REPRODUCING KERNEL

The projection $L^2(\mathbb{R}^+ \times \mathbb{R}) \to H_\psi$ is an integral operator with kernel:

$$k(s',a',s,a) = \langle \psi_{s'a'} | \psi_{sa} \rangle$$

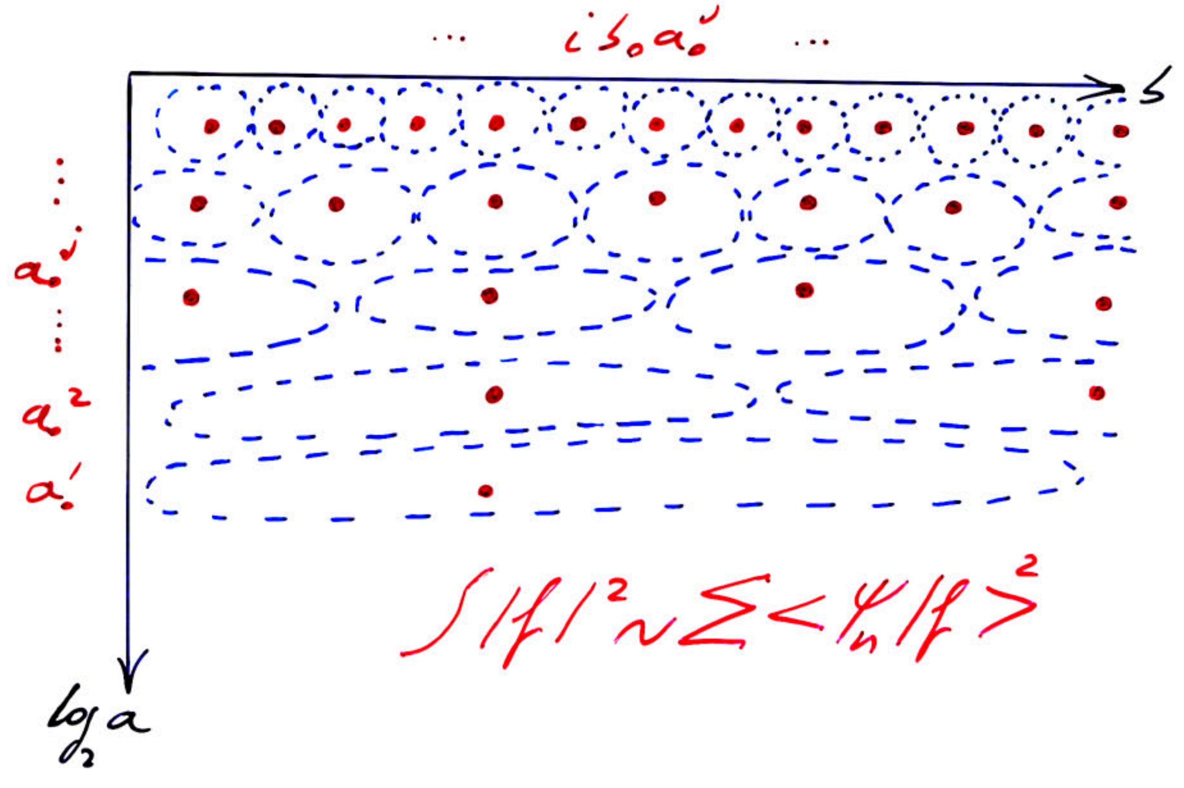
autocorrelation function of ψ.

Therefore $\tilde{f}(s,a) \in L^2(\mathbb{R}^+ \times \mathbb{R})$ is the continuous wavelet transform of a function f iff it satisfies the reproducing kernel equation:

$$\tilde{f}(s',a') = \int_{0^+}^{+\infty} \int_{-\infty}^{+\infty} k(s',a',s,a)\, f(s,a)\, \frac{da\, ds}{a^2}$$

DISCRETIZATION OF THE WAVELET COEFFICIENT PHASE SPACE

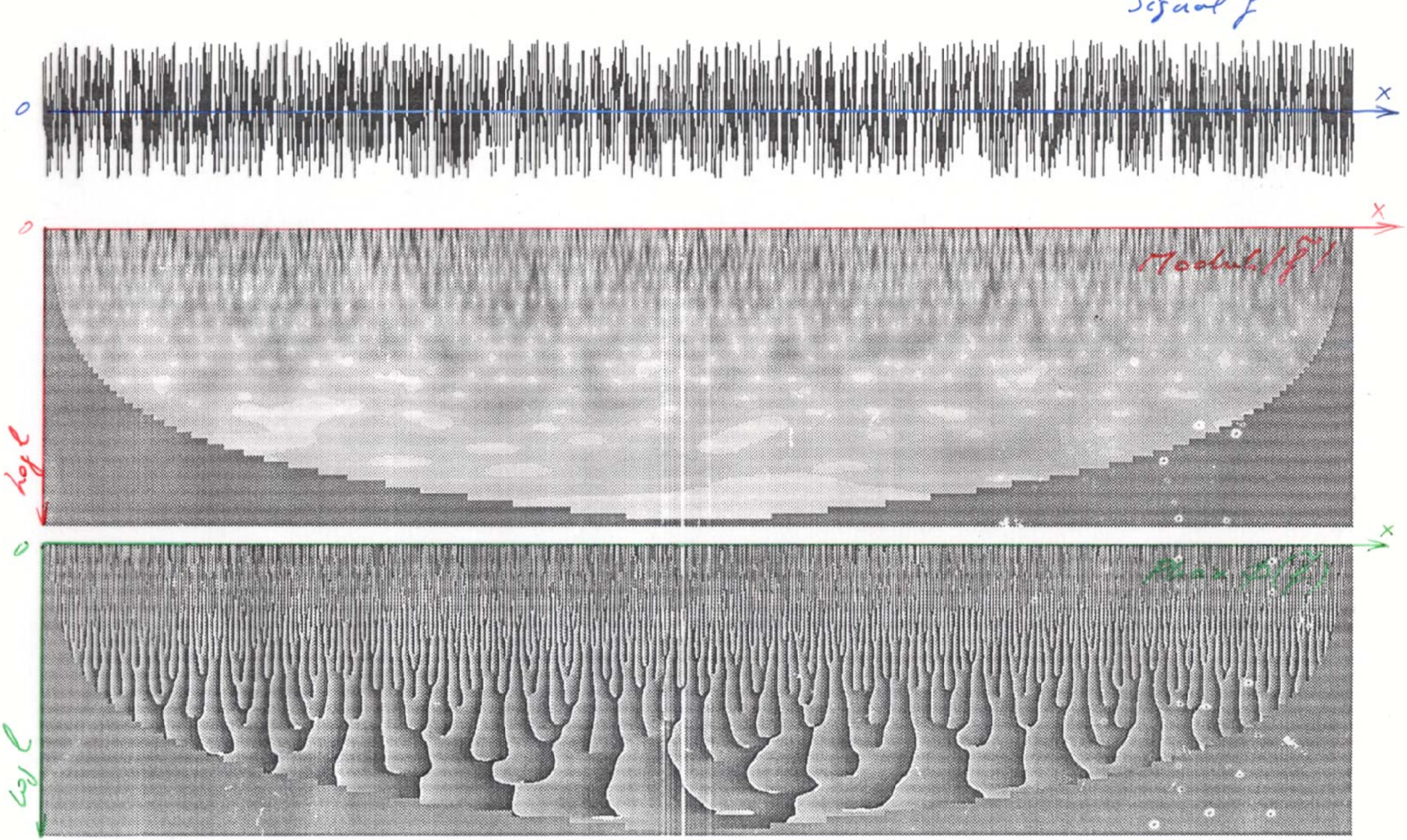
The continuous wavelet transform has a reproducing kernel $K\left(\frac{a'}{a}, \frac{b-b'}{a}\right)$ corresponding to the redundancy between the wavelet coefficients.



$$\int |f|^2 \sim \sum \langle \psi_n | f \rangle^2$$

We can then restrict the wavelet coefficients to a discrete grid optimally chosen such that the wavelet family associated to this grid constitutes quasi-orthonormal basis ('Frame').

RANDOM



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PHASE - SPACE DISCRETIZATION

It is possible to select a subset F out the family of wavelets ψ_{as} such that:

$$a = a_0^{j}$$
$$s = i s_0 a_0^{j}$$

$a_0 \in \mathbb{Z}^+$ dilation unit
$s_0 \in \mathbb{Z}$ translation unit

We obtain the set of discretized wavelets

$$\psi_{ji}(x) = a_0^{-j/2}\, \psi(a_0^{-j} x - i s_0)$$

and the wavelet analysis is

$$\tilde{f}_{ji} = \langle \psi_{ji} | f \rangle = \int_{-\infty}^{+\infty} f(x)\, \psi_{ji}\, dx$$

FRAMES

Painless nonorthogonal expansions
Ingrid Daubechies, Alex Grossmann
and Yves Meyer
J. Math. Phys. 27 (5), 1986

A frame is a discrete subset of independent wavelet coefficients such that the representation is complete

$\Rightarrow$ we can reconstruct any function $f \in \mathcal{H}$, Hilbert space of square integrable functions, from the discrete subset of wavelet coefficients.

The family $\{\psi_j\}$ is an orthogonal family of $\mathcal{H}$ iff and only if f is completely characterized by the numbers $\langle \psi_j | f \rangle$ such that $\forall j \; \langle \psi_j | f \rangle = \langle \psi_j | g \rangle$

$$\Rightarrow f = g.$$

TIGHT FRAMES

If $A = B$ we have a tight frame and the reconstruction becomes then

$$f = \frac{1}{A} \sum_{ji} \langle \psi_{ji} | f \rangle \psi_{ji} + \text{Residue}$$

In the case of an orthogonal basis then $A = 1$ and the residue becomes zero.

The optimal frame is the one which minimizes the residue.

For the Mexican hat it corresponds to: $a_0 = 2^{1/2}$ $b_0 = 1/2$

For Morlet with $k_0 = 3$: $a_0 = 2$ $b_0 = 1$ which corresponds to the dyadic grid.

FRAME BOUNDS

In order to be able to reconstruct the function f out of the discrete set of wavelet coefficients $\tilde{f}_{ji}$ we need the condition

$$A \|f\|^2 \leq \sum_{ji} |\langle \psi_{ji} | f \rangle|^2 \leq B \|f\|^2$$

$\forall f \in \mathcal{H}$
Hilbert space

The set of vectors $\{\psi_{ji}\} \in L^2(R)$ satisfying this condition with $A, B > 0$ is called a frame and A and B are the frame bounds.

ORTHOGONAL WAVELETS

There exist special wavelets

$$\psi_{ji}(\vec{x}) = 2^{j/2}\,\psi(2^j\vec{x} - i) \qquad j \in \mathbb{N},\ i \in \mathbb{Z}$$

which are orthogonal to all their translates by discrete steps $x = 2^{-j} i$ and to all their dilates by $l = 2^{-j}$.

In addition the family ψ_{ji} is complete in $L^2(\mathbb{R}^n)$ $\longrightarrow$ they constitute an orthogonal basis for $L^2(\mathbb{R}^n)$.

Analysis:

$$\tilde{f}_{ji} = \langle \psi_{ji} | f \rangle = 2^{j/2} \int_{-\infty}^{+\infty} f(\vec{x})\,\psi(2^j\vec{x} - i)\,d\vec{x}$$

Synthesis:

$$f(\vec{x}) = \sum_{j=-\infty}^{+\infty} \sum_{i=-\infty}^{+\infty} \tilde{f}_{ji}\,\psi_{ji}(\vec{x})$$

Energy conservation:

$$\int_{-\infty}^{+\infty} |f(\vec{x})|^2\,d\vec{x} = \sum_{j=-\infty}^{+\infty} \sum_{i=-\infty}^{+\infty} |\tilde{f}_{ji}|^2$$

'MOTHER' WAVELET

$\psi(x)$ should have the following properties:

1. Smoothness
   it should have $(r-1)$ continuous derivatives and a bounded derivative of order r,
2. Rapid decay at infinity
   for ψ and its derivatives of order $\leq r$,
3. $\psi_{ji}(x) = 2^{j/2}\,\psi(2^j x - i)$ should be an orthonormal basis of $L^2(\mathbb{R})$.

1909, Alfred Haar proposed:

$$\psi(x) = \begin{cases} 1 & \text{for } x \in [0, 1/2] \\ -1 & \text{for } x \in [1/2, 1] \\ 0 & \text{elsewhere} \end{cases}$$

but it is not smooth $(r=0)$.

'About 80 years were needed until Ingrid Daubechies proved that for $r \geq 1$ one can construct $\psi(x)$ of class C^r with compact support satisfying the above conditions [...] If $\psi(x)$ satisfies them and is compactly supported, P. G. Lemarié has proven that there exists multiresolution analysis behind this orthogonal basis!'

Y. Meyer

MULTIRESOLUTION ANALYSIS (MRA)

Orthogonal wavelets are the mathematical tool to describe the addition of details needed to go from a coarse approximation $\bar{f}_j$ to a higher approximation $\bar{f}_{j+1}$.

For this we need two functions

the scaling function $\phi(\vec{x})$ such that $\int_{-\infty}^{+\infty} \phi(\vec{x})\, d\vec{x} = 1$

which will be used to compute the approximations $\bar{f}_j$ at different scales

and

the mother wavelet $\psi(\vec{x})$ such that $\int_{-\infty}^{+\infty} \psi(\vec{x})\, d\vec{x} = 0$

which will be used to compute the details $\tilde{f}_j$ to go from scale j to scale $j+1$.

NESTED ⊥ SUBSPACES

We generate the family

$\phi_{ji}(\vec{x}) = 2^{j/2}\phi(2^j\vec{x} - i) \qquad j \in N,\ i \in Z$

of orthogonal functions (for fixed j), which are also orthogonal to $\psi_{j'i'}$ for $j' \geq j$.

Having met this orthogonality condition, the scaling functions ϕ_{ji} generate a set of nested subspaces $V_0 \subset V_1 \subset \ldots \subset V_j \subset V_{j+1}$ which correspond to smooth approximations $\bar{f}_j(\vec{x})$ of $f(\vec{x})$ at different octaves $l = 2^{-j}$, while the associated wavelets ψ_{ji} generate their orthogonal complementary subspaces

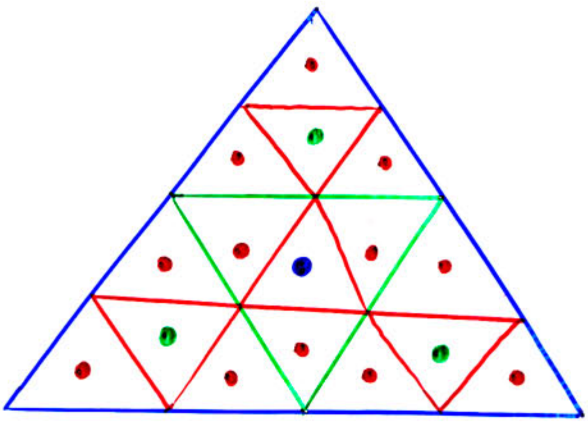
$W_0 \perp W_1 \perp \ldots \quad W_j \perp W_{j+1}$

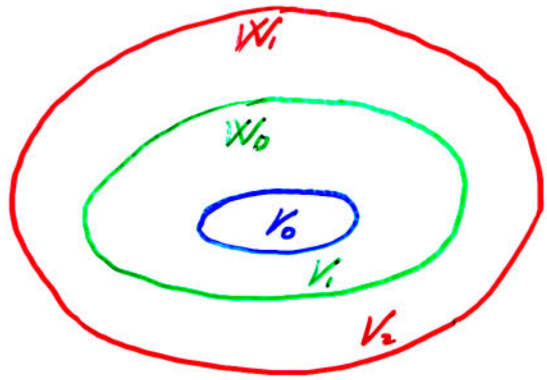
such that $V_{j+1} = V_j \oplus W_j$

corresponding to the details $\tilde{f}_j(\vec{x})$

EXAMPLE

Multiresolution analysis is a natural concept since mesh refinement provide a trivial example:





V approximation spaces

$$V_0 \subset V_1 \subset V_2$$

$$V_1 = V_0 \oplus W_0$$

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1$$

W orthogonal complement spaces

$\bar{f}_0 = f * F_\phi$

$\bar{f}_0 \in V_0$

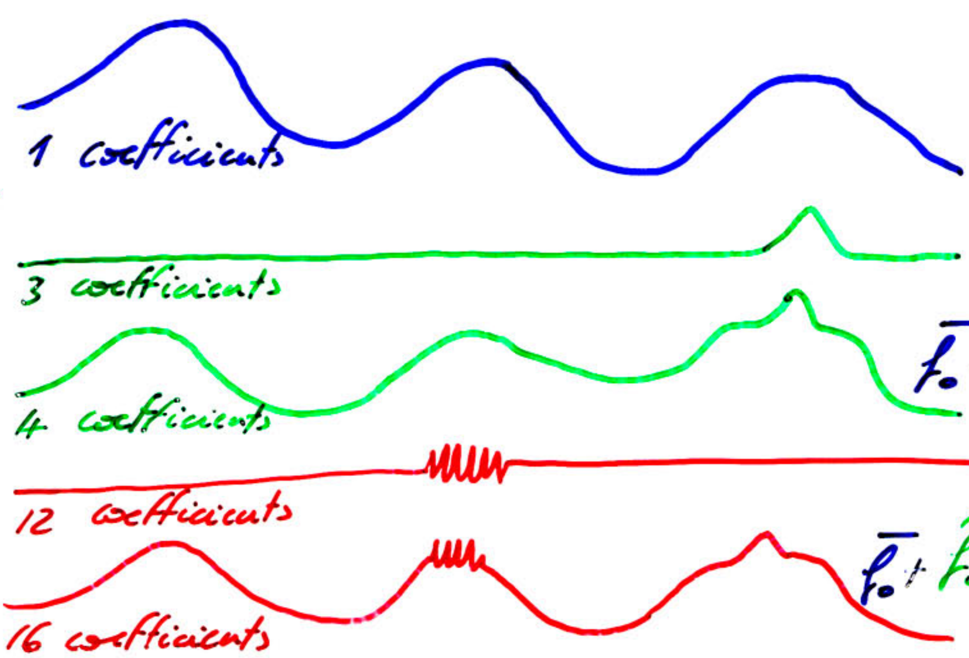
$\tilde{f}_0 = f * F_\psi$

$\tilde{f}_0 \in W_0$

$\bar{f}_1 \in V_1$

$\tilde{f}_1 \in W_1$

$\bar{f}_2 \in V_2$



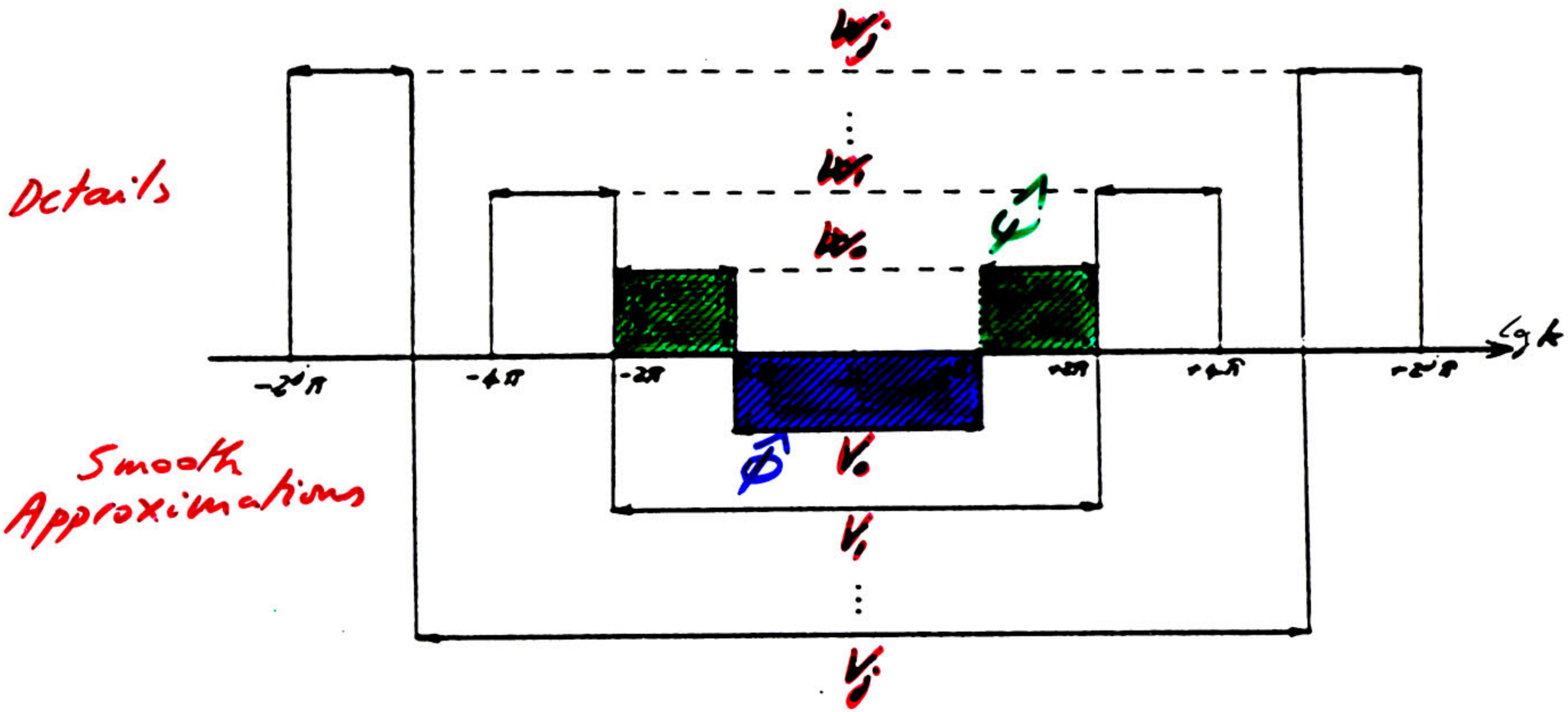
$\bar{f}_0 + \tilde{f}_0$

$\bar{f}_0 + \tilde{f}_0 + \tilde{f}_1$

FILTERS ASSOCIATED TO SCALING FUNCTIONS AND WAVELETS

ψ Wavelets

Discrete filters associated to wavelets ψ



Discrete filters associated to scaling functions ϕ

ϕ Scaling functions

QUADRATURE MIRROR FILTERS

In fact $\phi(\vec{x})$ and $\psi(\vec{x})$ are not known explicitly but computed using a set of quadrature mirror filters,

namely

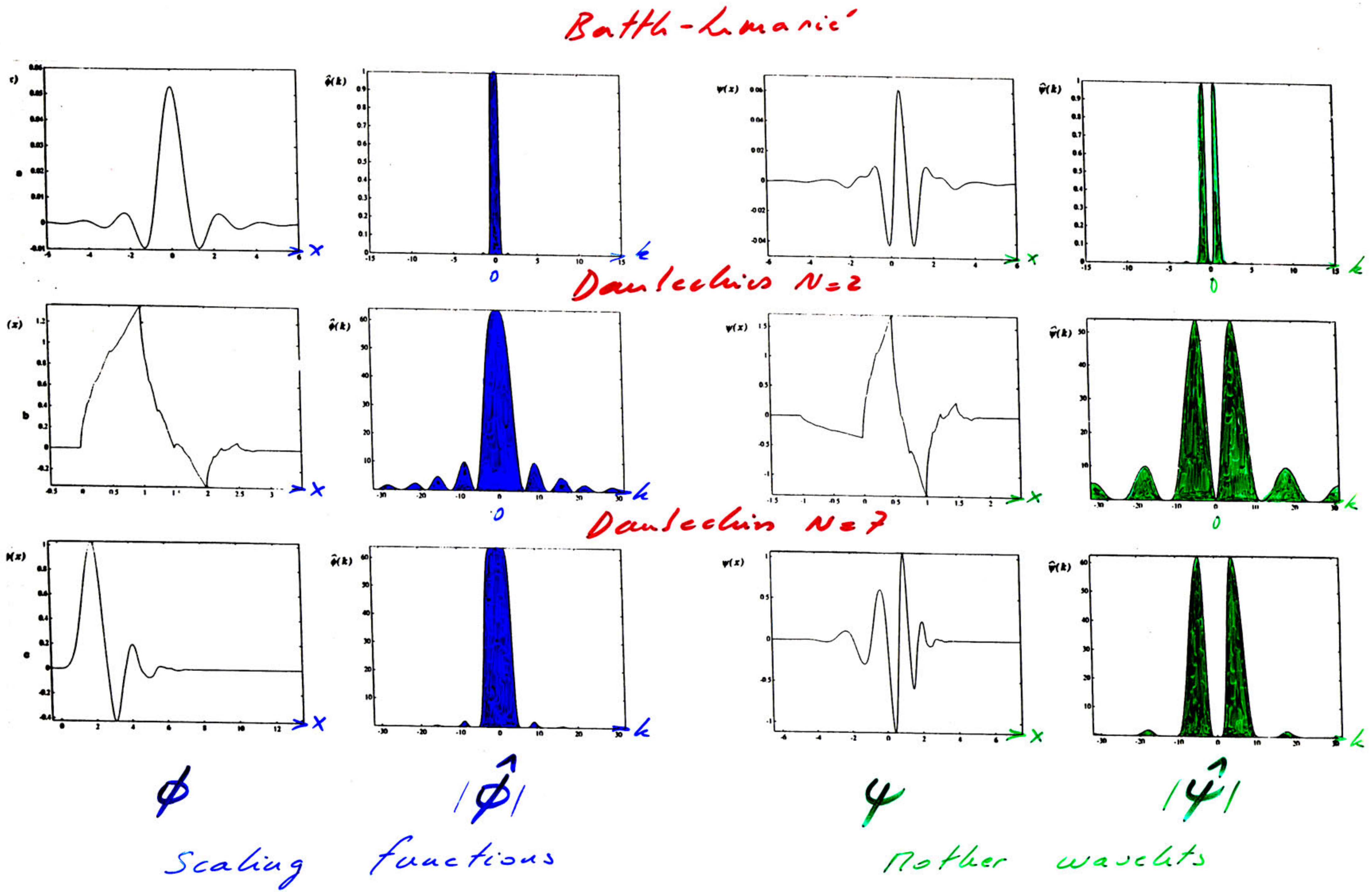
the discrete low-pass filter F_ϕ to obtain $\phi(\vec{x}) = \sqrt{2} \sum_{i=-\infty}^{+\infty} F_\phi \, \phi(2\vec{x}-i)$

and

the discrete high-pass filter F_ψ to obtain $\psi(\vec{x}) = \sqrt{2} \sum_{i=-\infty}^{+\infty} F_\psi \, \psi(2\vec{x}-i)$.

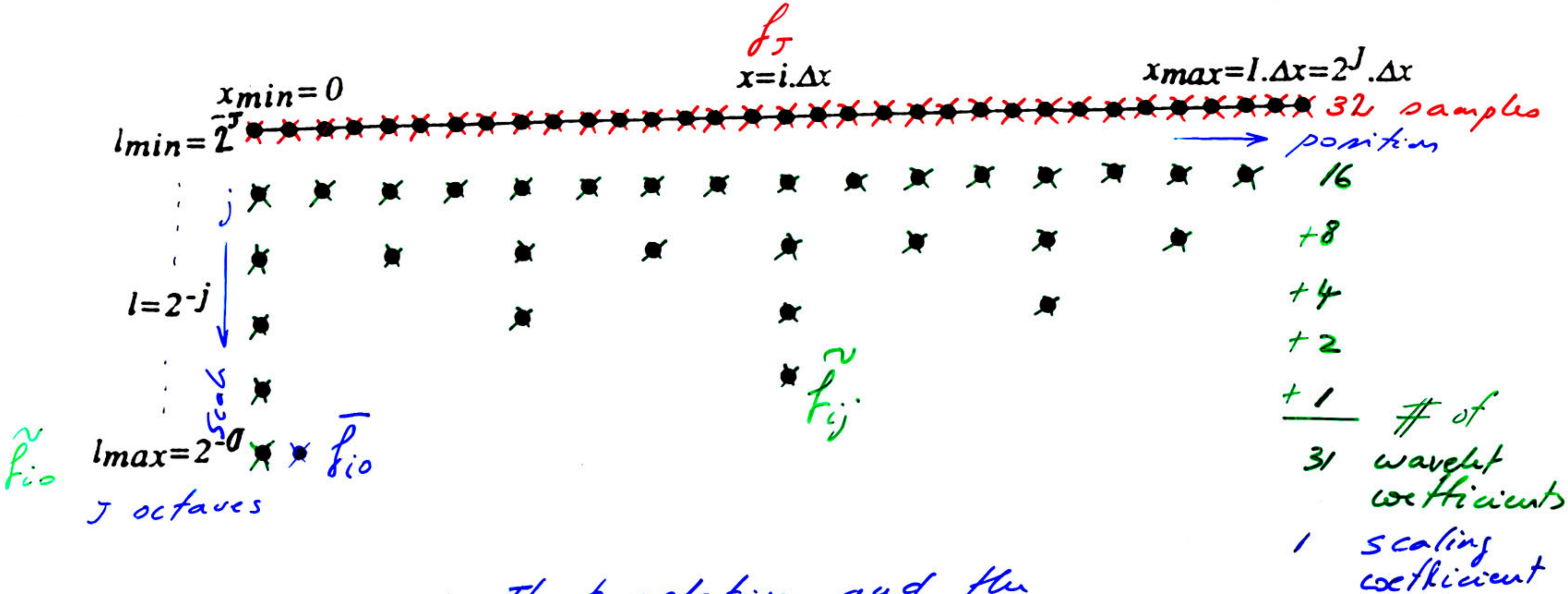
The orthogonality condition between the dilated of $\phi(\vec{x})$ and $\psi(\vec{x})$ lead to the quadrature relation between the filters:

$$F_{\psi i} = (-1)^i F_{\phi(1-i)}$$



Battle-Lemarié
Daubechies N=2
Daubechies N=7
ϕ
$|\hat{\phi}|$
ψ
$|\hat{\psi}|$
Scaling functions
Mother wavelets

DYADIC GRID



The translation and the dilation invariance are lost.

ORTHOGONAL WAVELET DECOMPOSITION

$\ell = 2^{-j}$:

$\ell = 2^{-J} = 2^{-3} = \frac{1}{8}$ sampling

Signal f_3

$* F_\phi$ → Trend $\bar{f}_2$ $\qquad$ $* F_\psi$ → Fluctuation $\tilde{f}_2$

$\ell = 2^{-2} = \frac{1}{4}$ smallest scale

$* F_\phi$ → $\bar{f}_1$ $\qquad$ $* F_\psi$ → $\tilde{f}_1$

$\ell = 2^{-1} = \frac{1}{2}$

$* F_\phi$ → $\bar{f}_0$ $\qquad$ $* F_\psi$ → $\tilde{f}_0$

$\ell = 2^{0} = 1$

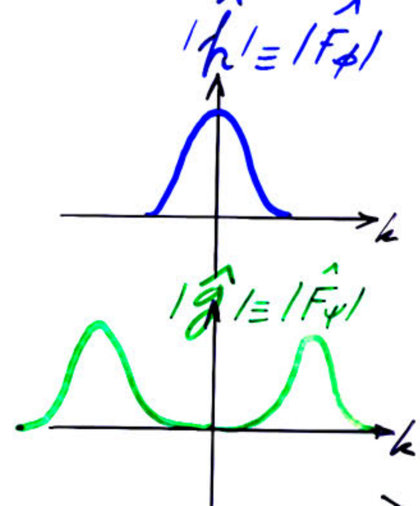
Signal f_0

$\equiv \{\bar{f}_0, \tilde{f}_0, \tilde{f}_1, \tilde{f}_2\}$

$\downarrow$ Subsampling

$\swarrow$ $F_\phi \equiv h$ QMF corresponding to the scaling function ϕ_j

$\searrow$ $F_\psi \equiv g$ QMF corresponding to the wavelet ψ_j



RECONSTRUCTION

Therefore we have:

$$\bar{f}_{j+1}(\vec{x}) = \bar{f}_j(\vec{x}) + \tilde{f}_j(\vec{x})$$

with

$$\bar{f}_{ji} = \langle \phi_{ji} | f \rangle = 2^{j/2} \int_{-\infty}^{+\infty} f(\vec{x}) \phi(2^j x - i) \, d\vec{x}$$

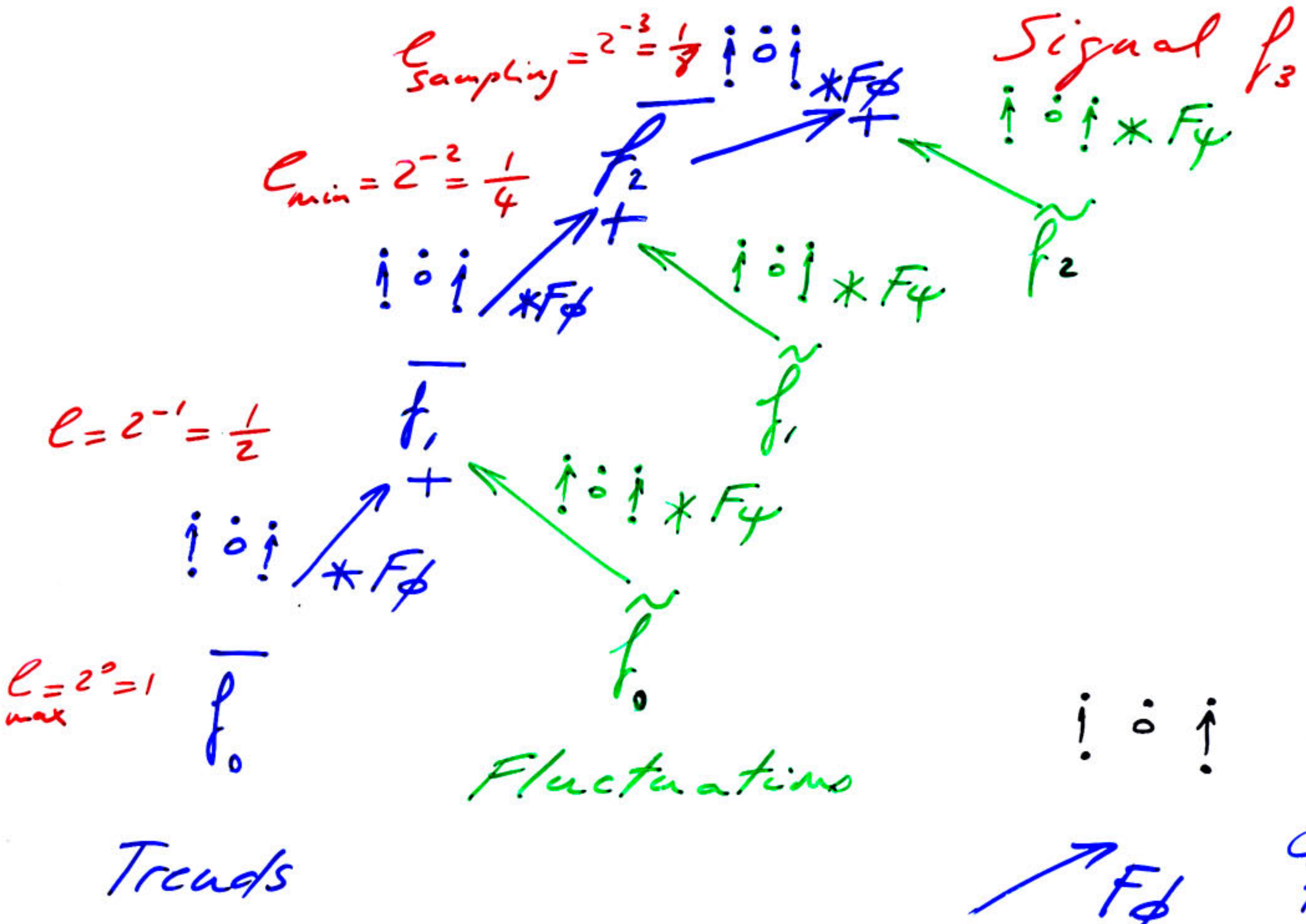
and

$$\tilde{f}_{ji} = \langle \psi_{ji} | f \rangle = 2^{j/2} \int_{-\infty}^{+\infty} f(\vec{x}) \psi(2^j x - i) \, d\vec{x}$$

Recursively we obtain the reconstruction formula:

$$f(\vec{x}) = \sum_{i=-\infty}^{+\infty} \bar{f}_{0i}(\vec{x}) + \sum_{j=0^+}^{+\infty} \sum_{i=-\infty}^{+\infty} \tilde{f}_{ji}(\vec{x})$$

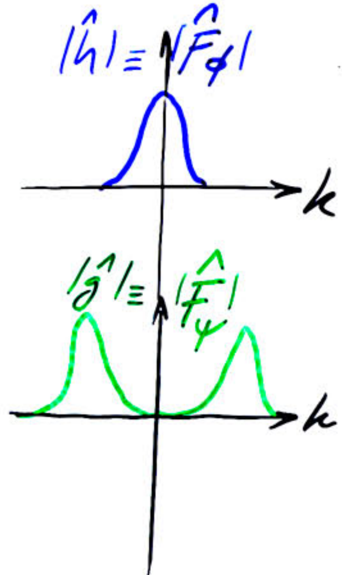
ORTHOGONAL WAVELET RECONSTRUCTION



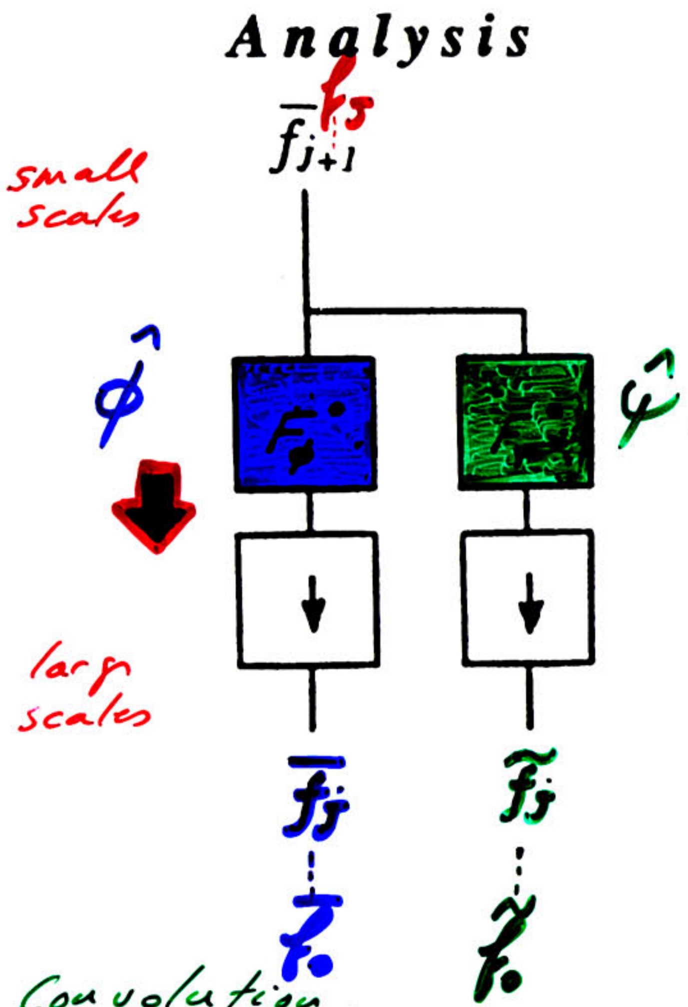
↑ o ↑ Oversampling

F_ϕ QMF corresponding to the scaling functions ϕ_j

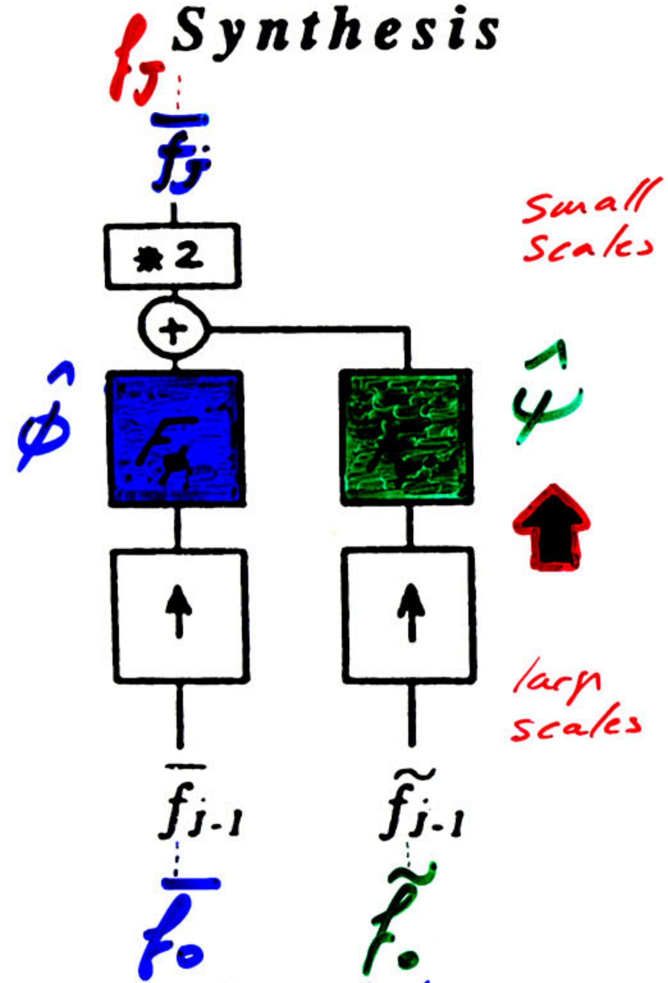
F_ψ QMF corresponding to the wavelets ψ_j



MULTIRESOLUTION ALGORITHM



- F_ϕ^* Convolution by the dual of the discrete filter associated to scaling functions ϕ
- F_ϕ Deconvolution by the discrete filter associated to scaling functions ϕ
- F_ψ^* Convolution by the dual of the discrete filter associated to wavelets ψ
- F_ψ Deconvolution by the discrete filter associated to wavelets ψ
- $\uparrow$ Oversampling by putting one zero between each sample
- $\downarrow$ Undersampling by taking one sample every two samples
- $*2$ Multiplication by 2



Convolution + undersampling:

$$\bar{f}_{j,i} = \sum_n h_{n-2i}\, \bar{f}_{j+1,n}$$

$$\tilde{f}_{j,i} = \sum_n g_{n-2i}\, \bar{f}_{j+1,n}$$

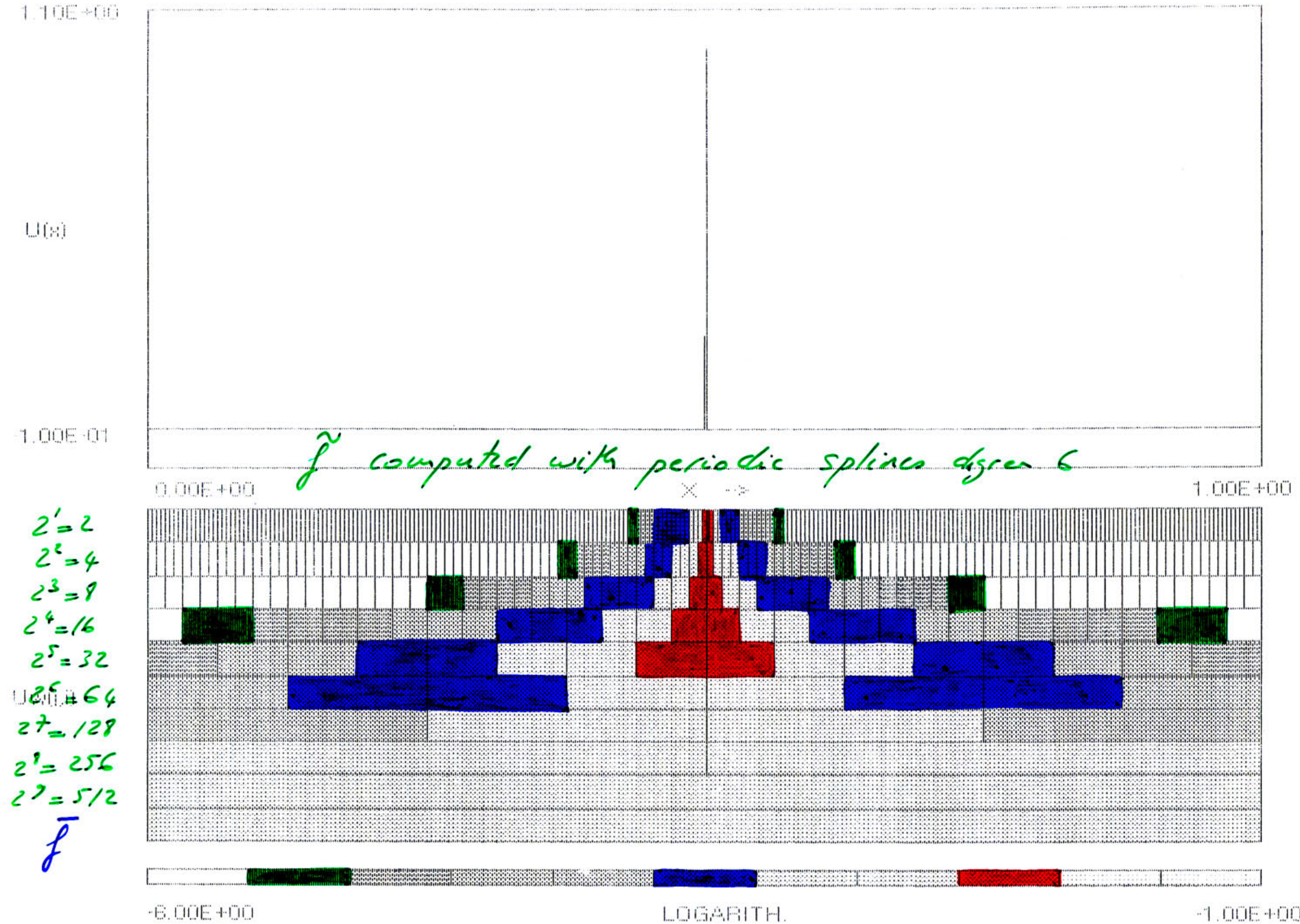
CN operations

Convolution + oversampling:

$$\bar{f}_{j,i} = \sum_n h_{2i-n}\, \bar{f}_{j-1,n} + \sum_n g_{2i-n}\, \tilde{f}_{j-1,n}$$

Filter WL. coefficients ?

Dirac



Periodic Spline Wavelets: IDEGR = 6 N = 512
Threshold EPS= 0.00E+00 # active WL-Coef. = 512

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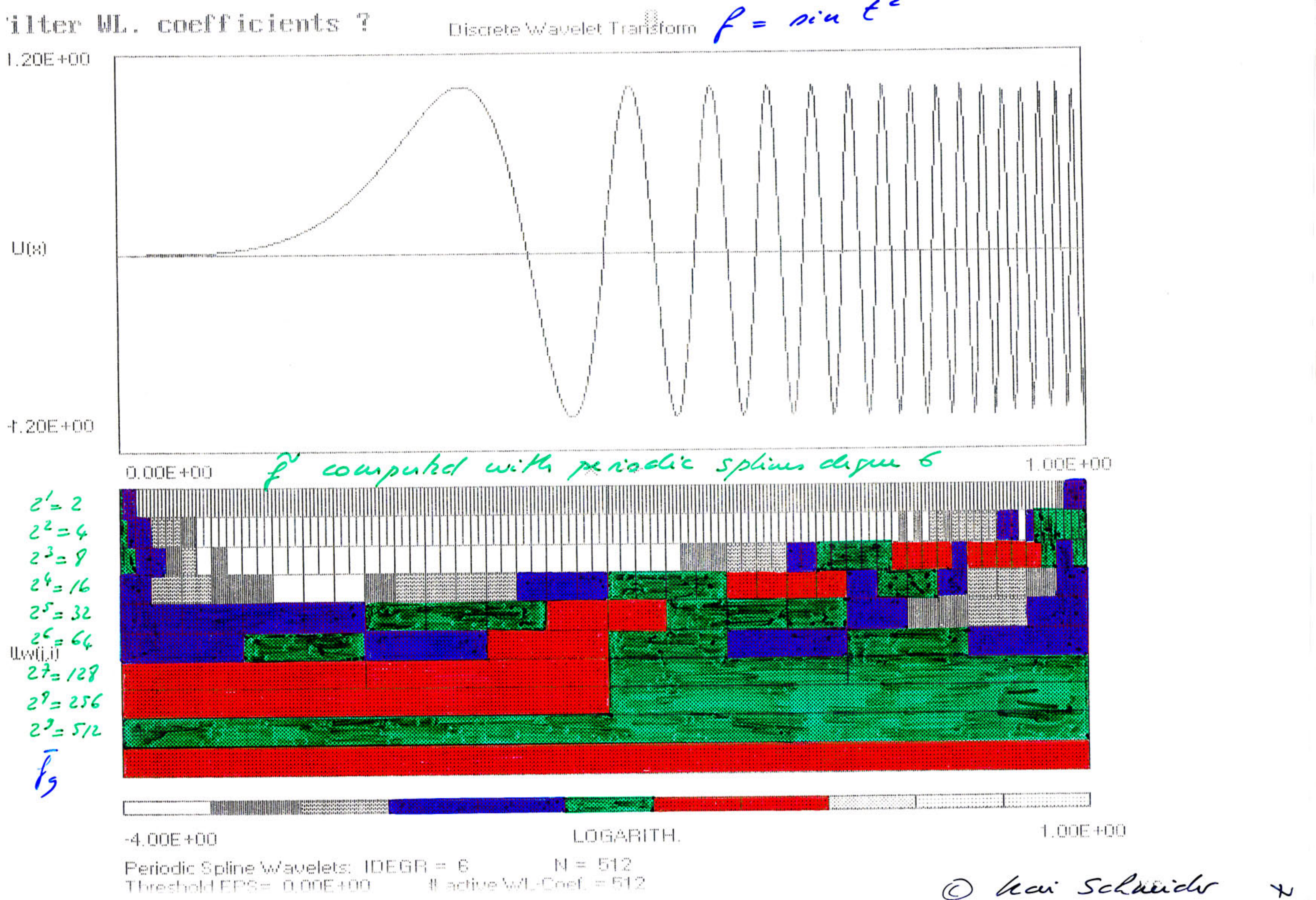
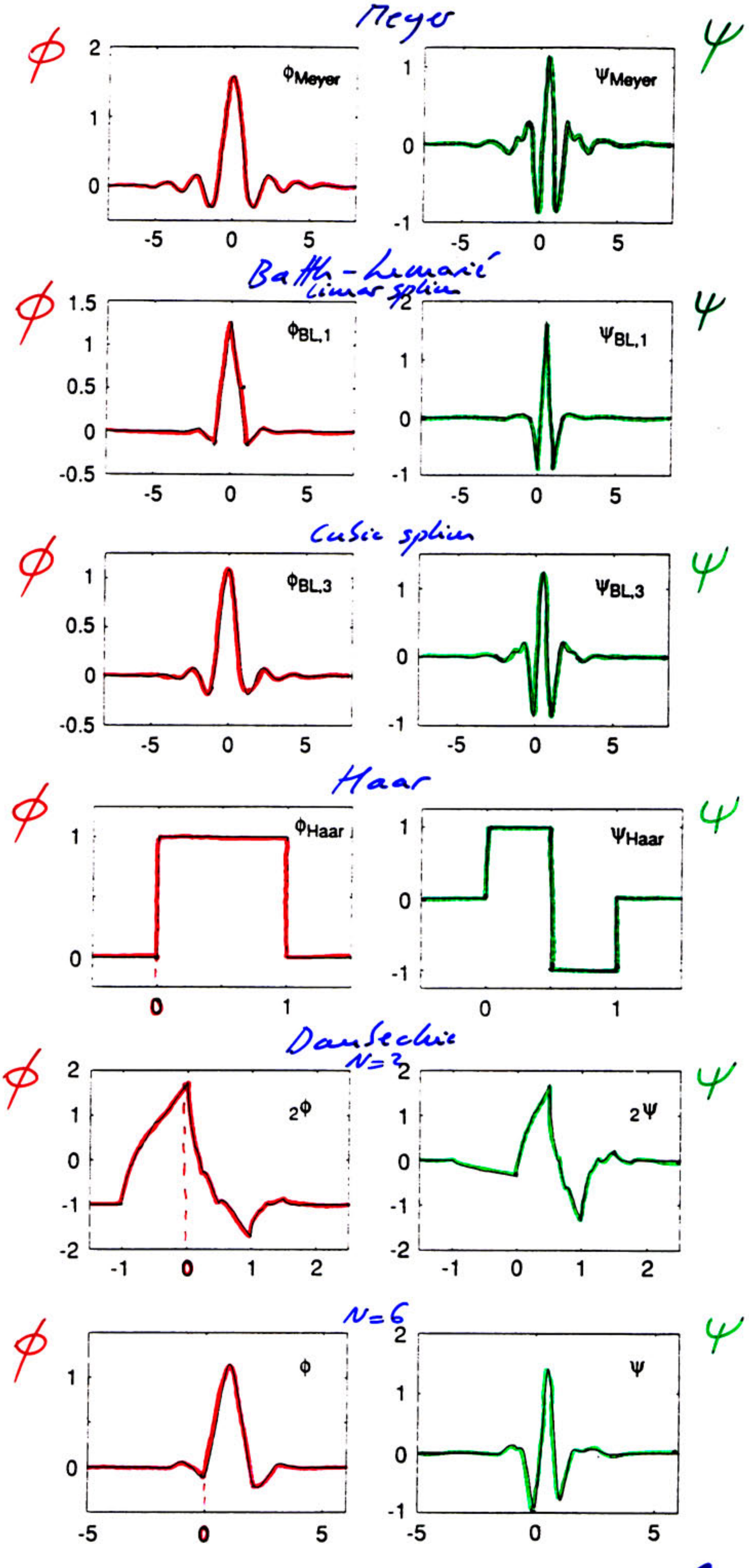


ilter WL. coefficients ?
Discrete Wavelet Transform
f = sin t²
1.20E+00
U(x)
-1.20E+00
0.00E+00
f computed with periodic splines degree 6
1.00E+00
2¹ = 2
2² = 4
2³ = 8
2⁴ = 16
2⁵ = 32
2⁶ = 64
Uw(j,i)
2⁷ = 128
2⁸ = 256
2⁹ = 512
f̄₉
-4.00E+00
LOGARITH.
1.00E+00
Periodic Spline Wavelets: IDEGR = 6
N = 512
Threshold EPS = 0.00E+00
active WL-Coef. = 512
© Kai Schneider



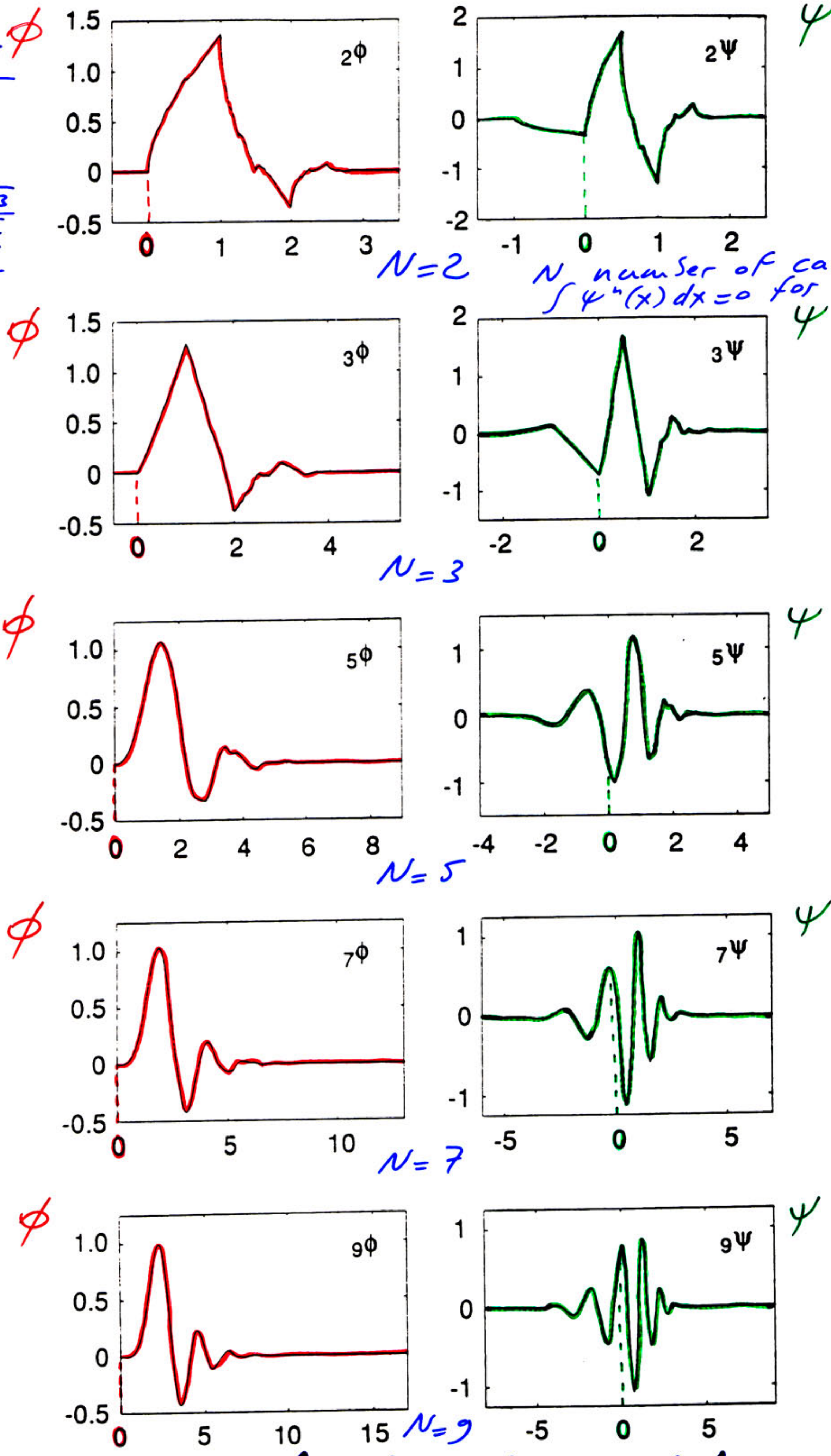
Orthogonal Wavelets

$F_\phi(0) = \frac{1+\sqrt{3}}{4\sqrt{2}}$

$F_\phi(1) = \frac{3+\sqrt{3}}{4\sqrt{2}}$

$F_\phi(2) = \frac{3-\sqrt{3}}{4\sqrt{2}}$

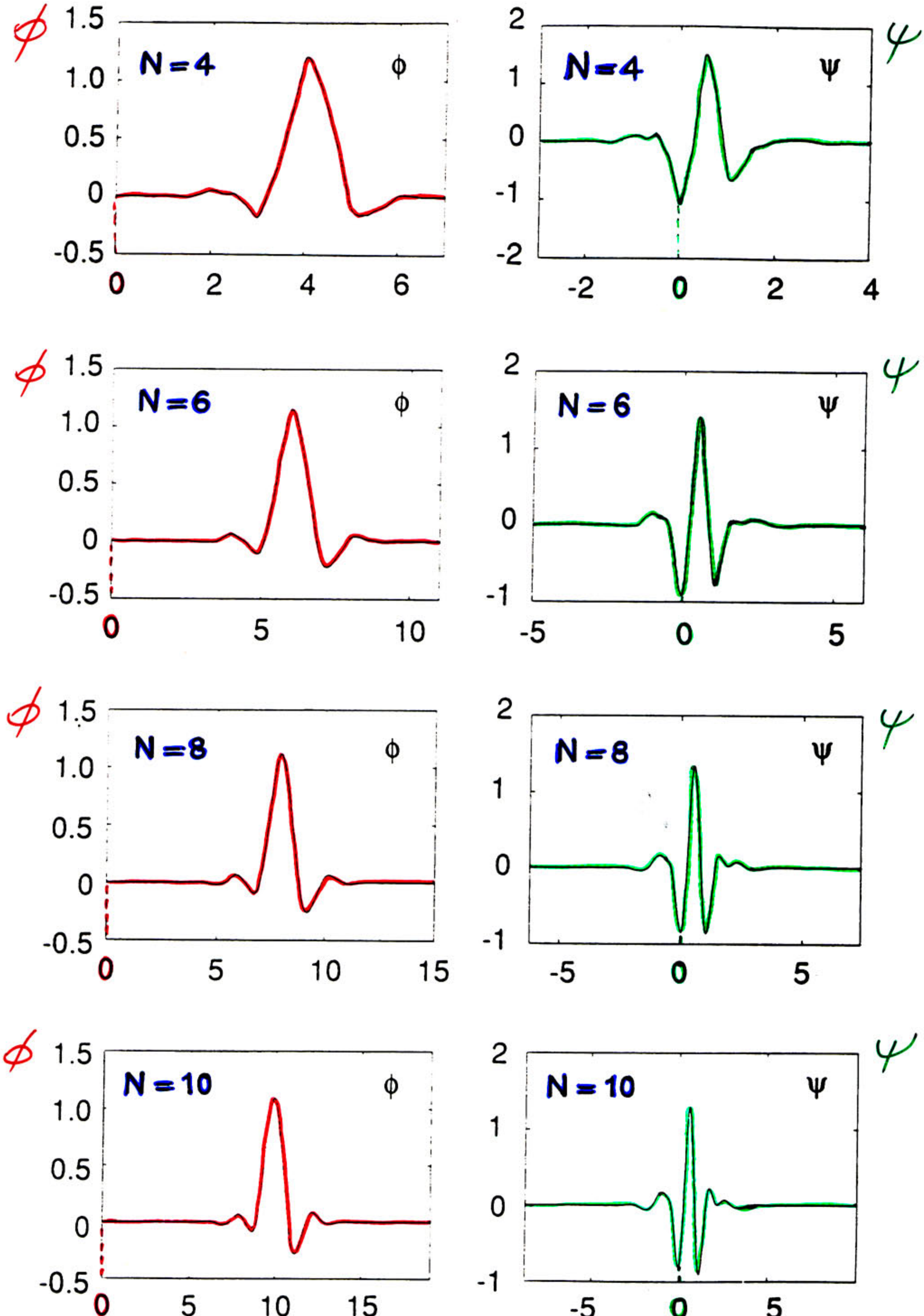
$F_\phi(3) = \frac{1-\sqrt{3}}{4\sqrt{2}}$



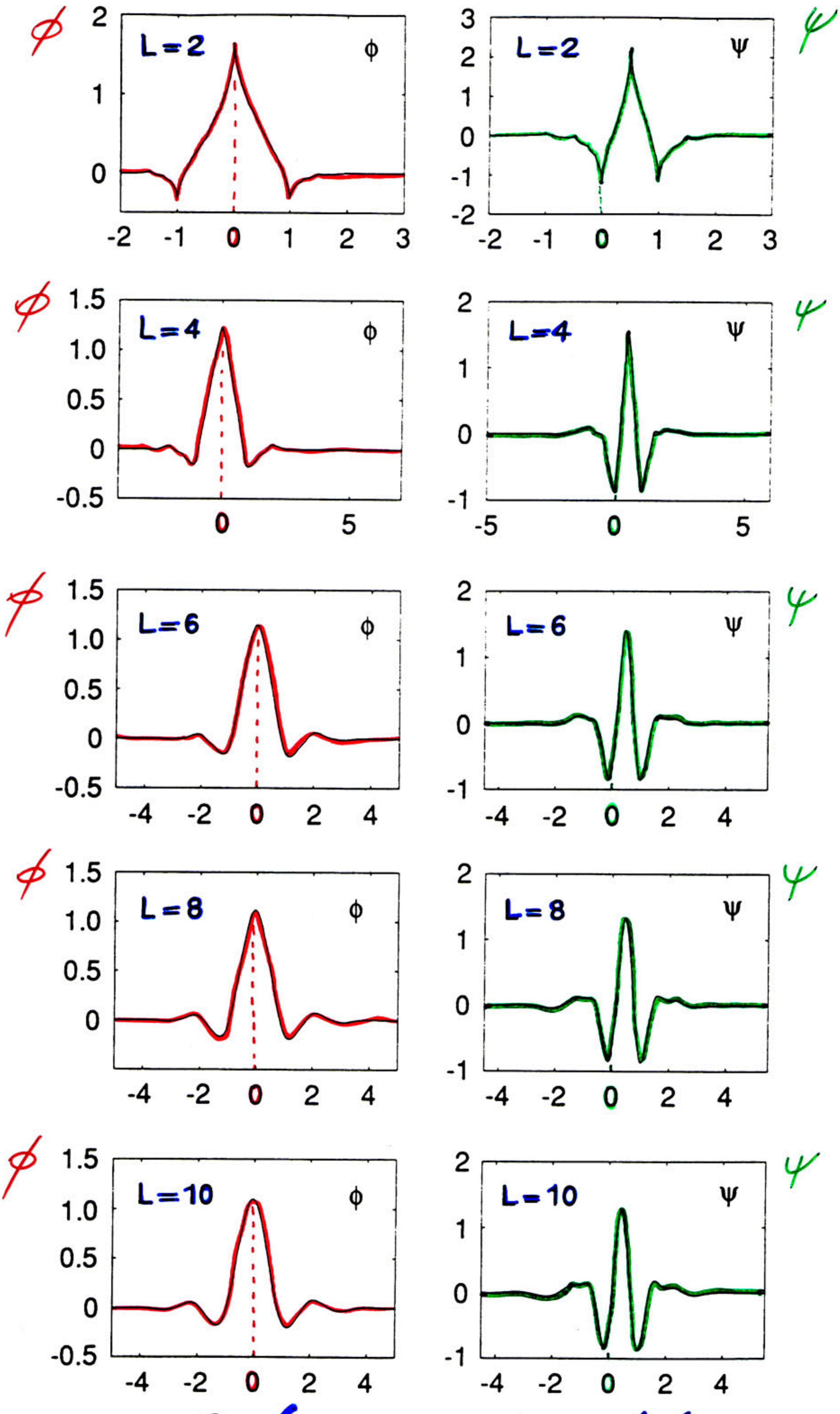
N number of cancellations $\int \psi^n(x)\,dx = 0$ for $n<N$

Daubechies Wavelets

Filter length $L = 2N$

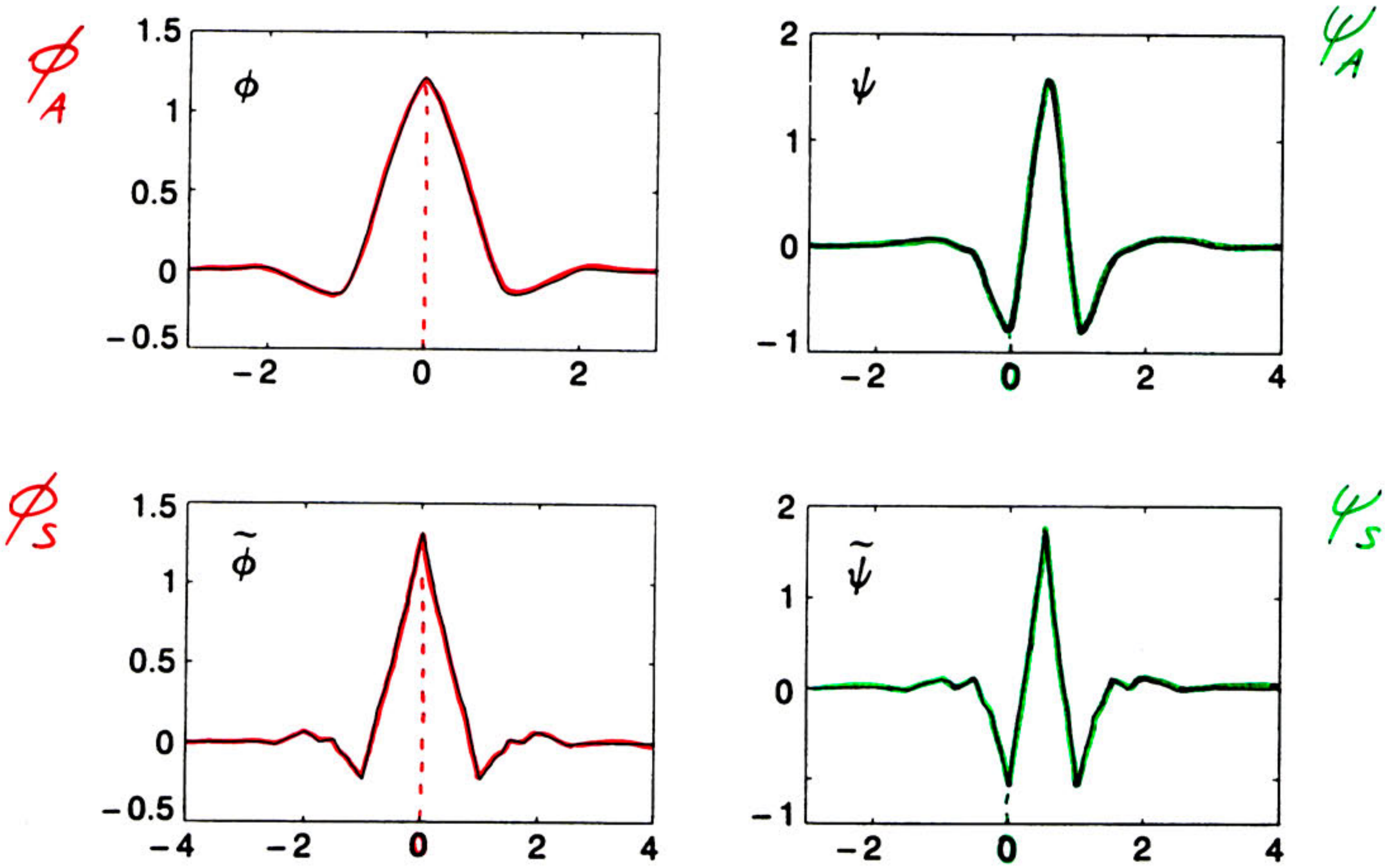


least asymmetric
compactly supported wavelets
$L = 3N$



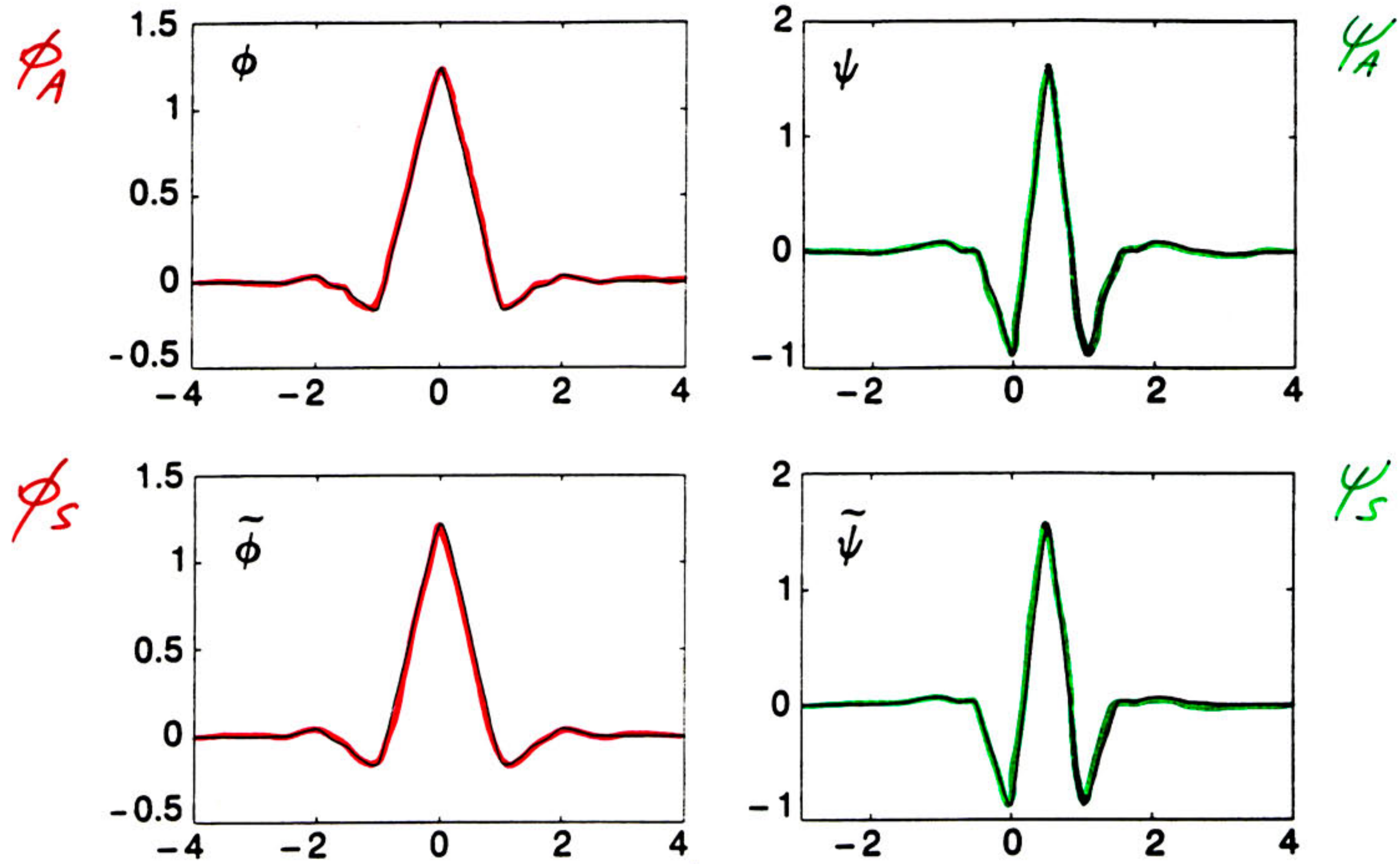
Coifman Wavelets

$L = 3N$

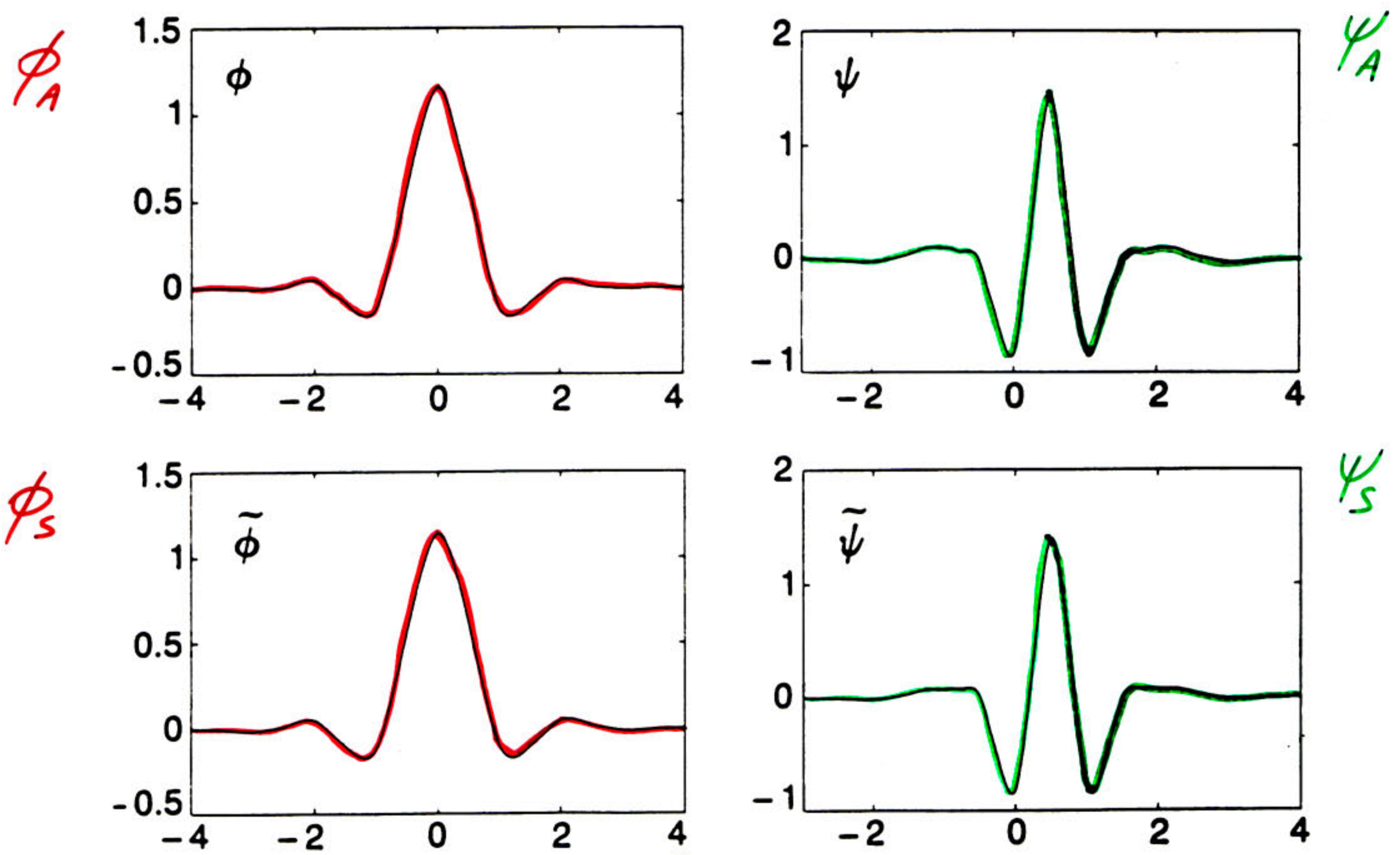


Biorthogonal Spline Wavelets

$N_A = N_S = 4$

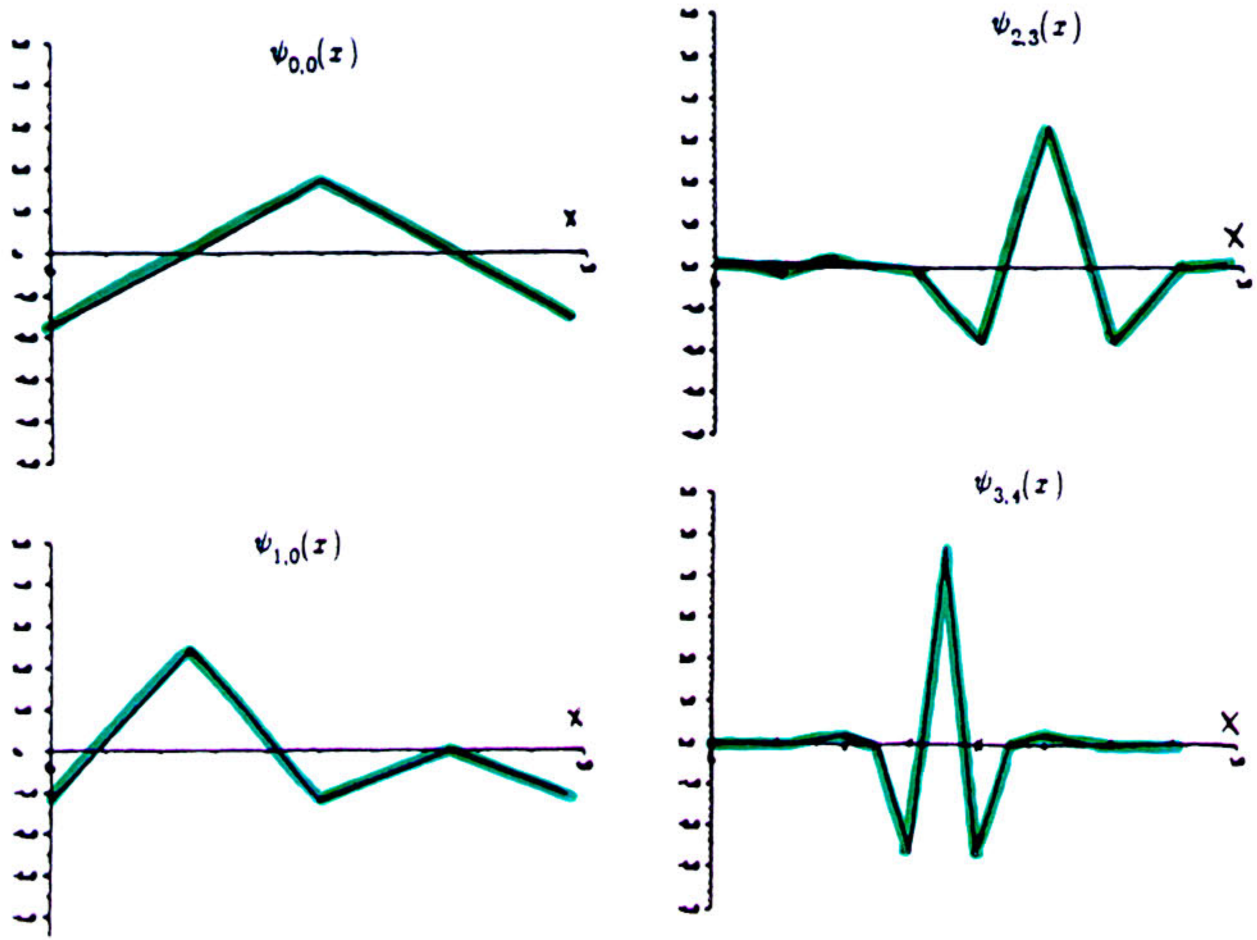


$L=4$



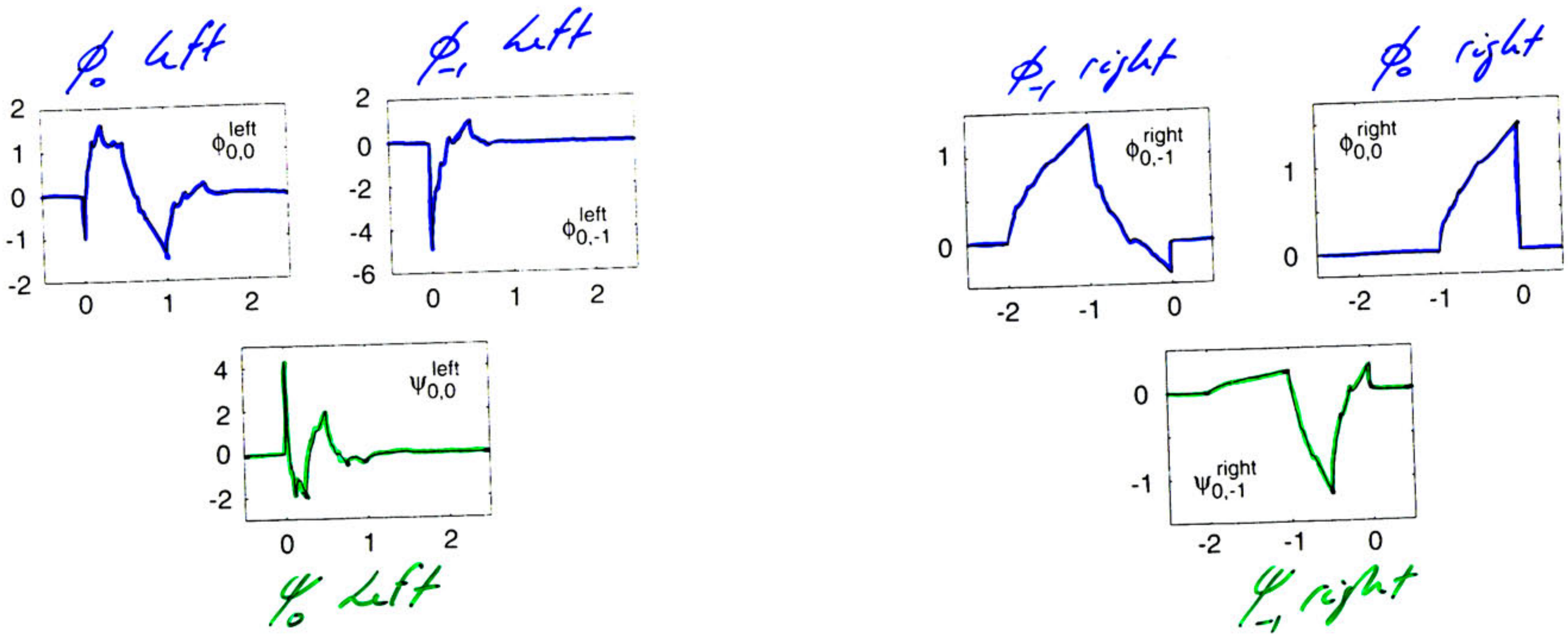
$L=10$

Biorthogonal Coifman Wavelets



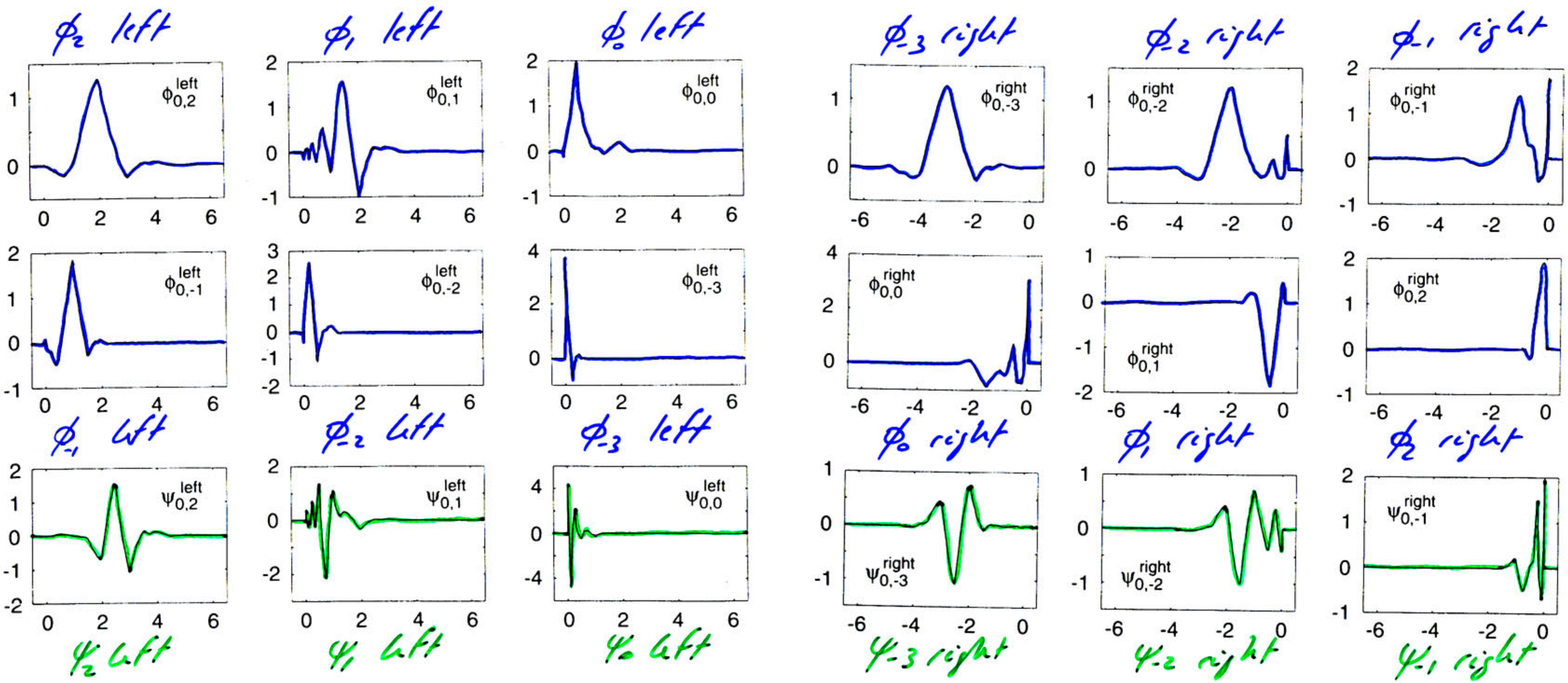
Periodic Wavelets

Built with order 2 spline



Daubechies

$N = 2$



Daubechies

$N=4$

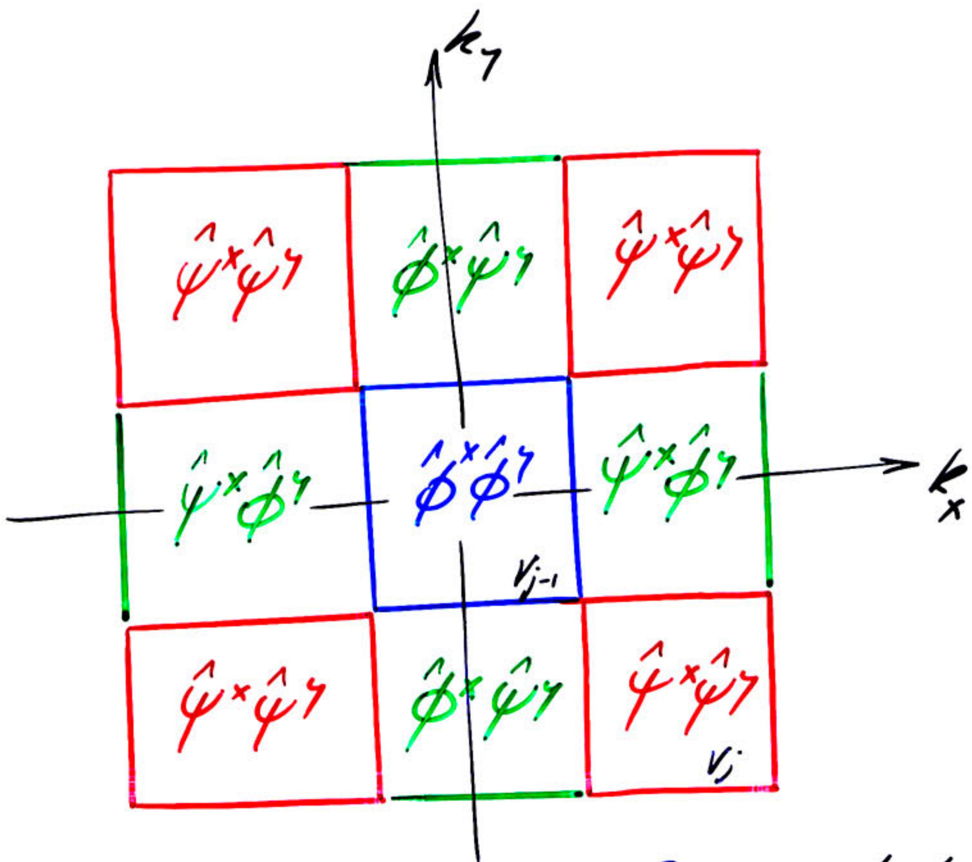
2D MULTIRESOLUTION

$$V_j = V_j^x \otimes V_j^y$$

$$= \left(V_{j-1}^x \oplus W_{j-1}^x\right) \otimes \left(V_{j-1}^y \oplus W_{j-1}^y\right)$$

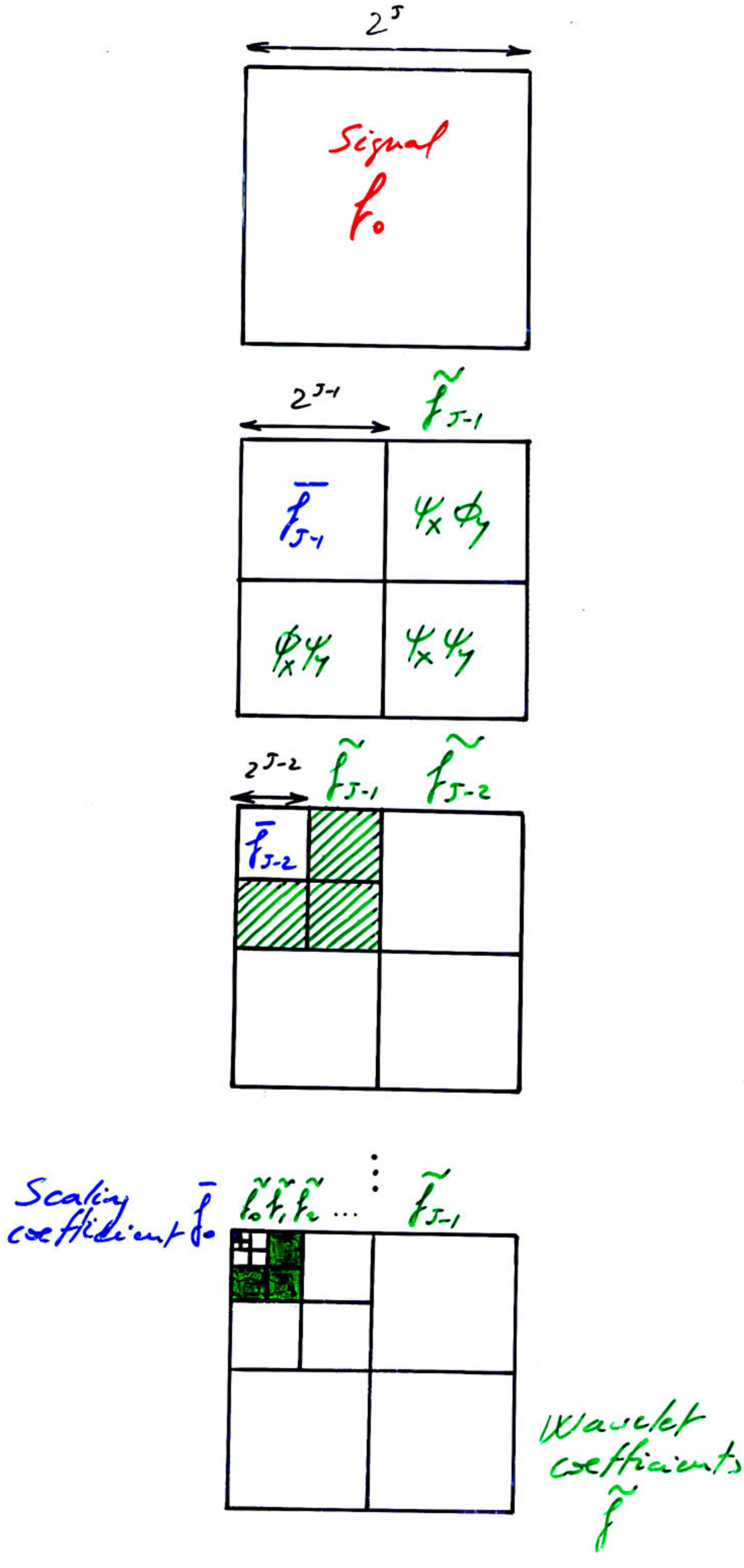
$$= \underbrace{\underbrace{V_{j-1}^x \otimes V_{j-1}^y}_{\phi^x \phi^y}}_{V_{j-1}} \oplus \underbrace{\underbrace{V_{j-1}^x \otimes W_{j-1}^y}_{\phi^x \psi^y} \oplus \underbrace{W_{j-1}^x \otimes V_{j-1}^y}_{\psi^x \phi^y} \oplus \underbrace{W_{j-1}^x \otimes W_{j-1}^y}_{\psi^x \psi^y}}_{W_{j-1}}$$

$$= V_{j-1} \oplus W_{j-1}$$



2D Fourier Space Representation

2D WAVELET REPRESENTATION



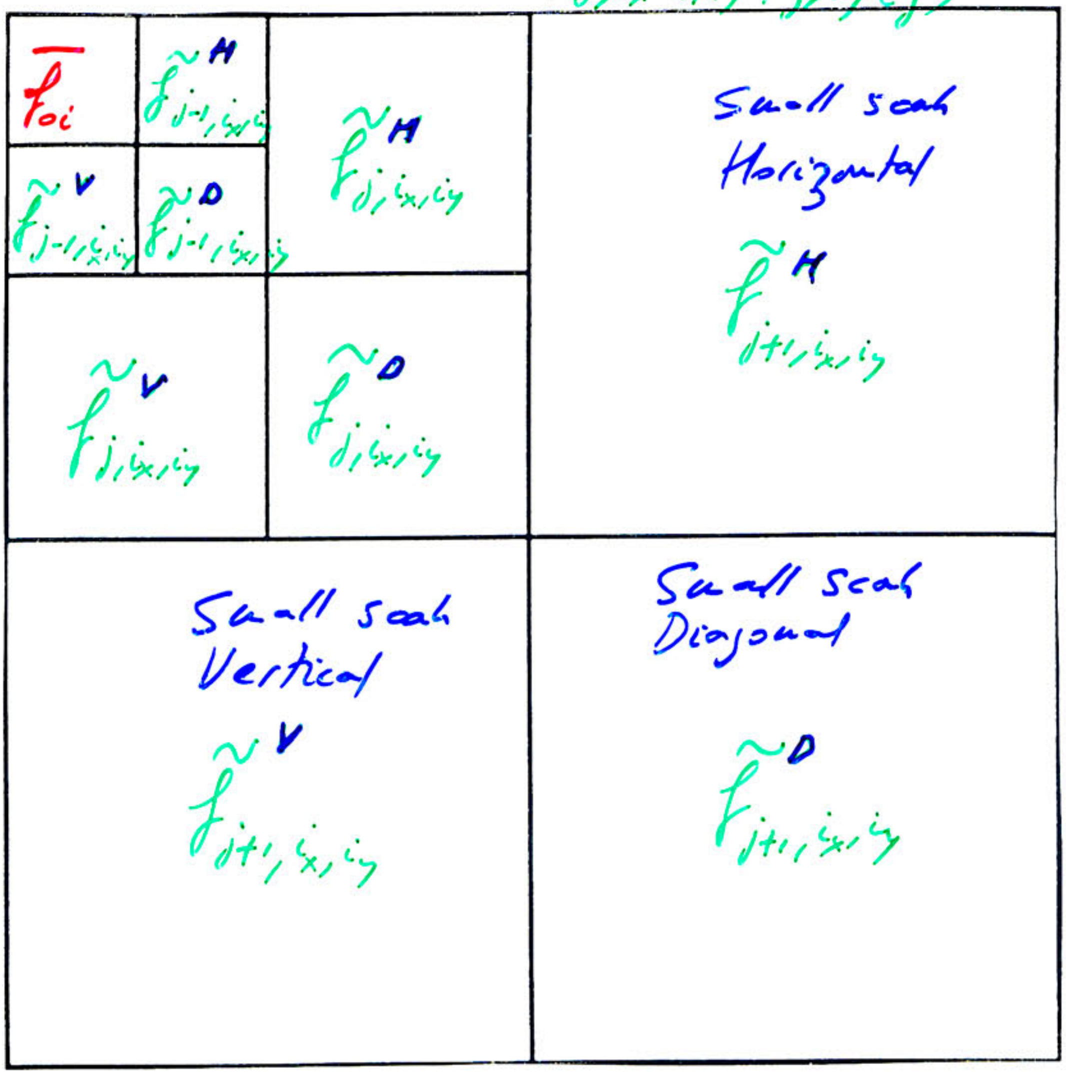
2D M.R.A.

ϕ:

ψ:

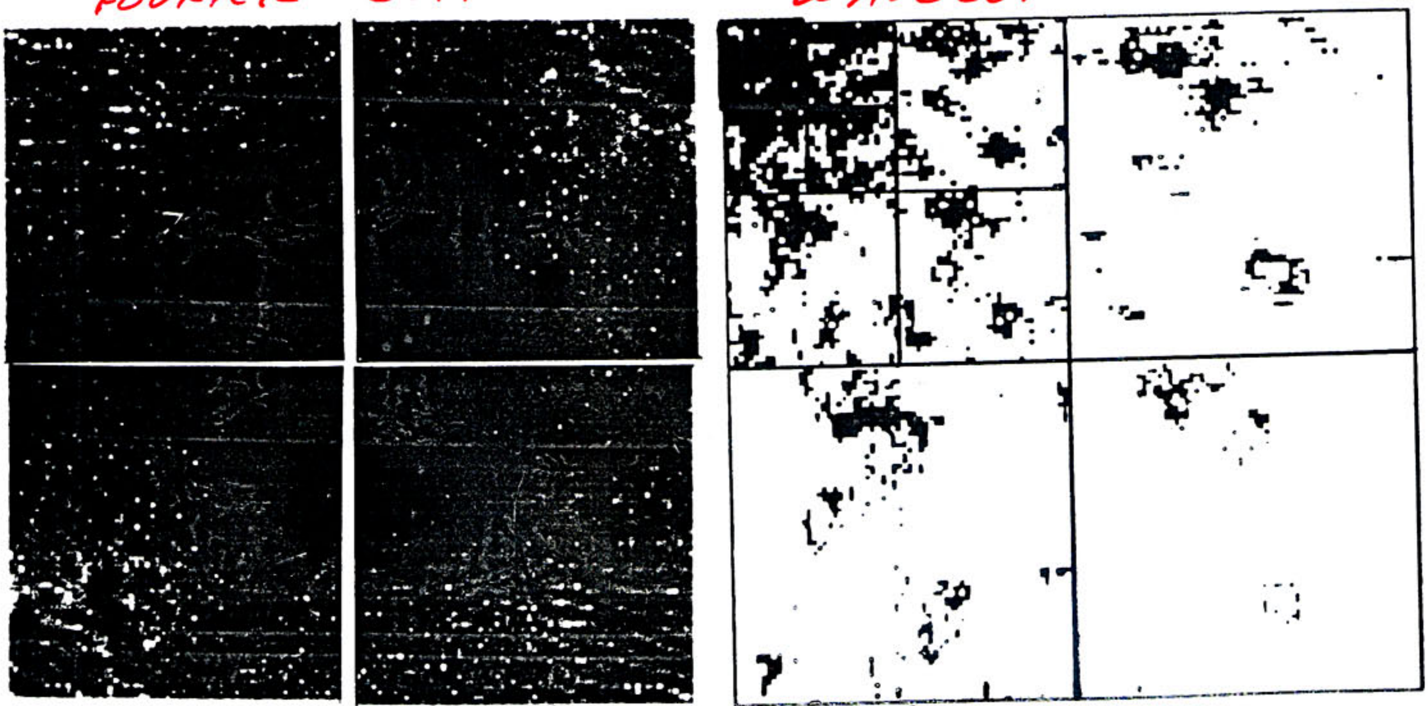
$$\psi^{\mu}_{j,i_x,i_y}(x,y)=\begin{cases}\psi_{j,i_x}(x)\,\phi_{j,i_y}(y) & \text{for } \mu=H\\ \phi_{j,i_x}(x)\,\psi_{j,i_y}(y) & \text{for } \mu=V\\ \psi_{j,i_x}(x)\,\psi_{j,i_y}(y) & \text{for } \mu=D\end{cases}$$

Largest Scale



ACTIVE FOURIER COEFFICIENTS

ACTIVE WAVELET COEFFICIENTS



Logarithmic scale

Color Scale

Active Coefficients