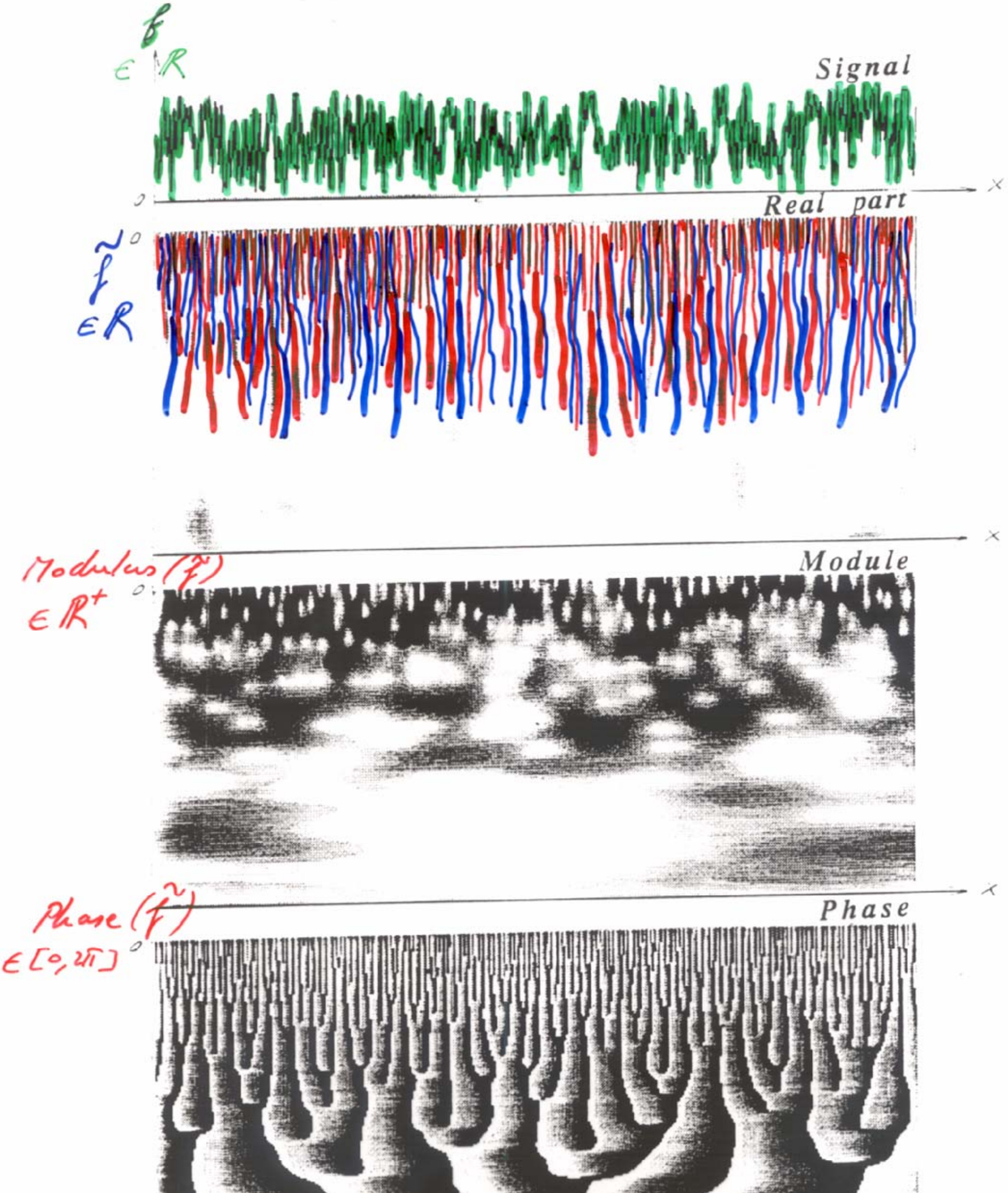
$$\tilde{f}(s, a) = \sqrt{a}\, e^{isk_0}\, \bar{\hat{\psi}}(a \vartheta_0)$$

- Modulus behaves as $\bar{\hat{\psi}}(a\vartheta_0)$ which is maximal for $\hat{\psi}(\vartheta_\psi) \Rightarrow a = \frac{k_\psi}{k_0}$
- Phase varies linearly with s and therefore unfolds the signal phase in space $\Rightarrow \frac{\partial \varphi}{\partial s} = \frac{k_0}{k_\psi}$



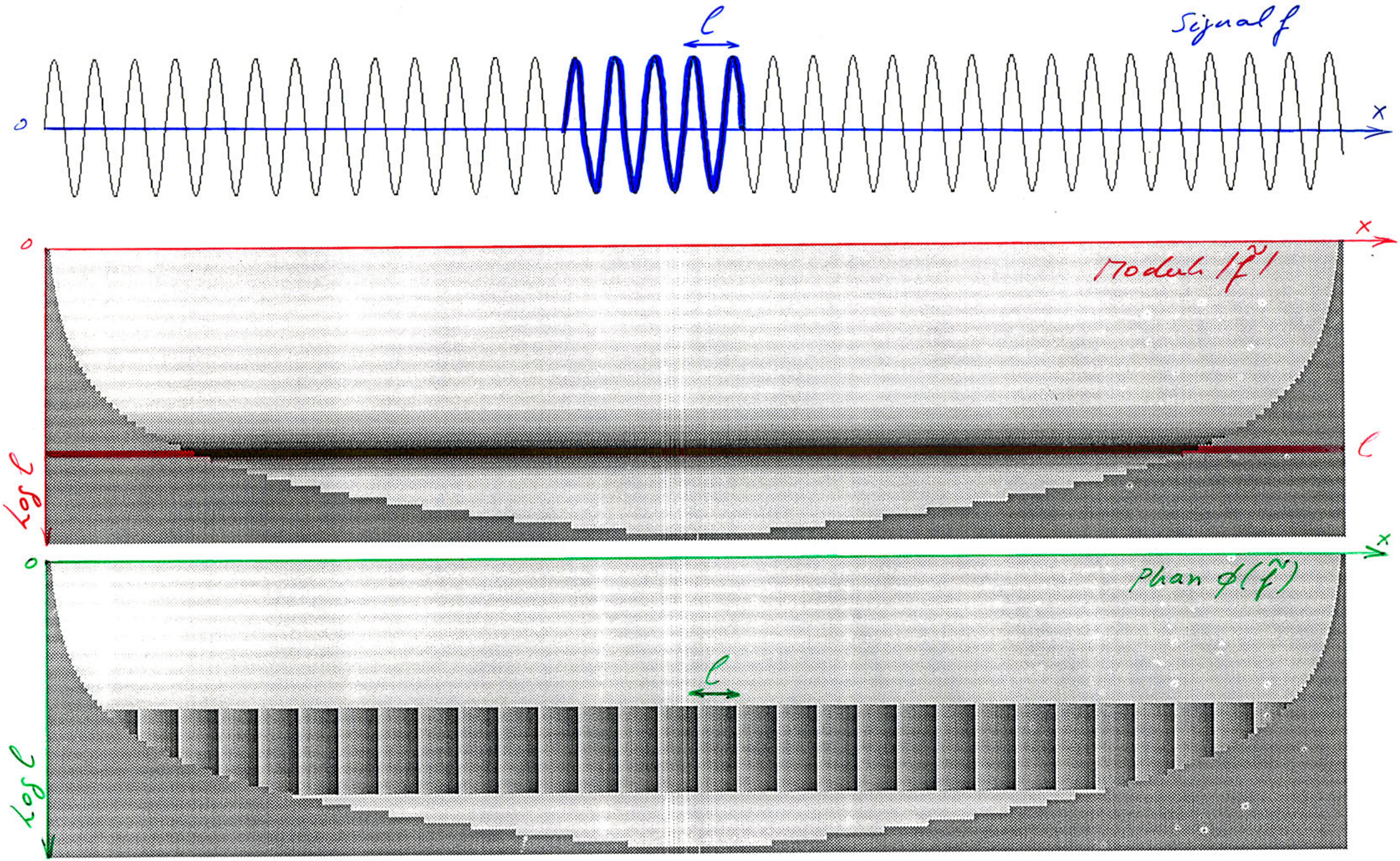
f ∈ R
Signal
Real part
f̃ ∈ R
Module
Modulus(f̃) ∈ R+
k_o = k_ψ / a
Phase
Phase(f̃) ∈ [0, 2π]
k_o = k_ψ ∂φ/∂s


f ∈ ℝ
Signal
Real part
f̃ ∈ ℝ
Module
Modulus(f̃) ∈ ℝ⁺
Phase
Phase(f̃) ∈ [0, 2π]

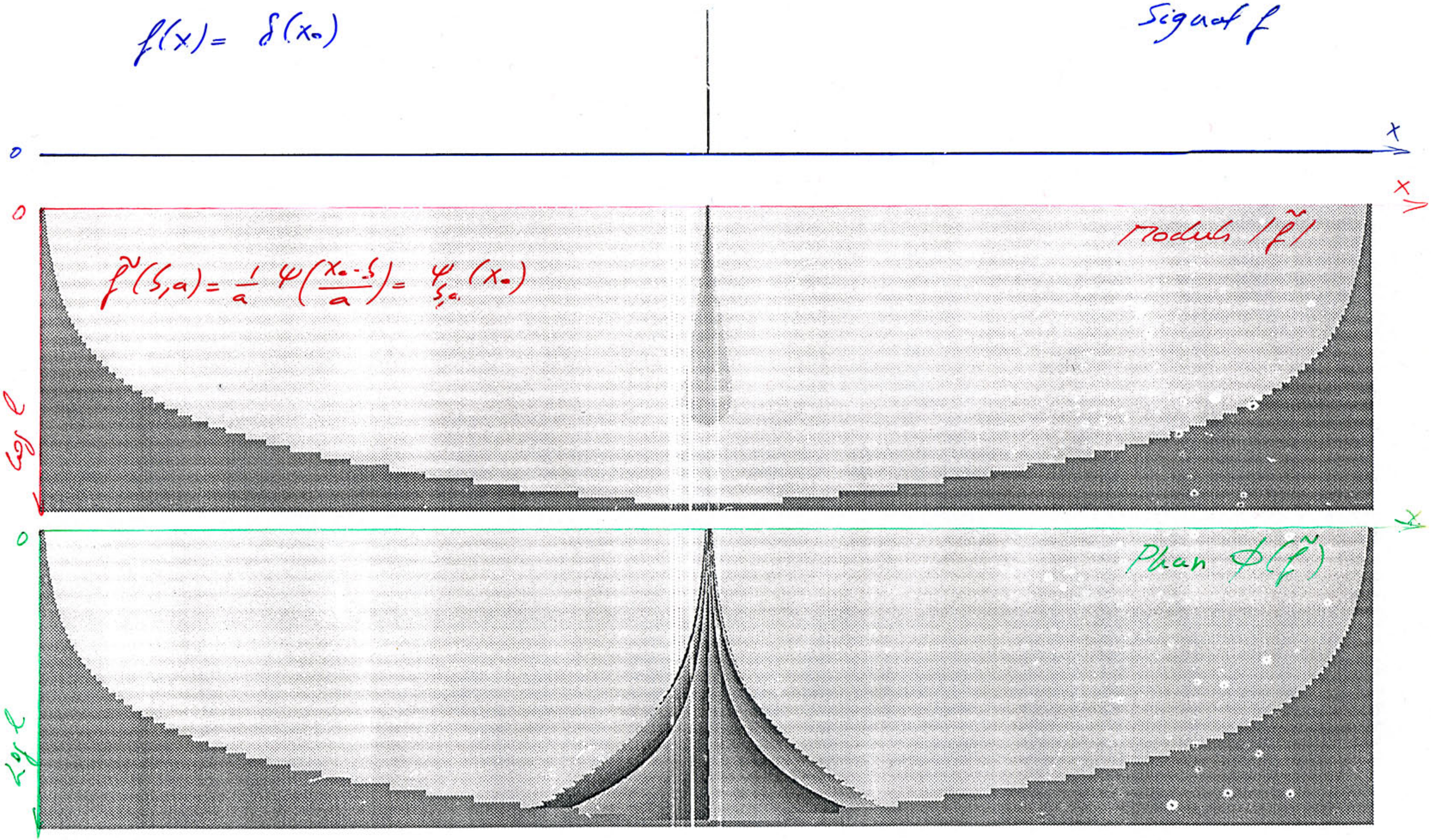


f ∈ R
Signal
Real part
f̃ ∈ R
Module
Modulus(f̃) ∈ R+
Phase
Phase(f̃) ∈ [0,2π]
x
0

SINUS

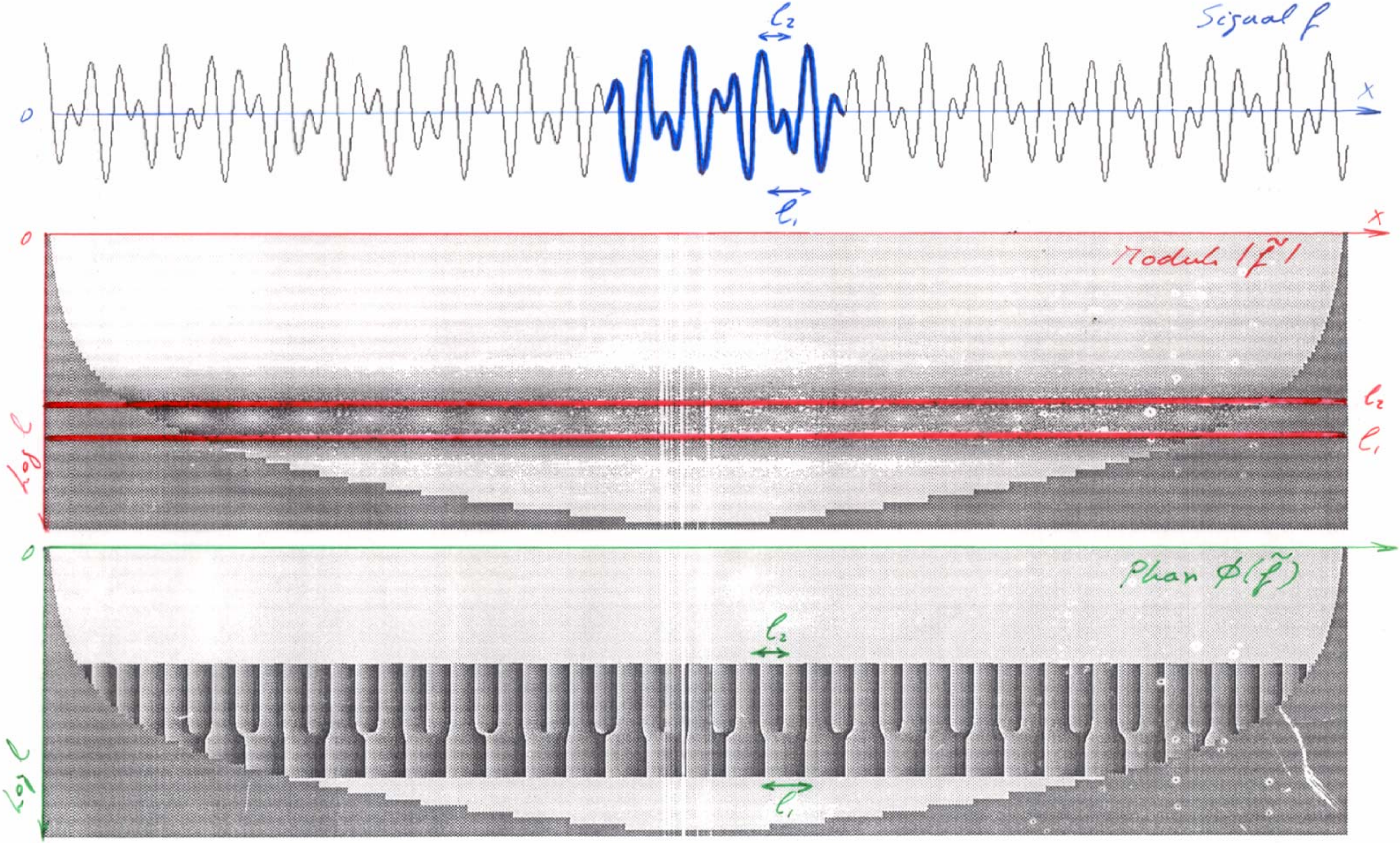


DELTA

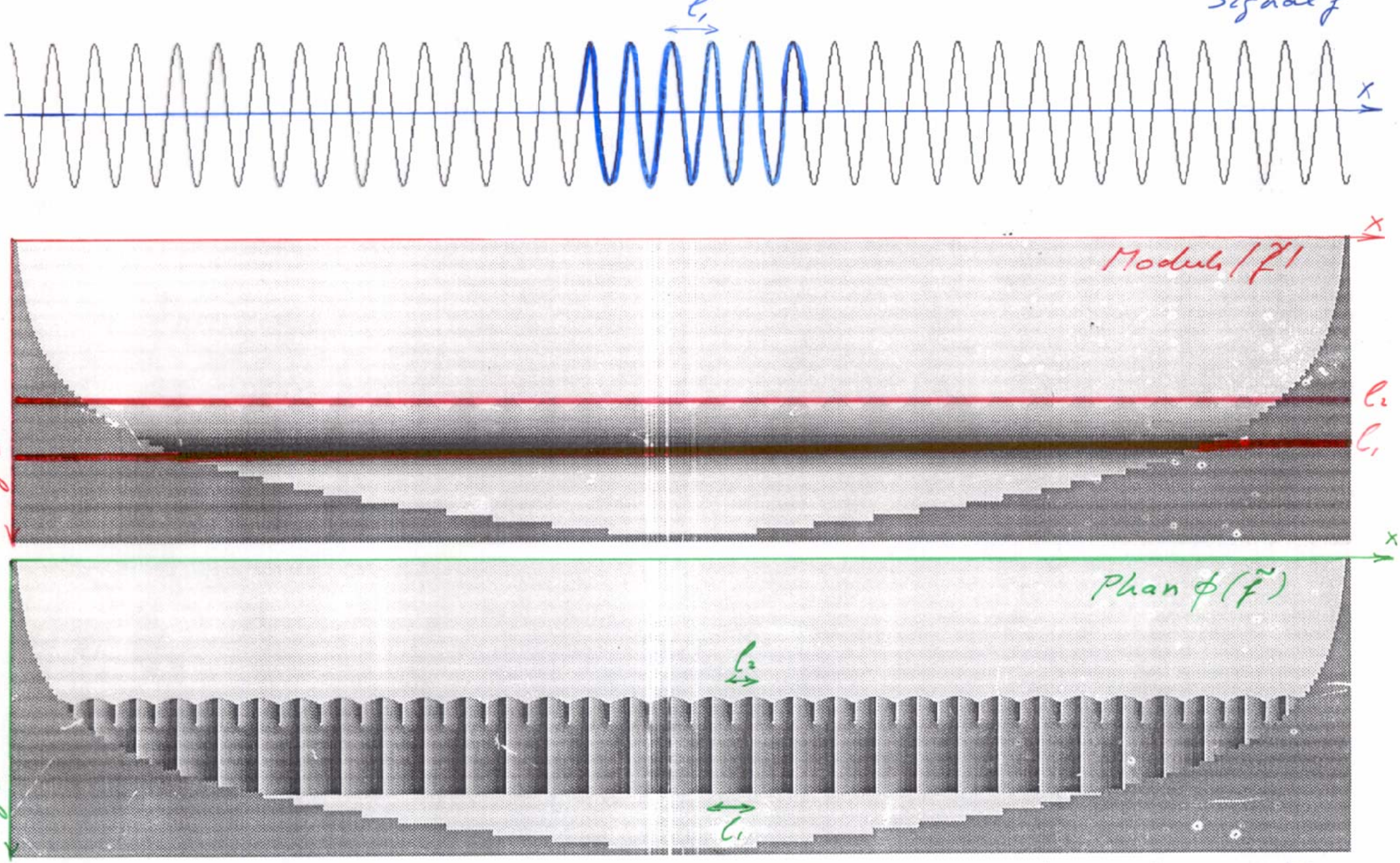


$f(x) = \delta(x_0)$
Signal f
0
x
0
x
Modulus $|\tilde{f}|$
$\tilde{f}(\xi,a) = \frac{1}{a}\,\psi\left(\frac{x_0-\xi}{a}\right) = \psi_{\xi,a}(x_0)$
log a
0
x
Phase $\Phi(\tilde{f})$
log a

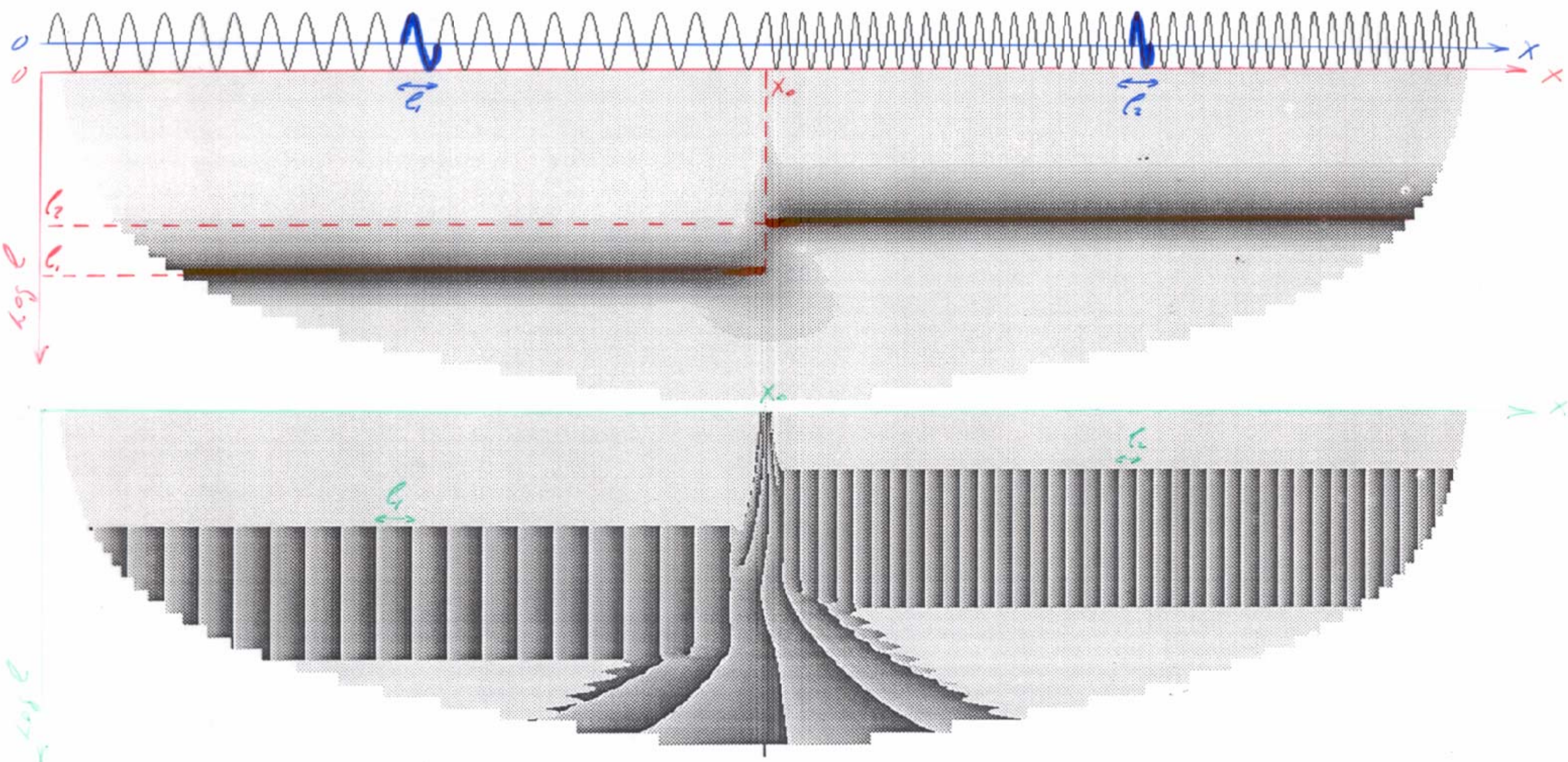
cos(t)+cos(1.68t)



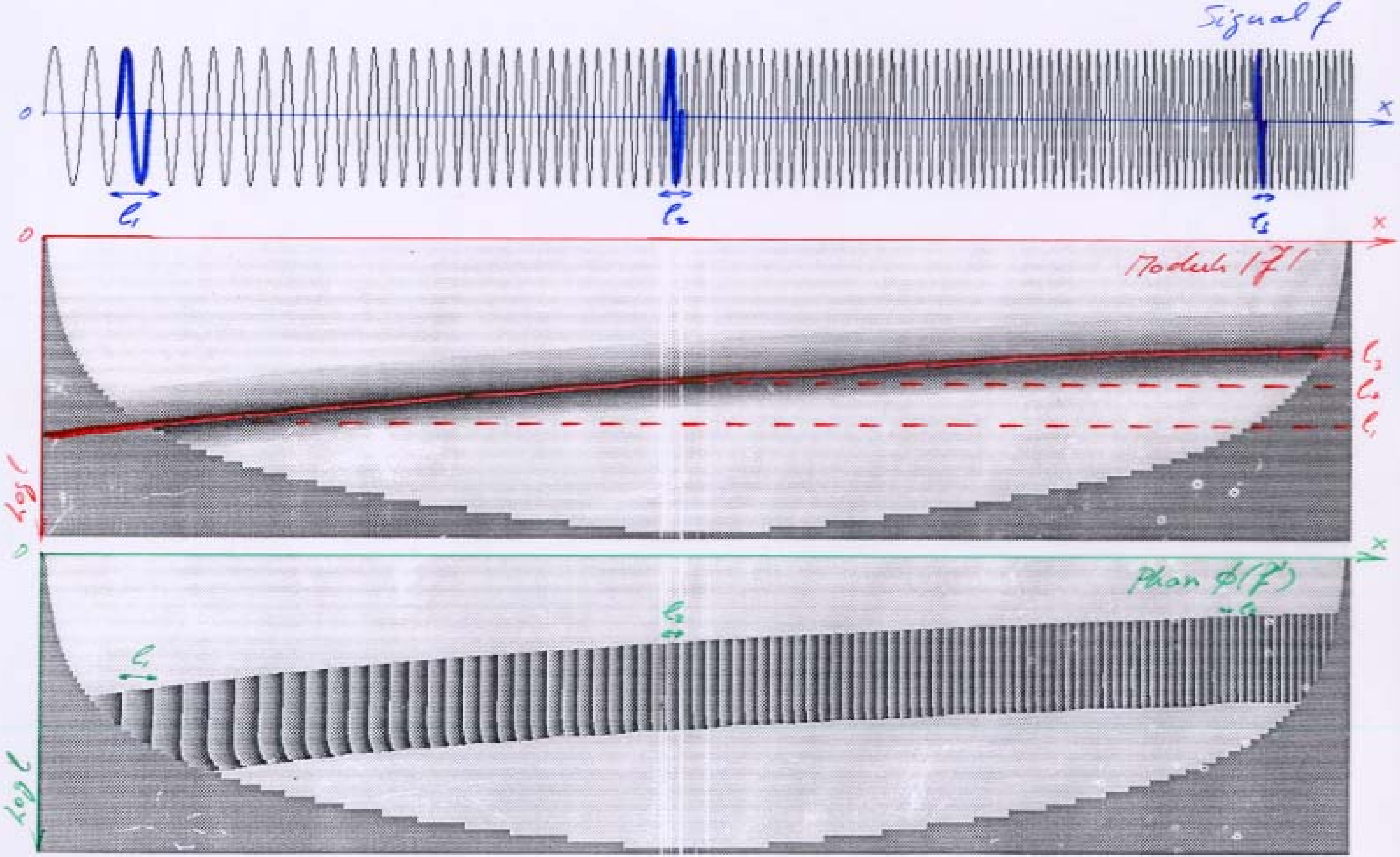
cos(t)+0.02cos(2t)



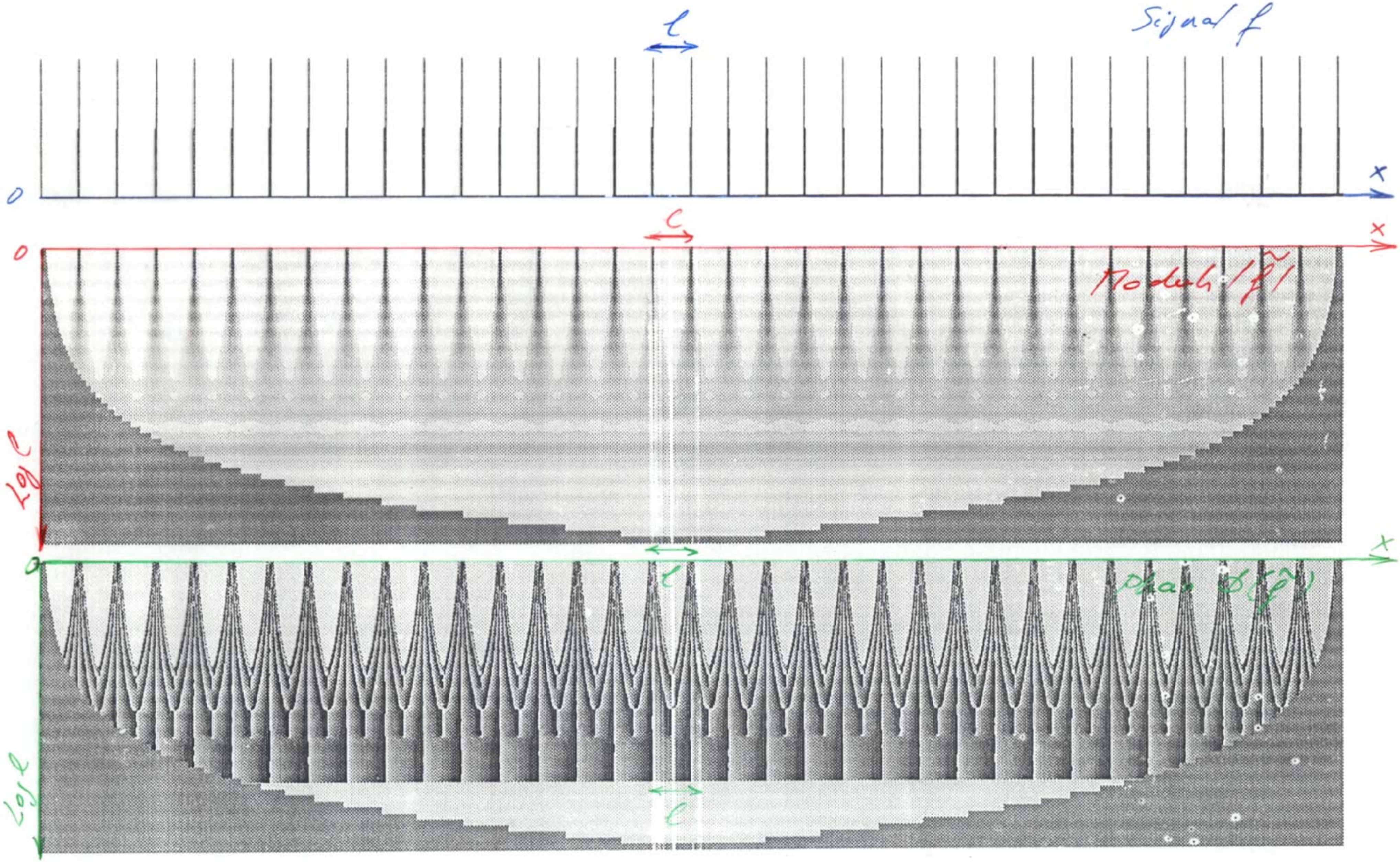
SIN(t) si t<512 SIN(2t) sinon



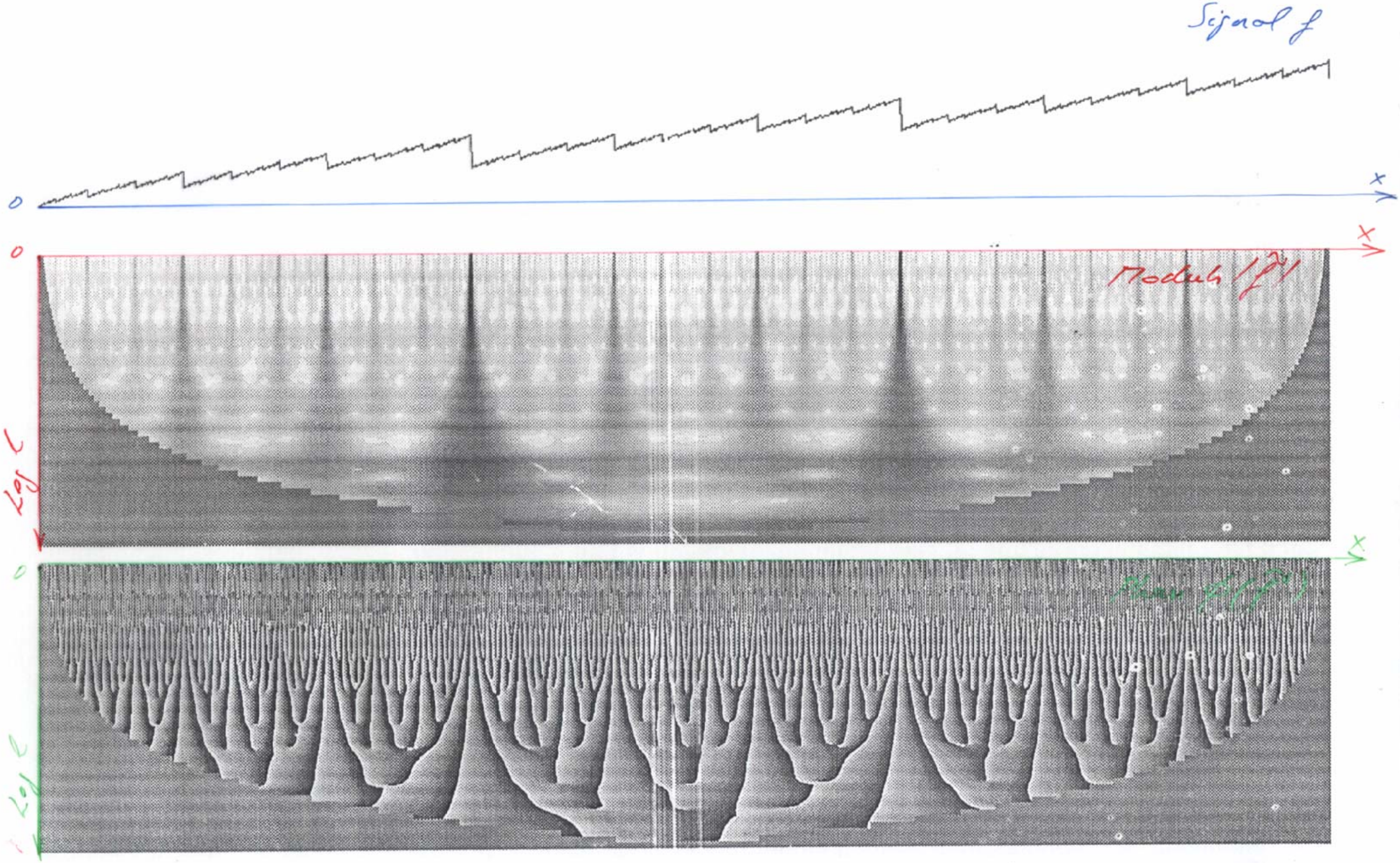
sin(t^2)



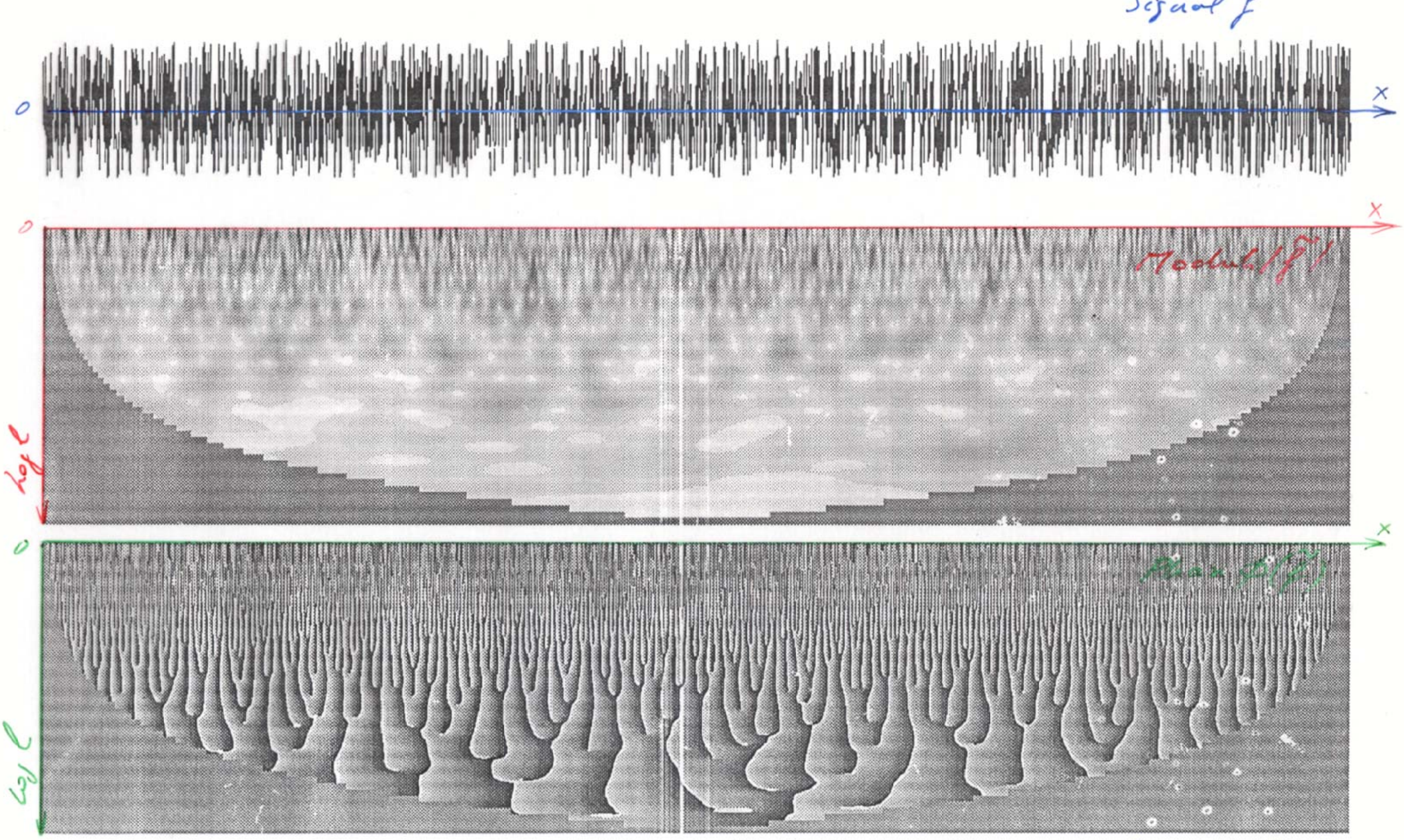
DELTA



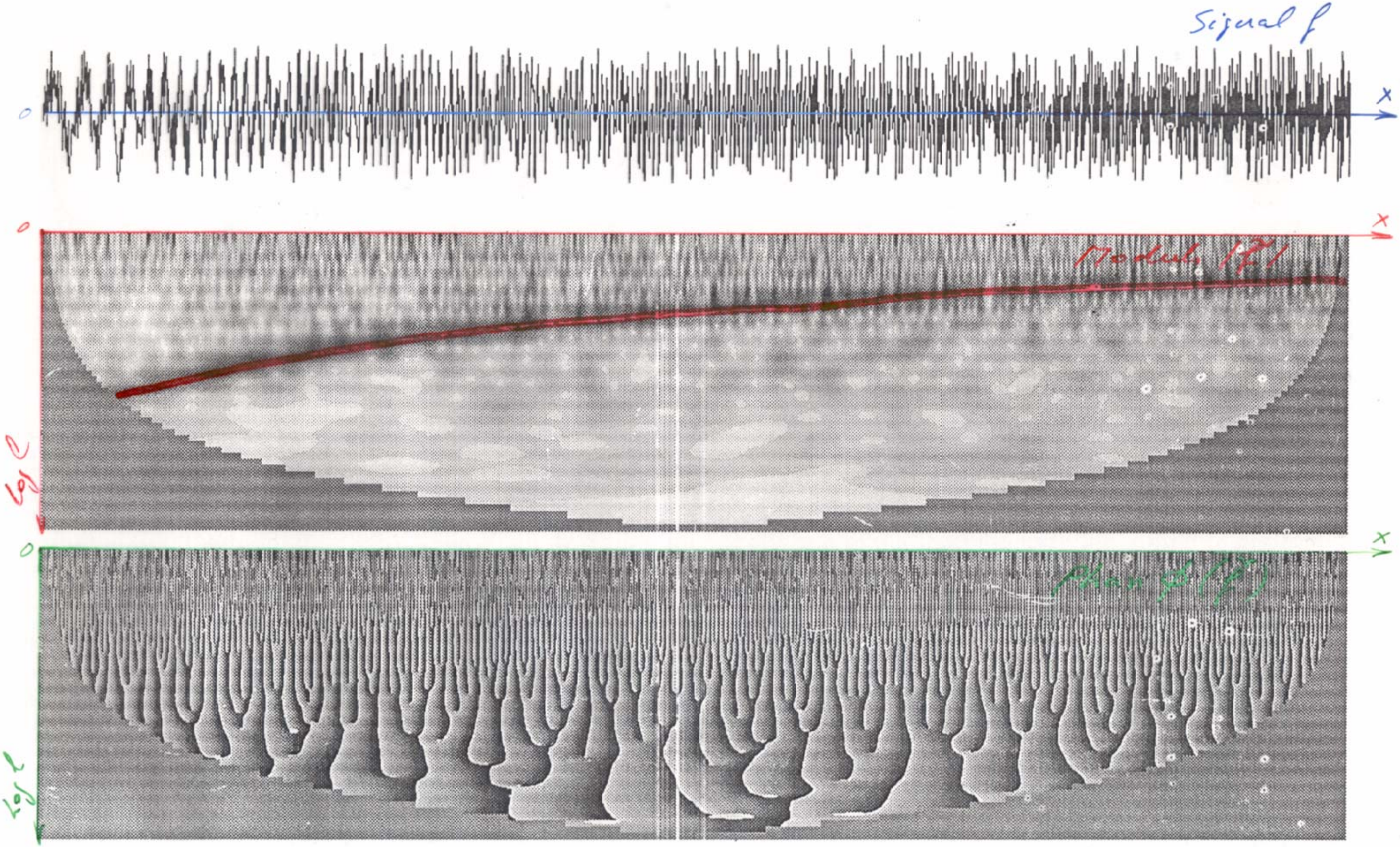
SAUTS



RANDOM



sinus(t^2)+random



LINEARITY

W is the continuous wavelet transform operator

$$W\left[f(x)+g(x)\right]=\tilde{f}(\varsigma,a)+\tilde{g}(\varsigma,a)$$

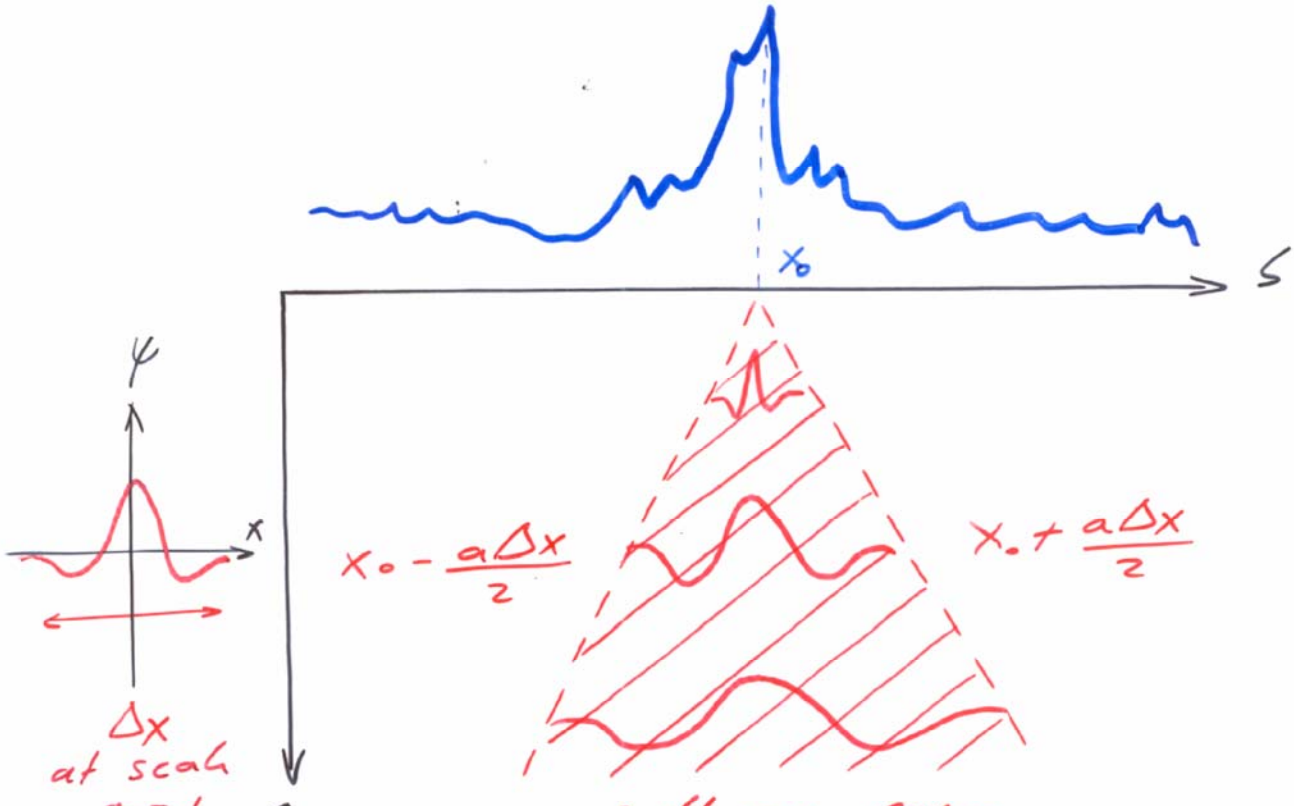
$$\Longrightarrow \quad W\left[\vec{V}\right] = \tilde{\vec{V}} \begin{vmatrix} \tilde{f}(x) \\ \tilde{f}(y) \\ \tilde{f}(z) \end{vmatrix}$$

with $\vec{V} \begin{vmatrix} f(x) \\ f(y) \\ f(z) \end{vmatrix}$

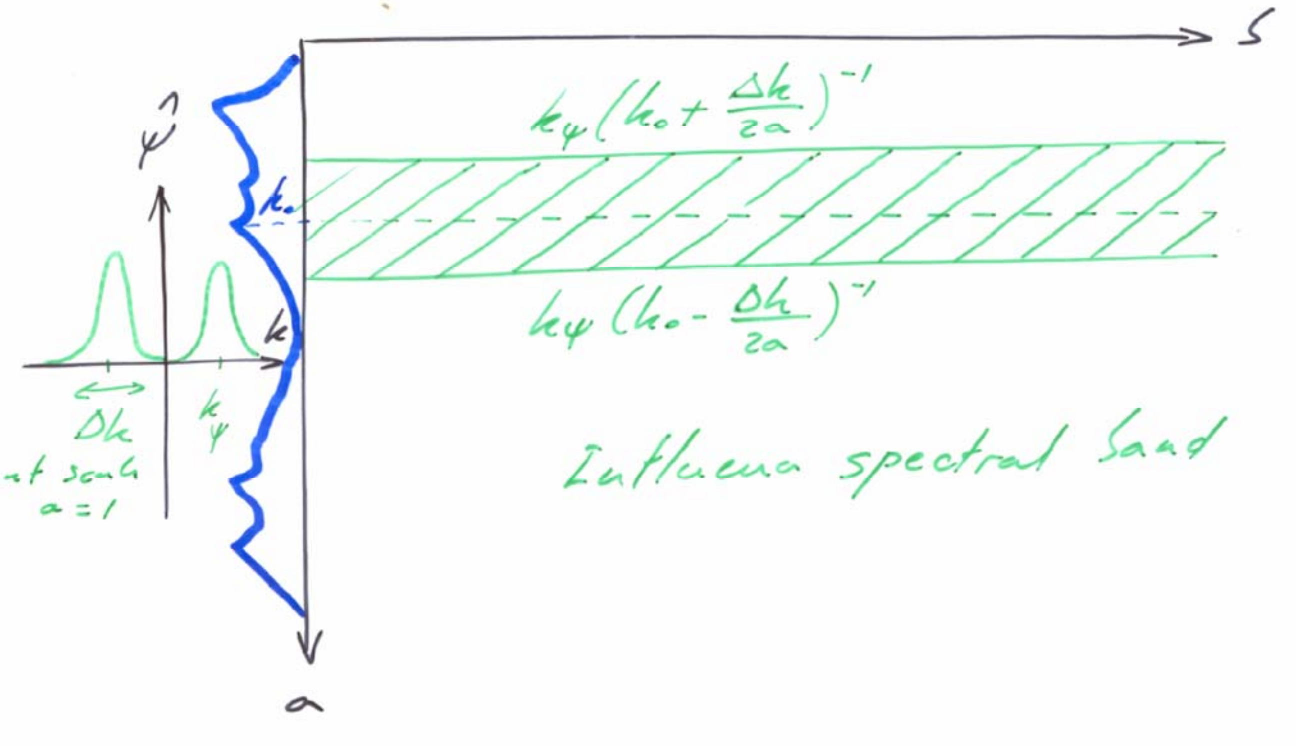
Differentiation

$$W\left[\frac{\partial^n f}{\partial x^n}\right]=(-1)^n\int_{-\infty}^{+\infty} f(x)\,\frac{\partial^n}{\partial x^n}\left[\overline{\psi_{\varsigma,a}}(x)\right]dx$$

CONSERVATION OF THE SPACE-SCALE LOCALITY



Influence cone



COVARIANCE BY TRANSLATION

$$W[f(x-x_0)] = \tilde{f}(a, b-x_0)$$

This implies that differentiation commutes with W:

$$\frac{\partial}{\partial x} W(f) = W\left(\frac{\partial f}{\partial x}\right)$$

$$\nabla[W(f)] = W(\nabla f)$$

$$\nabla \cdot [W(f)] = W(\nabla \cdot f)$$

A consequence of the covariance by translation is that the frequency of an harmonic signal can be read off from the phase of the wavelet coefficients. It corresponds to the number of zeros of the phase for a = cte.

COVARIANCE BY DILATION

$$W[f(\lambda x)] = \frac{1}{\lambda}\tilde{f}(\lambda a, \lambda b)$$

This is not the same as for the Fourier transform F:

$$F[f(\lambda x)] = \frac{1}{\lambda}\hat{f}\left(\frac{k}{\lambda}\right) = \frac{1}{\lambda}\hat{f}\left(\frac{2\pi}{\lambda \ell}\right)$$

ℓ wavelength

A consequence of the dilation covariance is that the wavelet transform of a power-law function is fully determined by its restriction to any line $a = $ Cste.

As a consequence, the lines of constant phase point out onto the singularities of the function f.

ENERGY CONSERVATION

The continuous wavelet transform is an isometry between

$L^2(\mathbb{R})$ and $H_\psi \subset L^2(\mathbb{R}^+ \times \mathbb{R})$

therefore it conserves energy:

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \frac{1}{C_\psi} \int_{0^+}^{+\infty} \int_{-\infty}^{+\infty} \tilde{f}(b,a) \overline{\tilde{f}}(b,a) \frac{da\, db}{a^2}$$

$$\text{with } C_\psi = 2\pi \int_{-\infty}^{+\infty} \frac{|\hat{\psi}(k)|^2 dk}{|k|}$$

It conserves energy both globally and locally for all coefficients inside the influence cone.

The total energy can also be splitted among contributions at different scales:

$$E(a) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} |\tilde{f}(b,a)|^2 \frac{db}{a^2}$$

REPRODUCING KERNEL

The projection $L^2(\mathbb{R}^+ \times \mathbb{R}) \longrightarrow H_\psi$ is an integral operator with kernel:

$$k(s', a', s, a) = \langle \psi_{s'a'} | \psi_{sa} \rangle$$

autocorrelation function of ψ.

Therefore $\tilde{f}(s, a) \in L^2(\mathbb{R}^+ \times \mathbb{R})$ is the continuous wavelet transform of a function f iff it satisfies the reproducing kernel equation:

$$\tilde{f}(s', a') = \int_{0^+}^{+\infty} \int_{-\infty}^{+\infty} k(s', a', s, a) \, \tilde{f}(s, a) \, \frac{da\, ds}{a^2}$$

INSTANTANEOUS FREQUENCY

Frequency is a characteristic of a wave-like (harmonic) signal, which can be measured experimentally, to describe the rapidity of the wave oscillations.

If this wave has a variable amplitude we can still define a frequency iff the amplitude varies much slower than the oscillations, stationary phase hypothesis, namely:

$$f(t) = \underbrace{A(t)}_{\text{amplitude}}\, e^{\underbrace{i\varphi(t)}_{\text{phase}}} + \underbrace{n(t)}_{\text{noise}}$$

with $\frac{\partial A}{\partial t}$ and $\frac{\partial \varphi}{\partial t} << |e^{i\varphi}| = 1$

The instantaneous frequency is

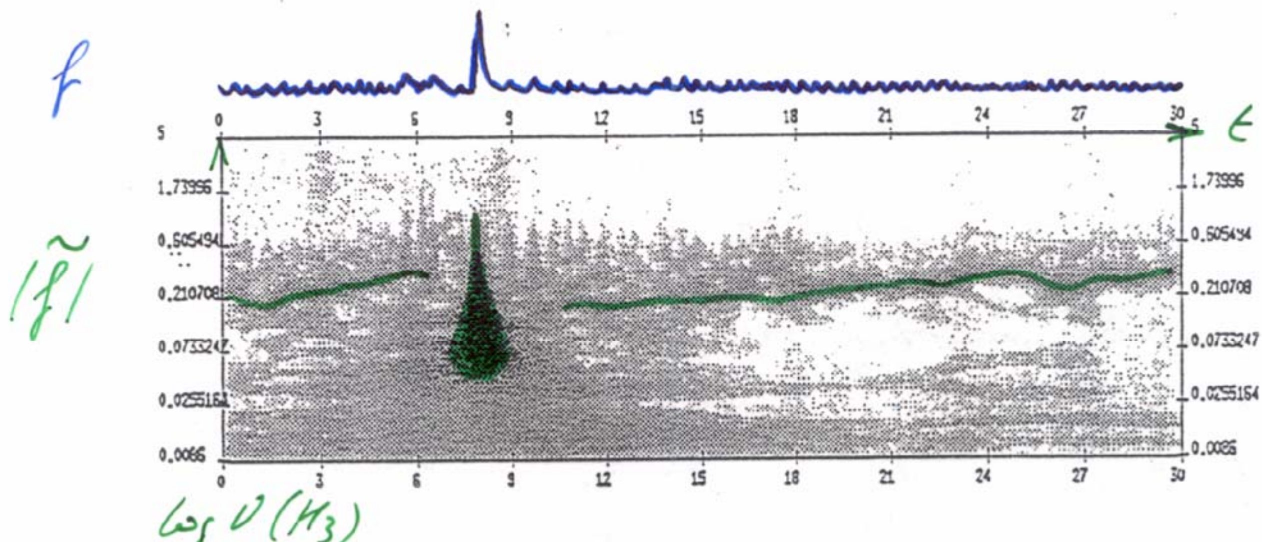
$\nu(t) = \frac{\nu_\varphi}{2\pi} \frac{\partial \varphi}{\partial t}$ which defines

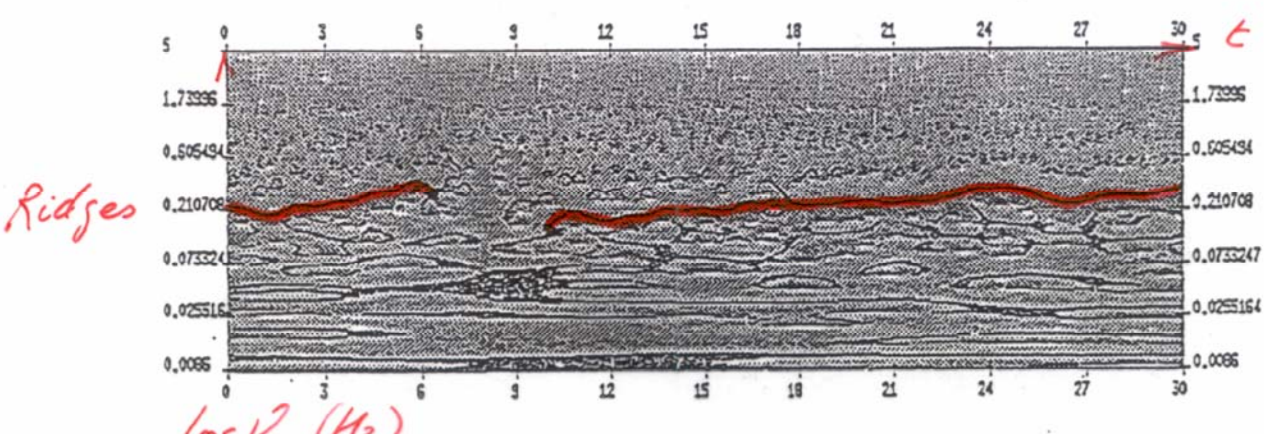
the ridge of the wavelet transform, curve in phase-space (b, a) where $\frac{\partial \varphi}{\partial t} = \text{cte}$

1) *Figure 1: Calcul de transformées en ondelettes (signal respiratoire)*

MEDICAL SIGNAL

De haut en bas, le signal, le module, et la fréquence instantanée de la transformée en ondelettes (les lignes continues noires représentent les arètes de la transformée) (On s'intéresse à l'évolution temporelle des fréquences instantanées).





Ex. Virgo: $10 \log_{10} \frac{P_S}{P_B} = -40\,dB$ soit $\frac{P_S}{P_B} = 10^{-4}$

Reference:

Carmona, Hwang and Torresani
Characterization of signals by the ridges of their wavelet transforms, 1995
anonymous ftp from chelsea.math.uci.edu
http://cpt.SX1.univ-mrs.fr

CHARACTERIZATION OF THE LOCAL SCALING OF A SIGNAL

The signal f can be a function, a distribution or a measure, that we want to study in x_0.

$$\|\tilde{f}(b,a)\|_{L^1} \leq M_1 |b-x_0|^\alpha \|\psi\|_{L^1} + M_1 M_\alpha a^\alpha$$

with $M_\alpha = \int_{-\infty}^{+\infty} x^\alpha \psi(x)\, dx$

α is the degree of differentiability of f at x_0 if $\alpha \geq 1$, i.e. f regular in x_0 or α is the Lipschitz exponent of the singularity in x_0 if $-1 < \alpha < 1$.

If we want to eliminate the most regular (polynomial) contribution of f we have to choose ψ with cancellations up to order m (then $M_m = 0$).
In this case $\|\tilde{f}(b,a)\|_{L^1}$ will only react to regions where f is less smooth as order m.

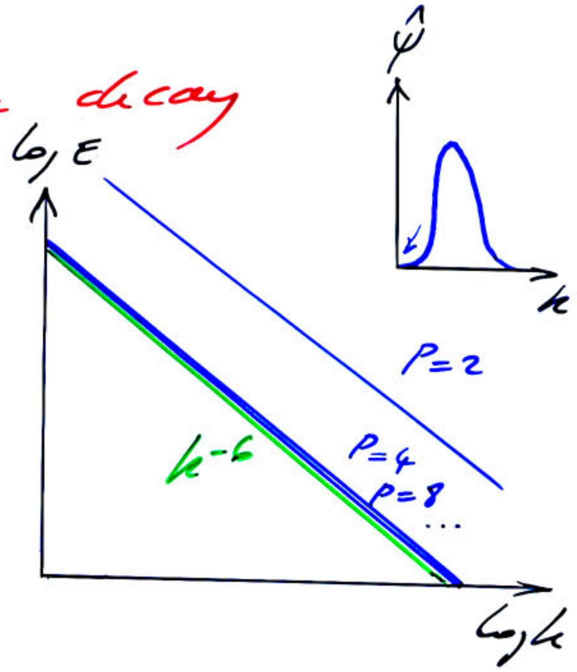
SPECTRAL ANALYSIS USING WAVELETS

$$\tilde{E}(k) = \frac{1}{C_\psi k}\int_0^{+\infty} \underbrace{E(k')}_{\text{Energy of the signal}} \; \underbrace{\left|\hat{\psi}\left(\frac{k_0 k'}{k}\right)\right|^2}_{\text{Energy of the wavelet}} dk'$$

- Signal with algebraic decay of its spectrum

$$E(k) = k^{-\alpha}$$
for $k \to \infty$

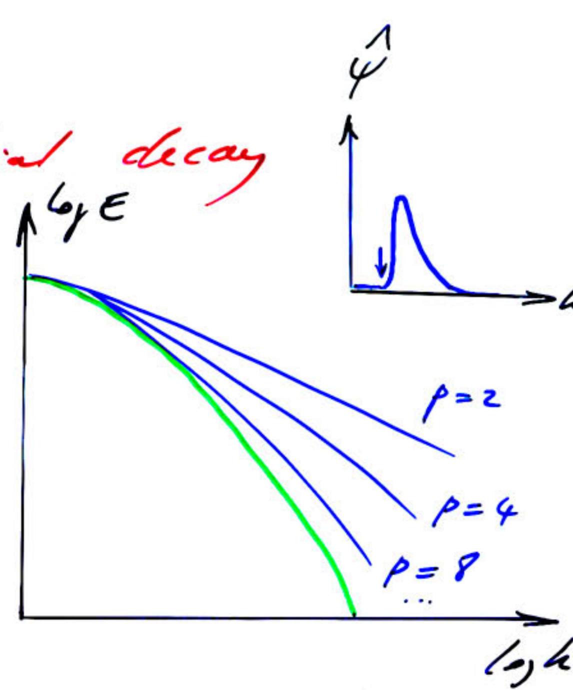
The wavelet should have at least
$p > \frac{\alpha - 1}{2}$ cancellations
to detect the algebraic scaling $k^{-\alpha}$.



- Signal with exponential decay of its spectrum

$$E(k) = e^{-k^2}$$

The wavelet should have ∞ly many cancellations:
ex: Meyer or Paul's wavelet.



CWT ALGORITHM (1)

1D periodic signal $f(x)$

1D periodic wavelet $\psi(x)$

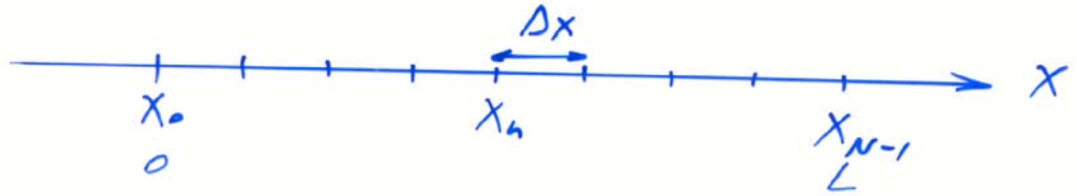
$x \in [0, L]$ with period L

Space discretization

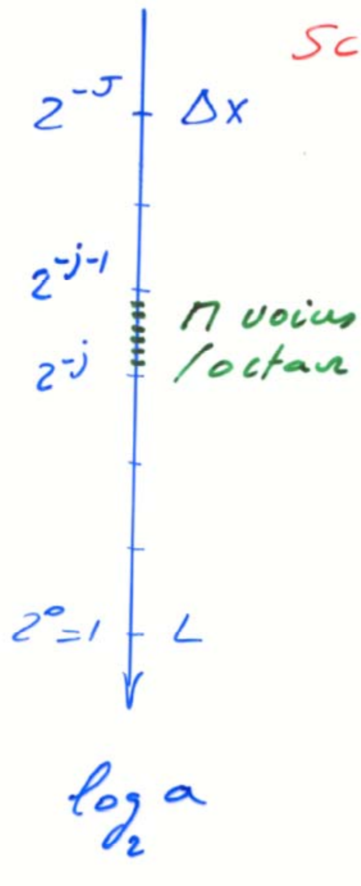
Sampling on a regular grid

$\Delta x = \frac{L}{N}$ with $N = 2^J$ samples

$\Longrightarrow$ $x_n = \frac{nL}{N}$ and $b_n = \frac{nL}{N}$



Scale discretization



Logarithmic sampling

$a_j = a_0^{-j}$ with $j \geqslant 0$

$a_0 = 2^{1/M}$ M number of voices/octave

$\Longrightarrow$ MJ samples for scale

Largest scale $L \longrightarrow a_0 = 1$

Smallest scale $\Delta x \longrightarrow a_J = 2^{-J}$

CWT ALGORITHM (2)

L2-norm wavelet family

$$\psi_{a,s}(x) = \frac{1}{\sqrt{a}} \psi\left(\frac{x-s}{a}\right)$$

$$\Longrightarrow \hat{\psi}_{a,s}(k) = \sqrt{a}\, \hat{\psi}(ak)\, e^{-2\pi i s k}$$

Wavelet coefficients

$$\tilde{f}(a,s) = \int_{-\infty}^{+\infty} f(x)\, \overline{\psi_{a,s}(x)}\, dx$$

$$\overset{\text{Parseval}}{=} \int_{-\infty}^{+\infty} \hat{f}(k)\, \overline{\hat{\psi}_{a,s}(k)}\, dk$$

$$= \sqrt{a} \int_{-\infty}^{+\infty} \hat{f}(k)\, \hat{\psi}(ak)\, e^{2\pi i k s}\, dk$$

Discrete wavelet coefficients

$$\tilde{f}(a_j, s_n) = a_j^{1/2} \sum_{k=0}^{N-1} \hat{f}_k\, \hat{\psi}(a_j k)\, e^{2\pi i s_n k}$$

where $\hat{f}_k = \frac{1}{N} \sum_{n=0}^{N-1} f(x_n)\, e^{-2\pi i x_n k}$

with $x_n = \frac{nL}{N}$, $s_n = \frac{nL}{N}$