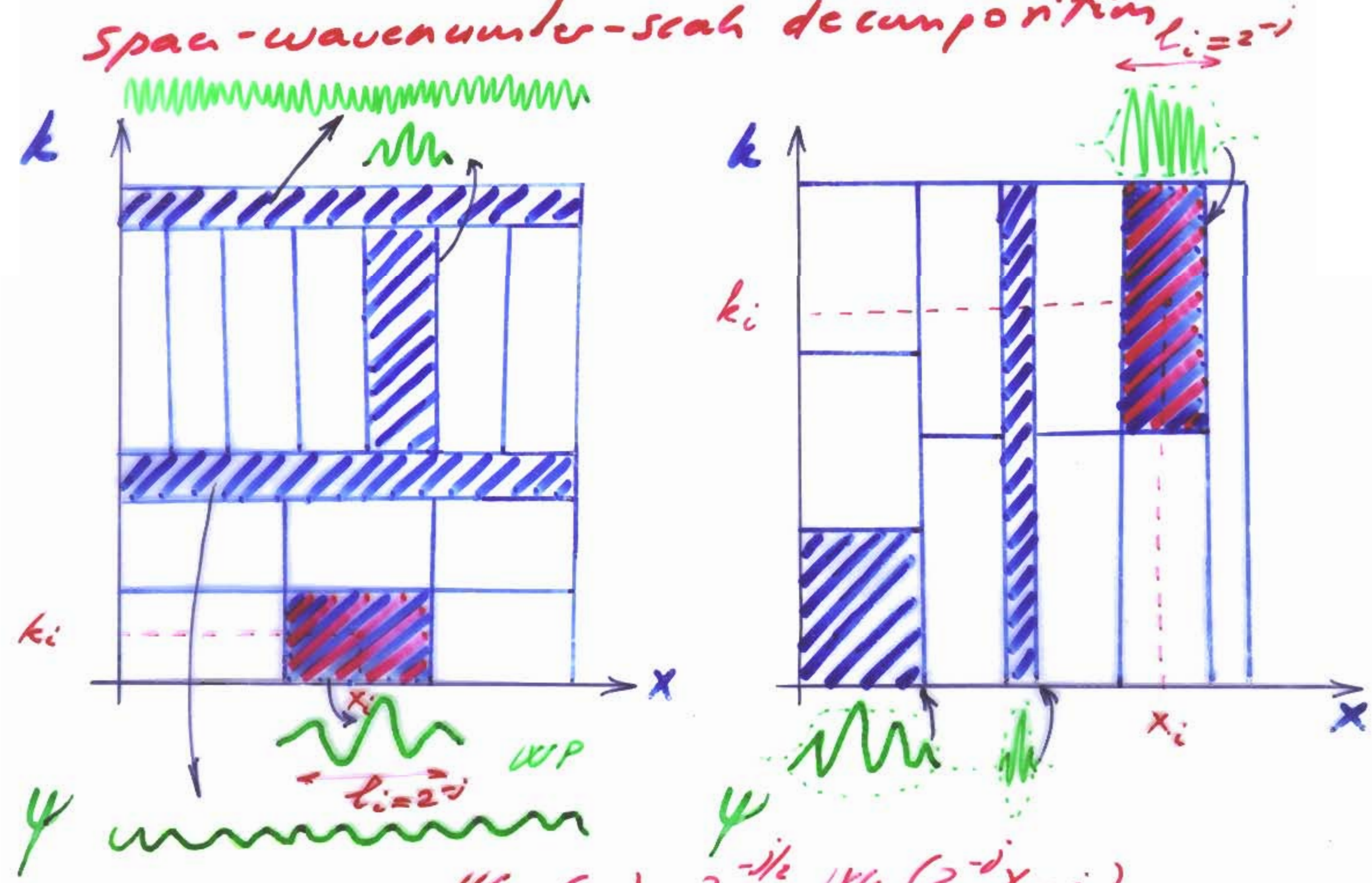
WAVELET PACKETS and LOCAL COSINES

Coifman, Meyer & Wickerhauser (1991)

Family of orthogonal bases which unifies Dirac, Fourier, Windowed Fourier (Malvar) and wavelets

$\longrightarrow$ large flexibility to tile phase space with phase-space 'atoms' and obtain space-wavenumber-scale decomposition



$$\psi_{jik}(x) = 2^{-j/2}\, W_k(2^{-j}x - i)$$

ORTHOGONAL BASES OF PHASE-SPACE 'ATOMS'

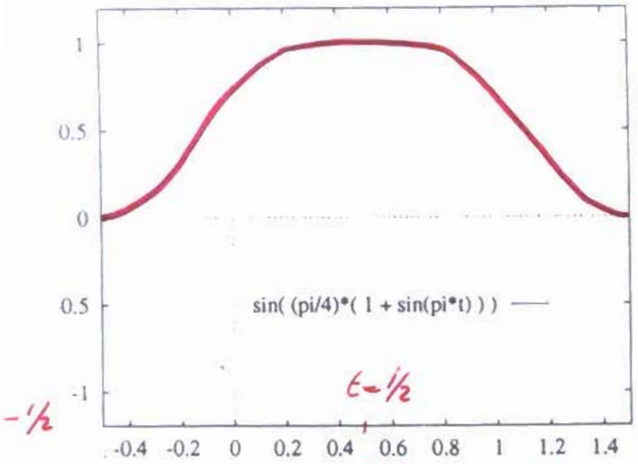
The Balian-Low obstruction theorem (C.R.A.S., 292, 1981) prevents the wave-packets to form an orthogonal basis

Coifman, Meyer and Wickerhauser (C.R.A.S., 312, 1991) circumvent this problem by:

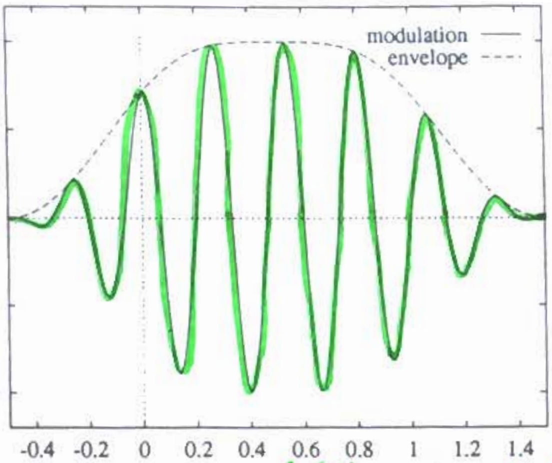
- replacing exponentials e^{ikx} by cosines $\cos \pi (k+\frac{1}{2})x$ and an appropriate window shape
  → adaptative local cosine transform

or - using quadrature mirror filters → wavelet packet transform.

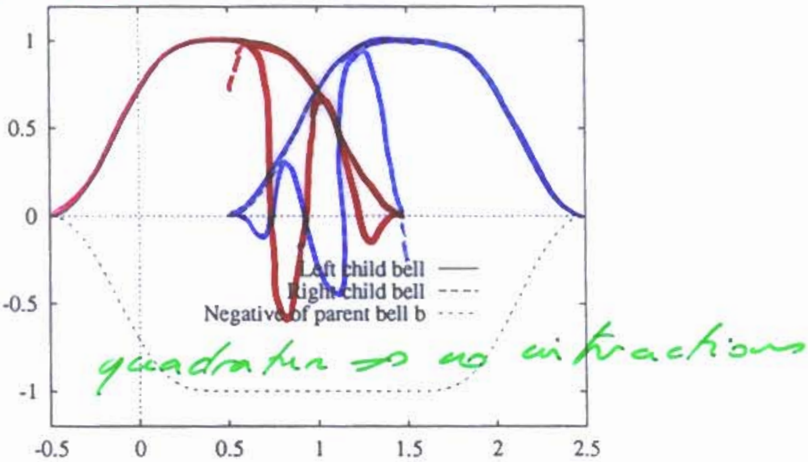
We have a library of modulated waveforms out of which various orthogonal bases can be extracted.



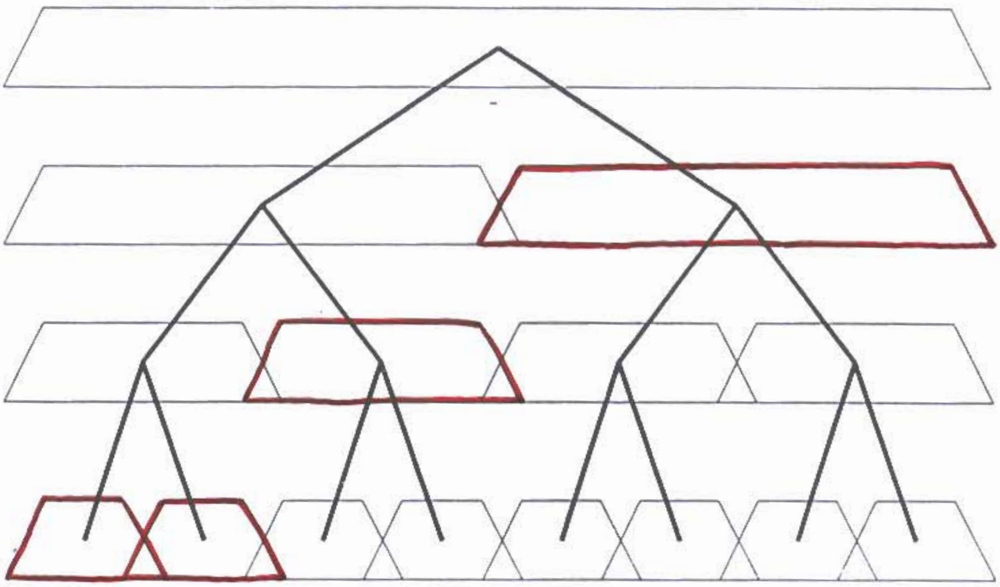
$$\beta(x) = \begin{cases} \sin\frac{\pi}{4}\left(1+\sin\frac{\pi}{2}t\right) & \\ 0 & \text{if } t<-1 \\ 1 & \text{if } t>1 \end{cases}$$



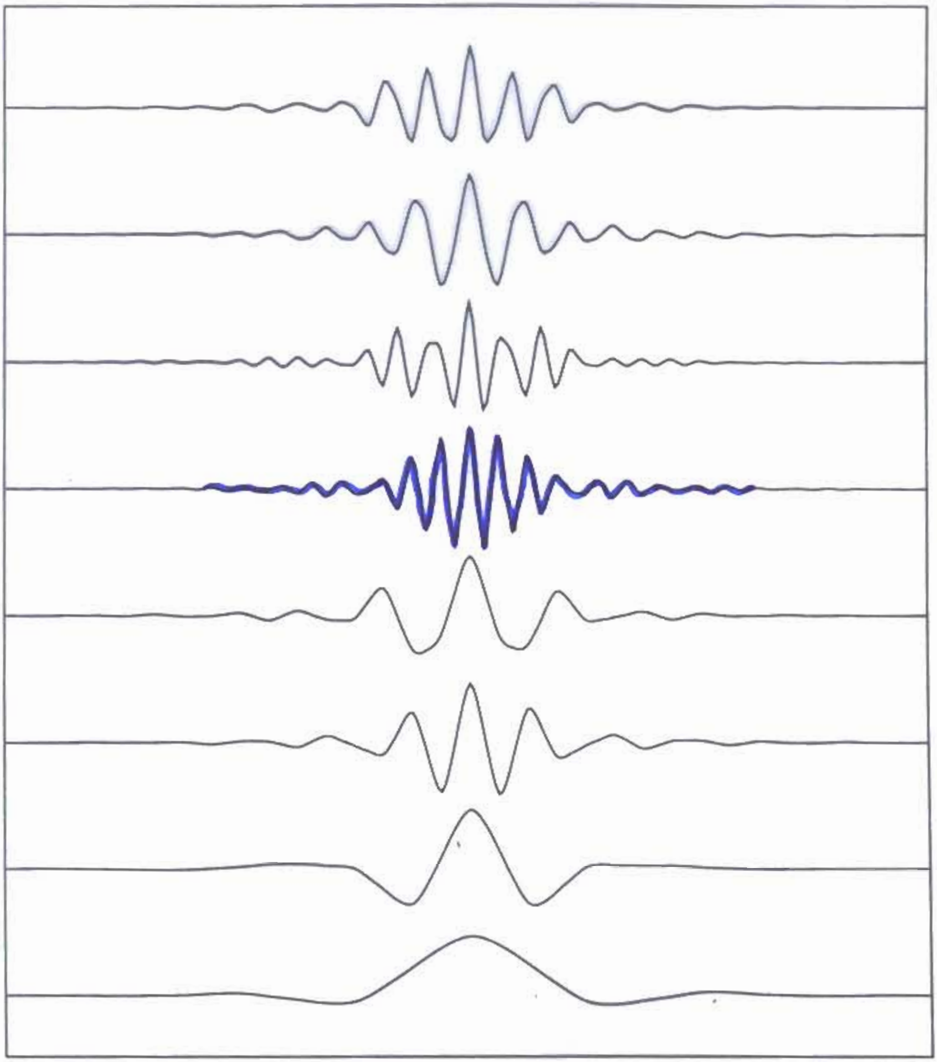
$\psi_{nk}(t) = b_k(t)\, m_k(n+\frac{1}{2}, t)$
with $m(n,t) = \sqrt{2}\cos \pi n t$



Adjacent orthogonal windows



Our possible basis made of Malvar wavelets (local adapted cosines)



Orthogonal Wavelet
Packets built with Coifman 18:

Each WP is orthogonal
to all of its translated, modulated
and dyadic rescaled versions.

LIBRARY OF PHASE-SPACE 'ATOMS'

We define 3 operators

translation $T_x\, \psi(x') = \psi(x'-x)$

modulation $M_k\, \psi(x') = e^{ikx}\, \psi(x')$

dilation $D_\ell\, \psi(x') = \ell^{1/2}\, \psi(\ell x')$

These transformations conserve energy so the wave-packets can be normalized to be unit vectors in L^2.

Analysis

$$\tilde{f}_{xk\ell} = \langle \psi_{xk\ell} | f \rangle$$

Synthesis

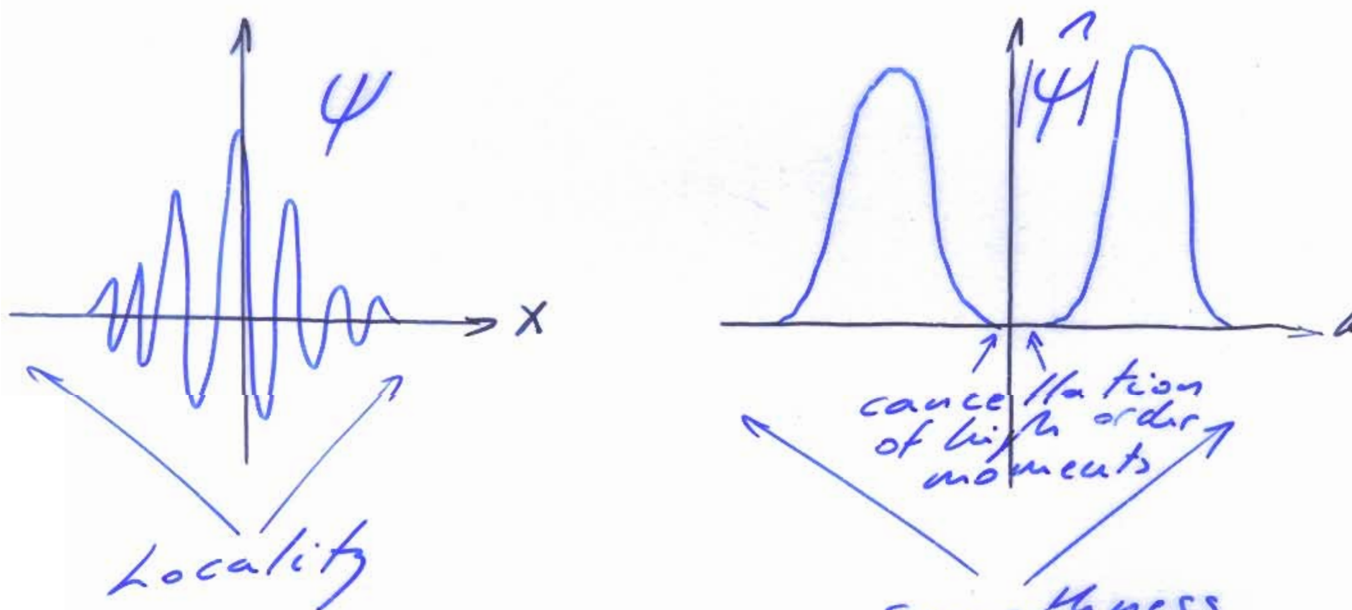
$$f(x) = \sum \tilde{f}_{xk\ell}\, \psi_{xk\ell}$$

Energy Conservation

$$\int |f(x)|^2\, dx = \sum |\tilde{f}_{xk\ell}|^2$$

LIBRARY OF ORTHOGONAL BASES

We build a library $\mathcal{B}$ of many orthogonal bases B_n made of elementary functions ψ which have the spatial (compactness) and spectral (smoothness) locality properties we wish:



$$\mathcal{B}\left\{ B_n : \{\psi_{ni}\} \right\} \text{ with}$$

$$\psi_\lambda = \psi_{j,i,k} = 2^{-j/2}\, W_k(2^{-j}x - i)$$

i position, j scale, k wavenumber
λ index of the wavepacket

WAVELET PACKET BASIS

A wavelet packet basis of $L^2(\mathbb{R})$ is any orthogonal basis selected from among the functions

$$\psi_{jik}(x) = 2^{-j/2} W_k(2^{-j}x - i).$$

The choice of the QMF W_k influences the properties of the wavelet packets in terms of smoothness, vanishing moments.

The computational complexity grows linearly with the filter's length.

We want to select the basis ψ which allows the most efficient representation of the signal f.

We obtain a new class of orthogonal bases of $L^2(\mathbb{R}^n)$ by constructing a library of modulated waveforms of prescribed smoothness, out of which various bases may be extracted, in particular the Walsh basis, the Windowed Walsh basis, the Wavelet basis and various Wavepacket bases.

QMF
PHASE-SPACE
'ATOMS'

Set of orthogonal functions:

$$\psi_{jik}(x) = 2^{-jk} W_k (2^{-j} x - i)$$

characterized by 3 parameters:

$$\begin{cases} j & \text{scale} \\ i & \text{position} \\ k & \text{wavenumber} \end{cases}$$

W_k are not constructed explicitly but are computed from two conjugate quadratic filters applied to the N sampled values of the field ω.

$\psi_{jik}(x)$ are square-integrable functions with prescribed smoothness depending on W.

Projection of ω:

$$\tilde{\omega}_{jik} = \langle \psi_{jik} / \omega \rangle$$

Reconstruction of ω:

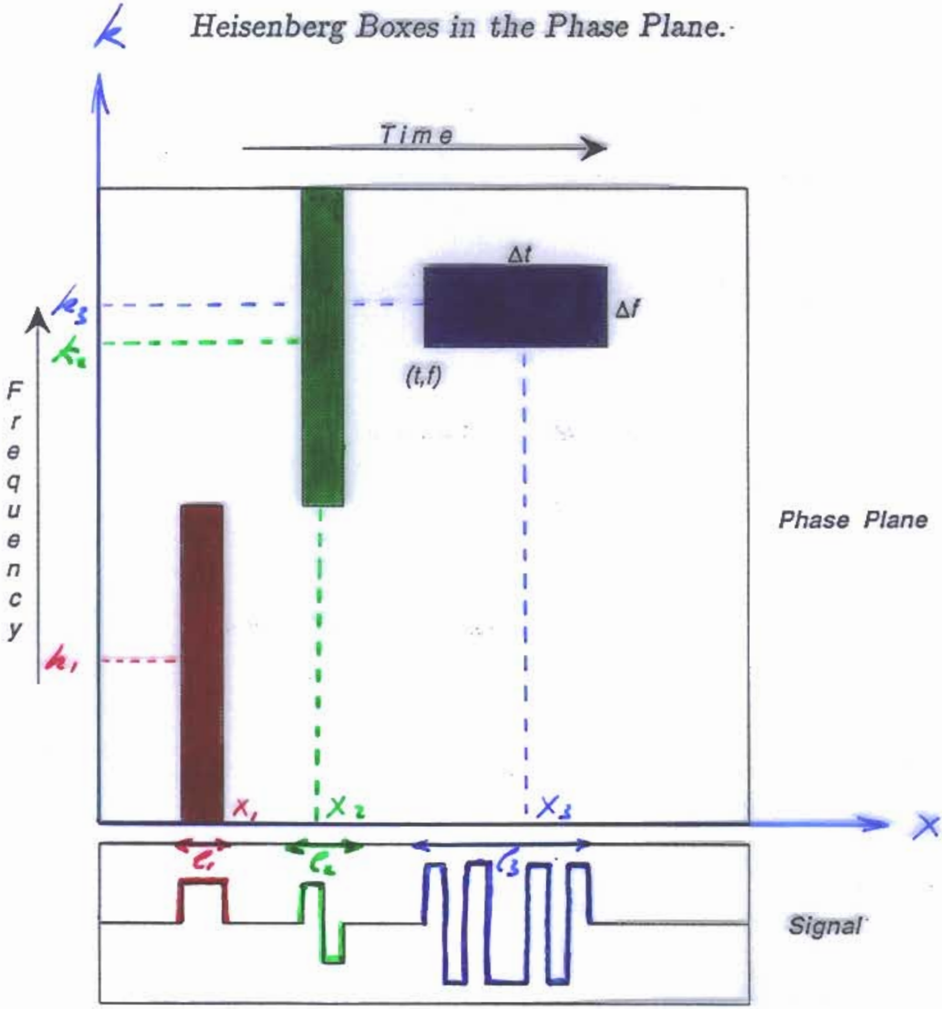
$$\omega(x) = \sum \tilde{\omega}_{jik} \, \psi_{jik}$$

Both require N operations.

In 2D:

$$\psi_{j\vec{i}\vec{k}}(\vec{x}) = 2^{-j} W_{k_x}(2^{-j}x - i_x) \, W_{k_y}(2^{-j}y - i_y)$$

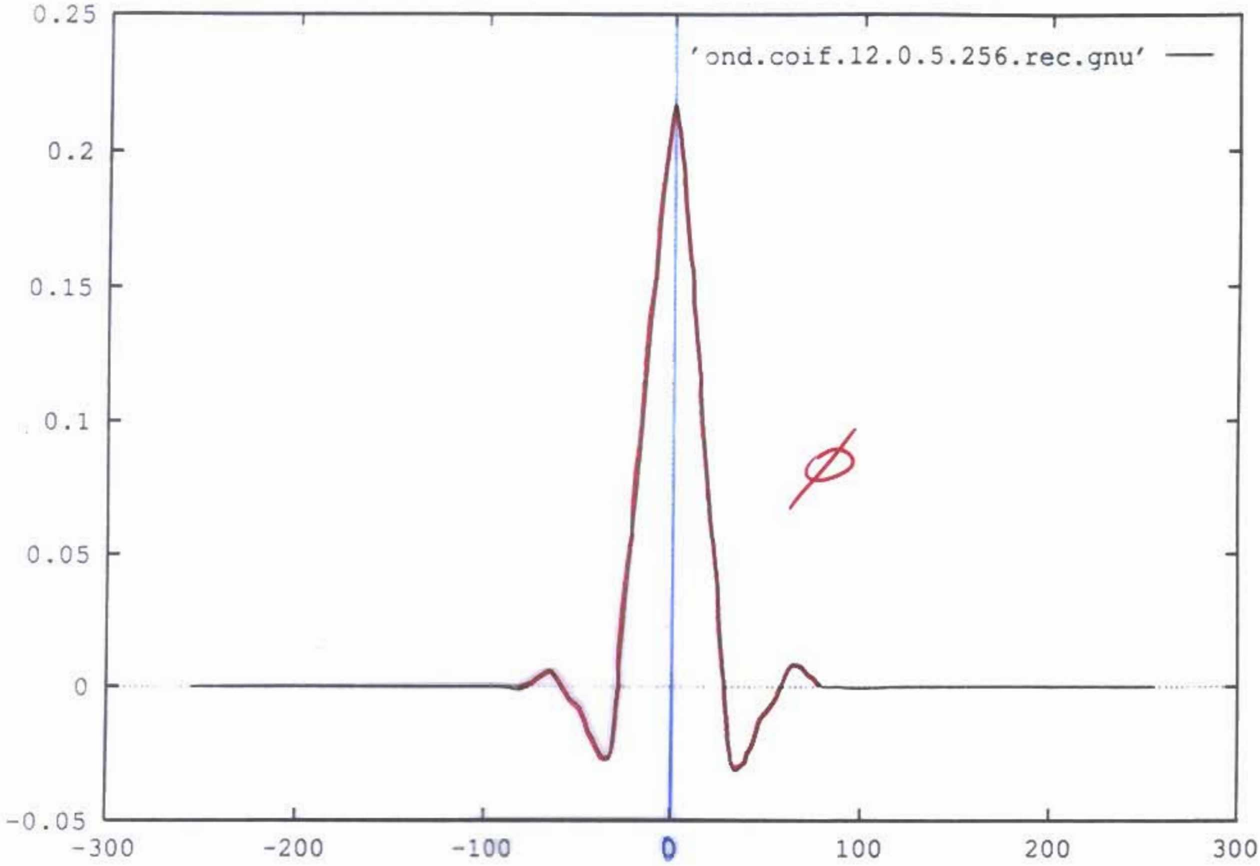
Heisenberg Boxes in the Phase Plane.



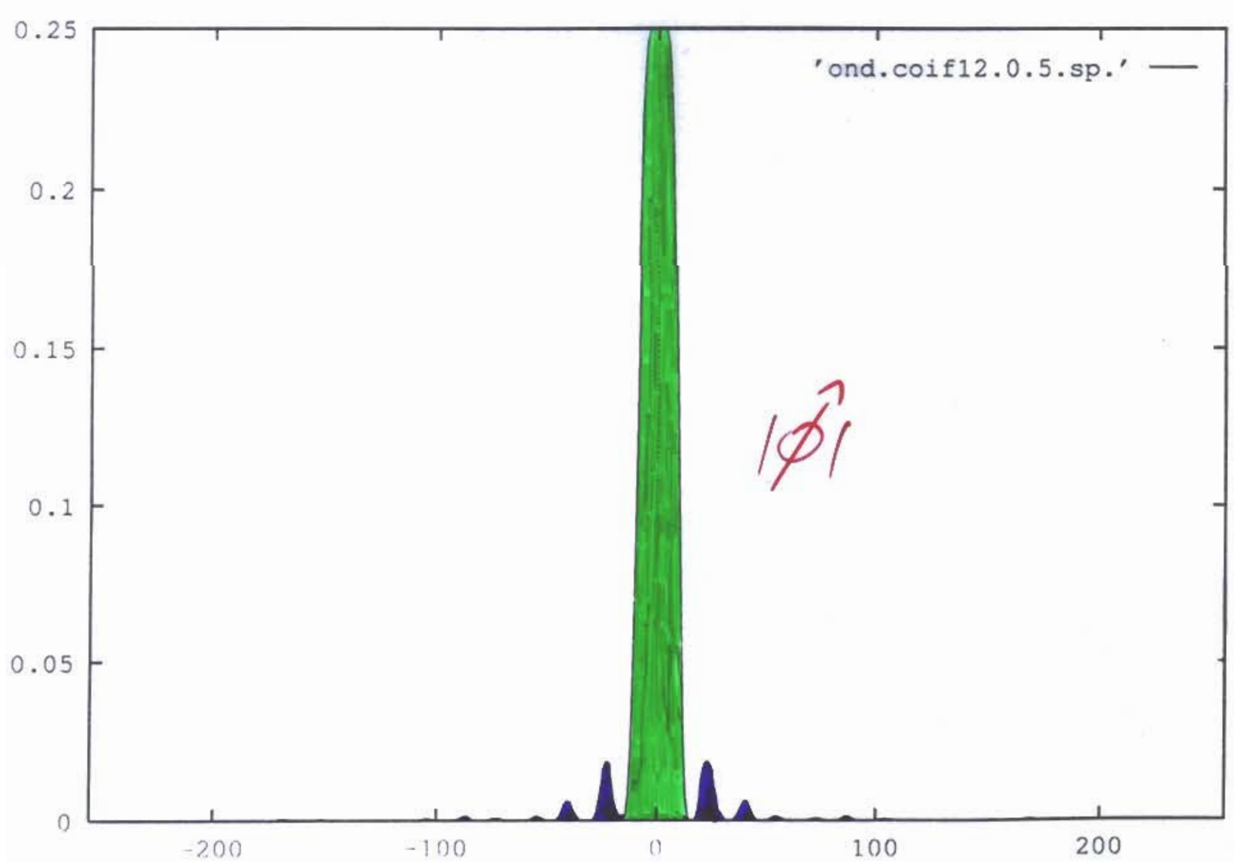
Examples of wavelet packets built with the Haar filters W.

$$\psi_{jik}(x) = 2^{-j/2} W_k(2^{-j}x - i)$$

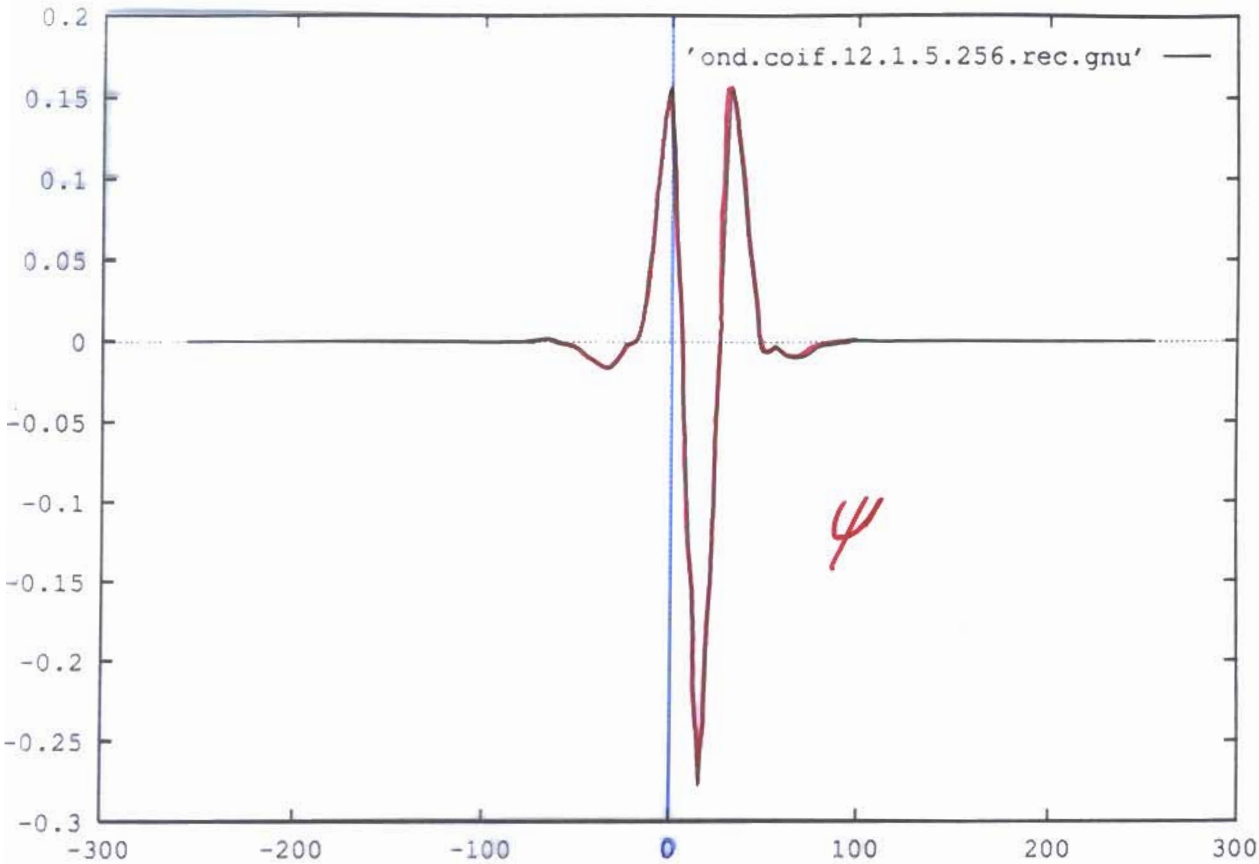
Coifman 12



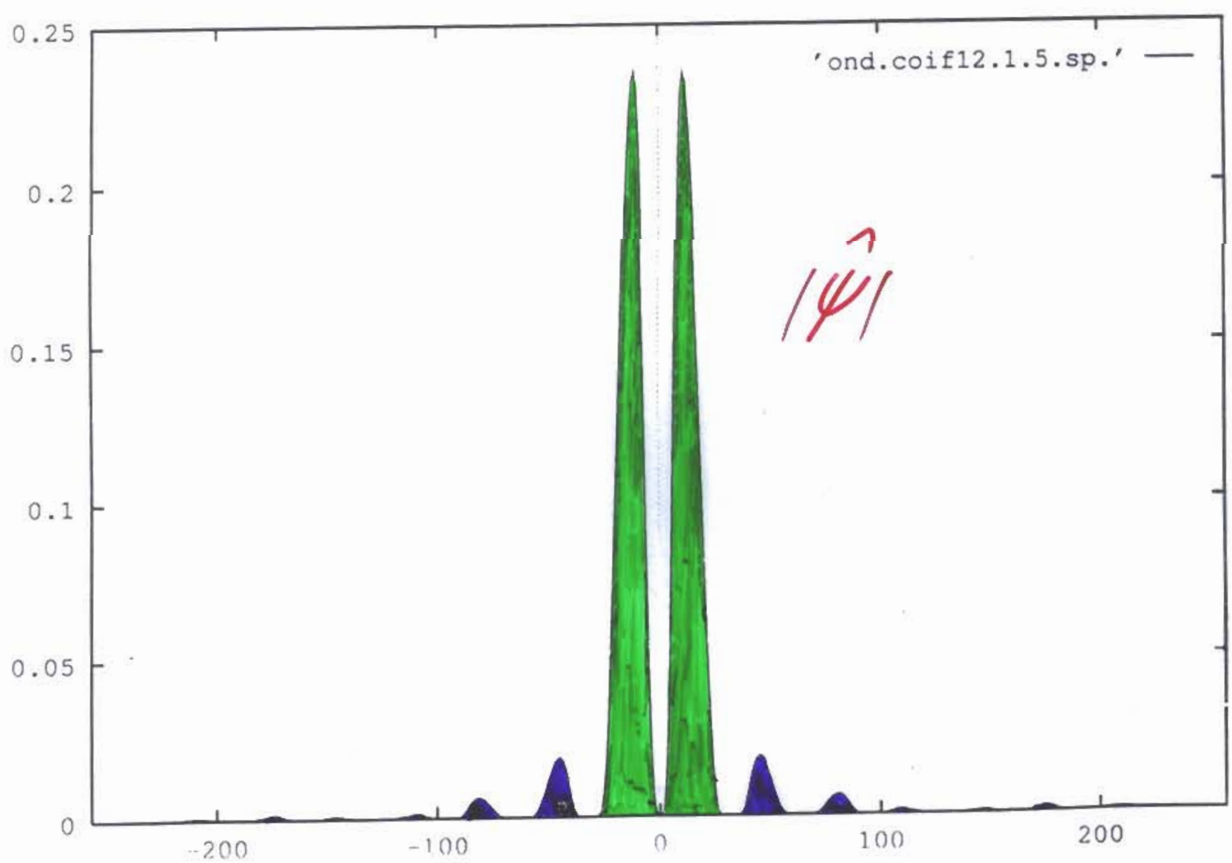
scale 0



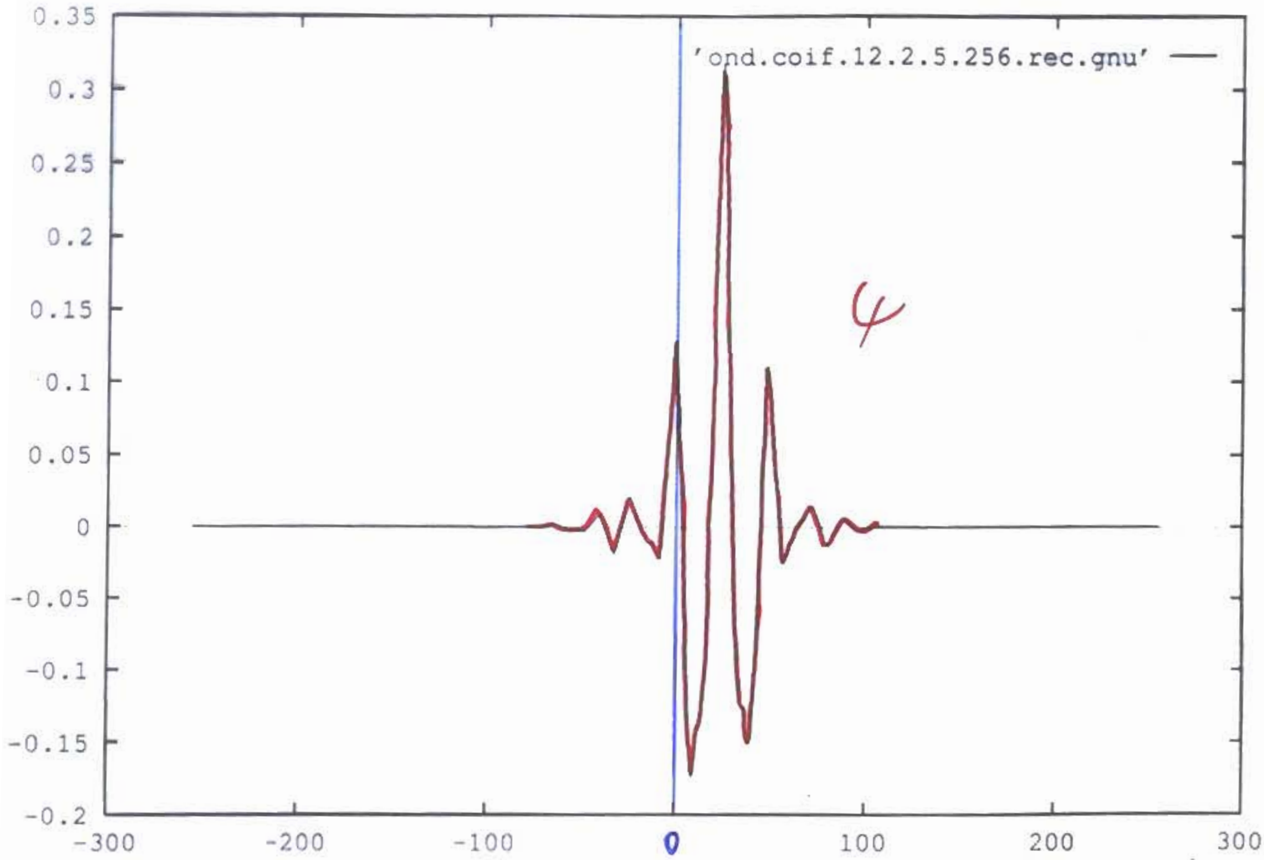
Coifman 12



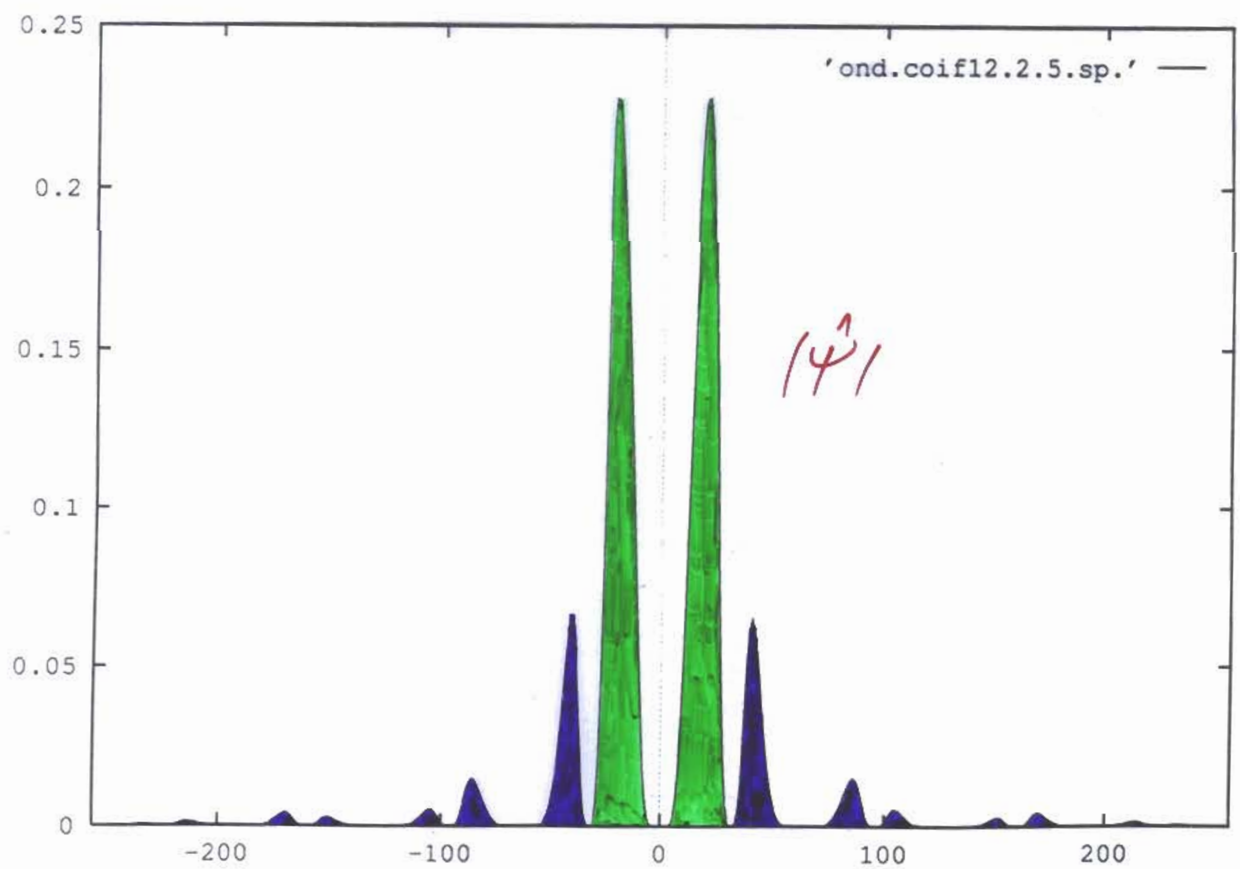
scale 1

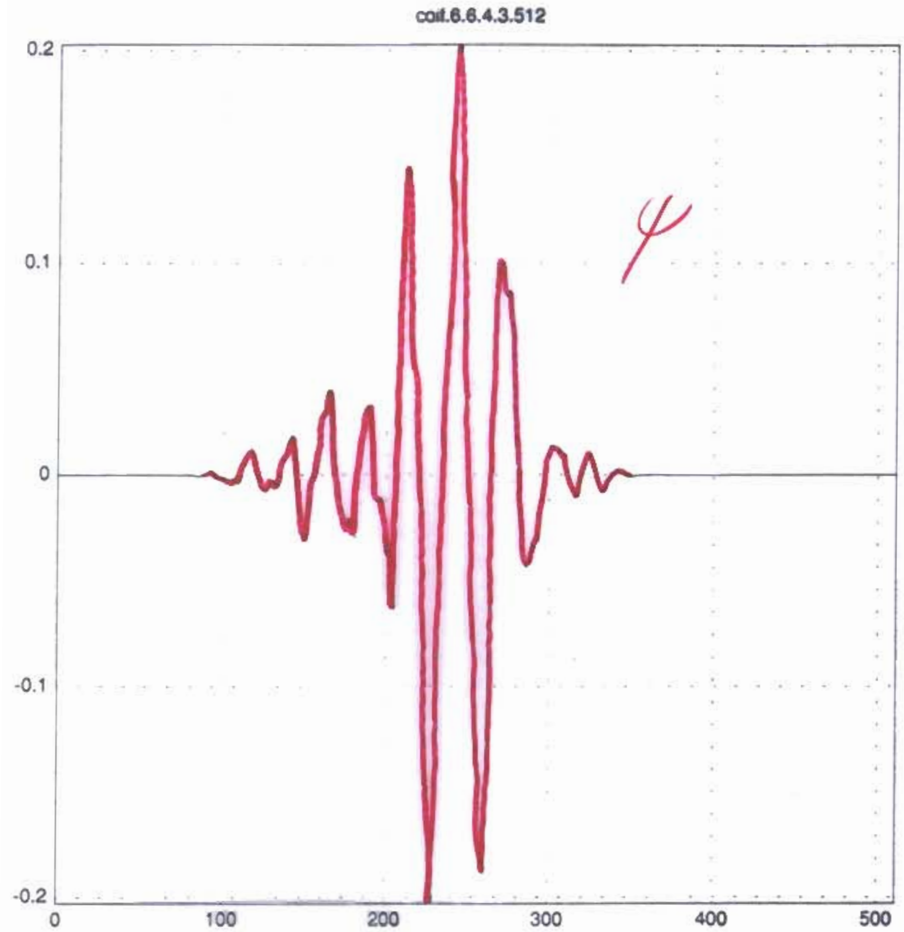


Coifman 12

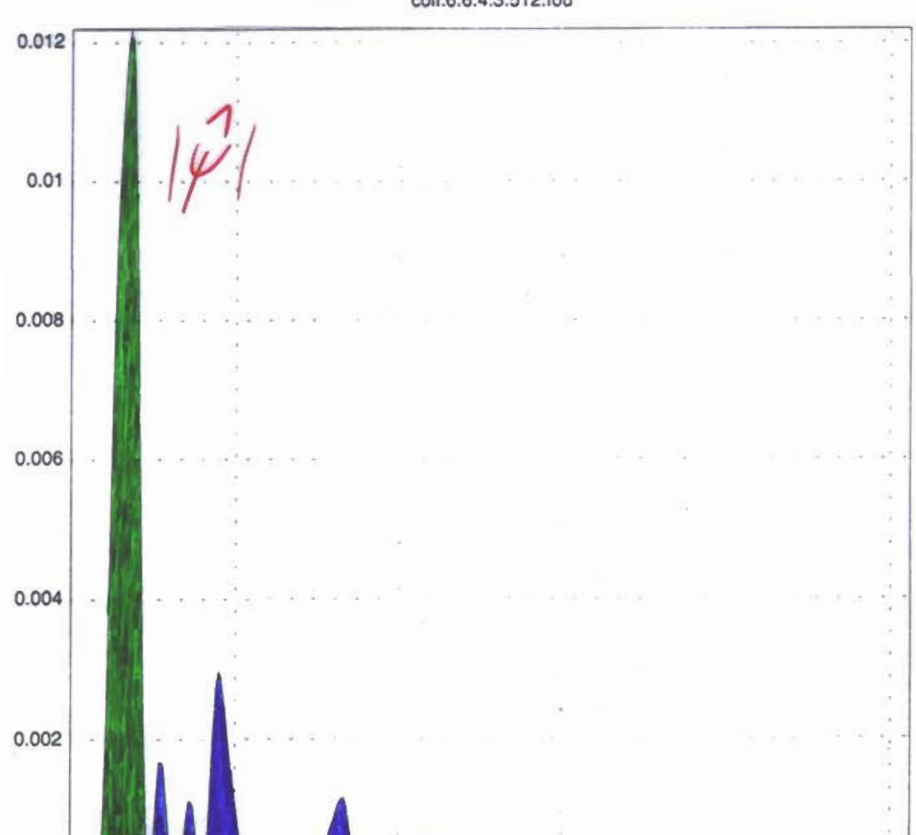


scale 2

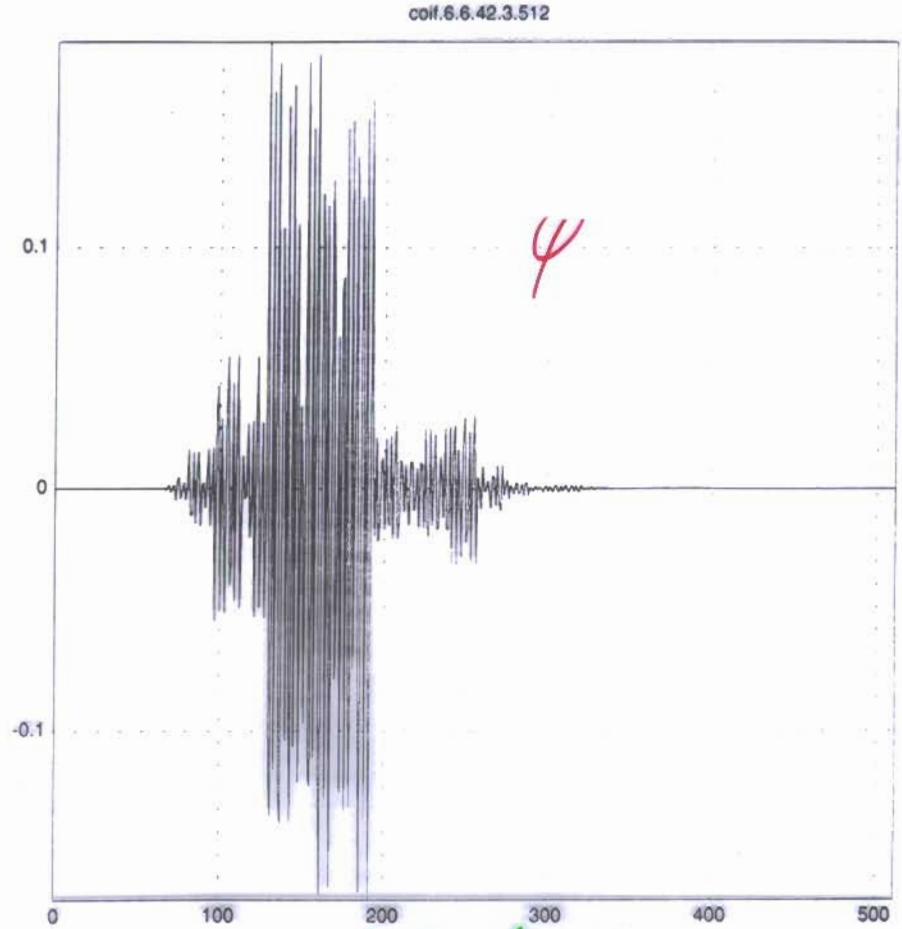




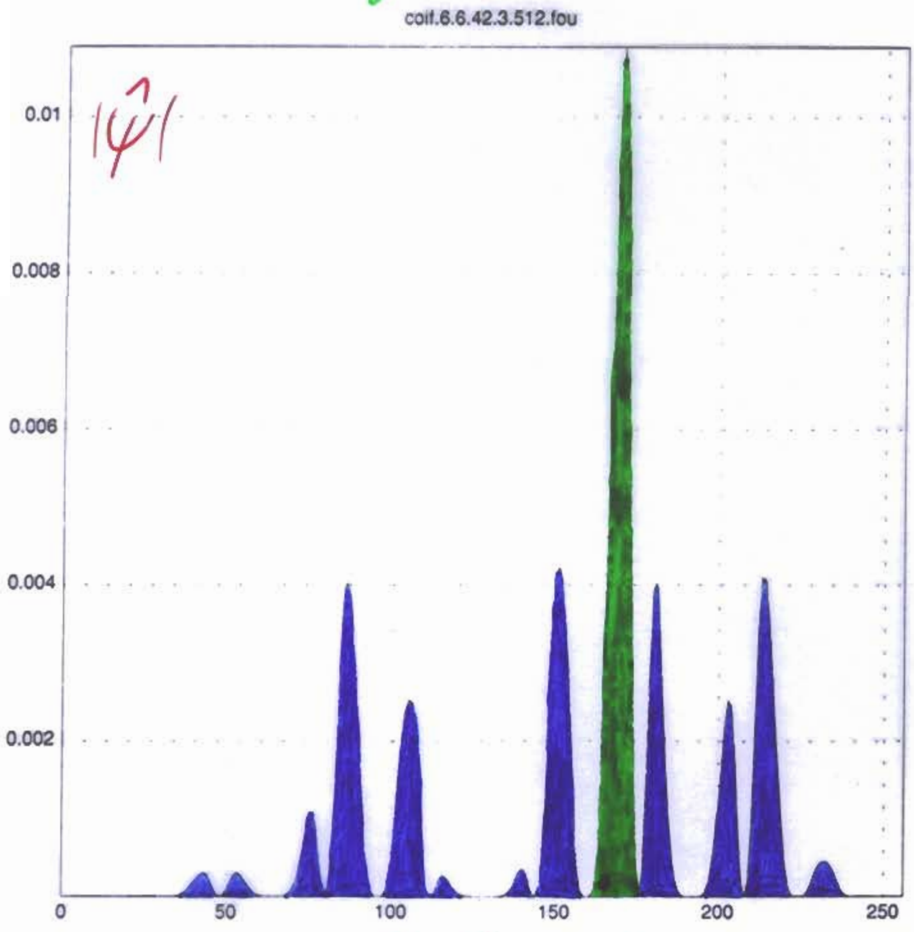
$j=6,\ k=4$



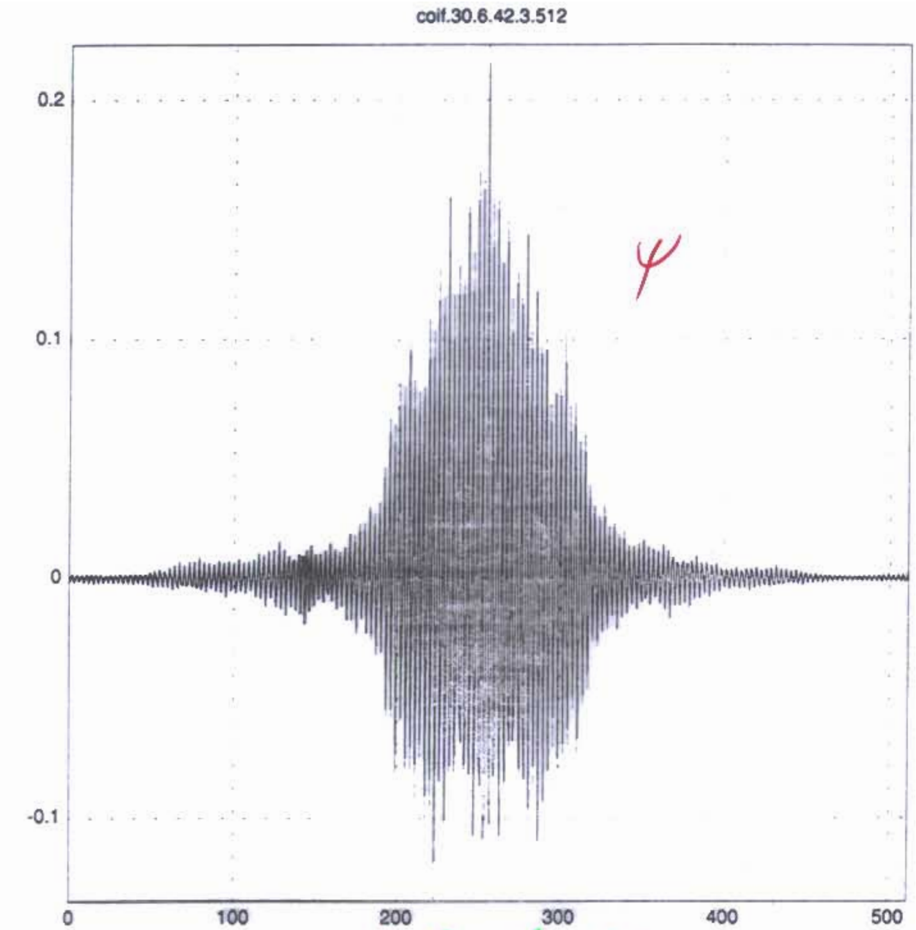
Coifman 6



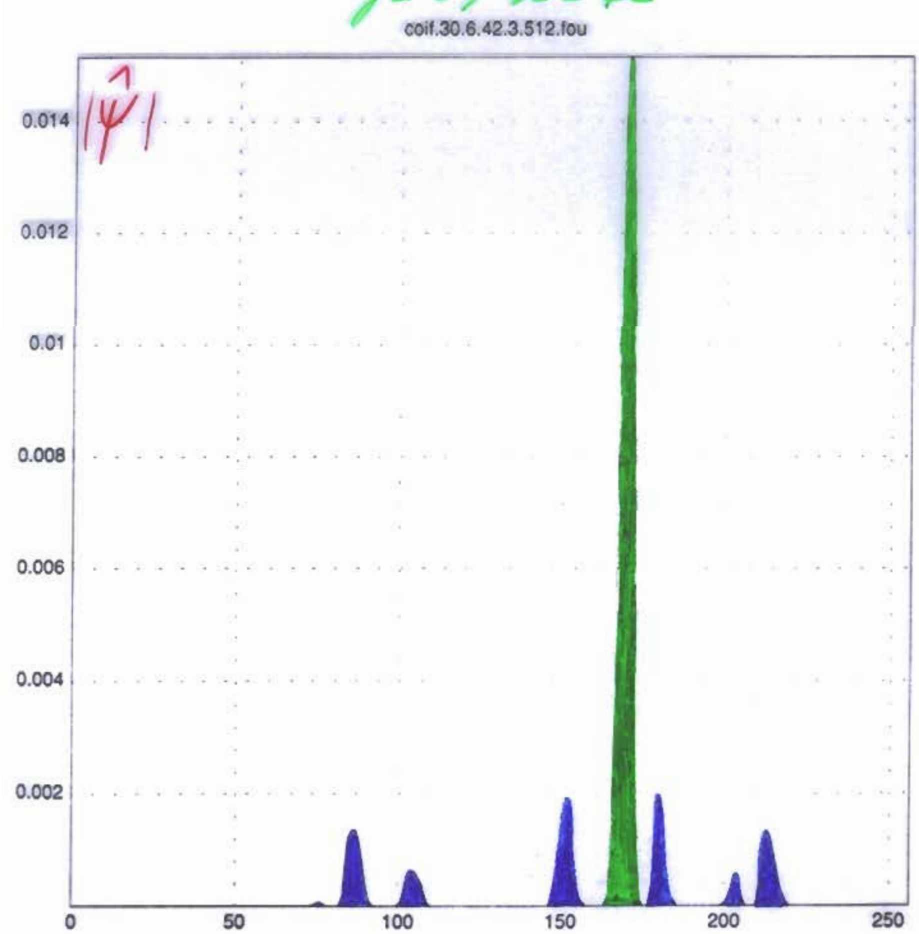
$j=6,\ k=42$



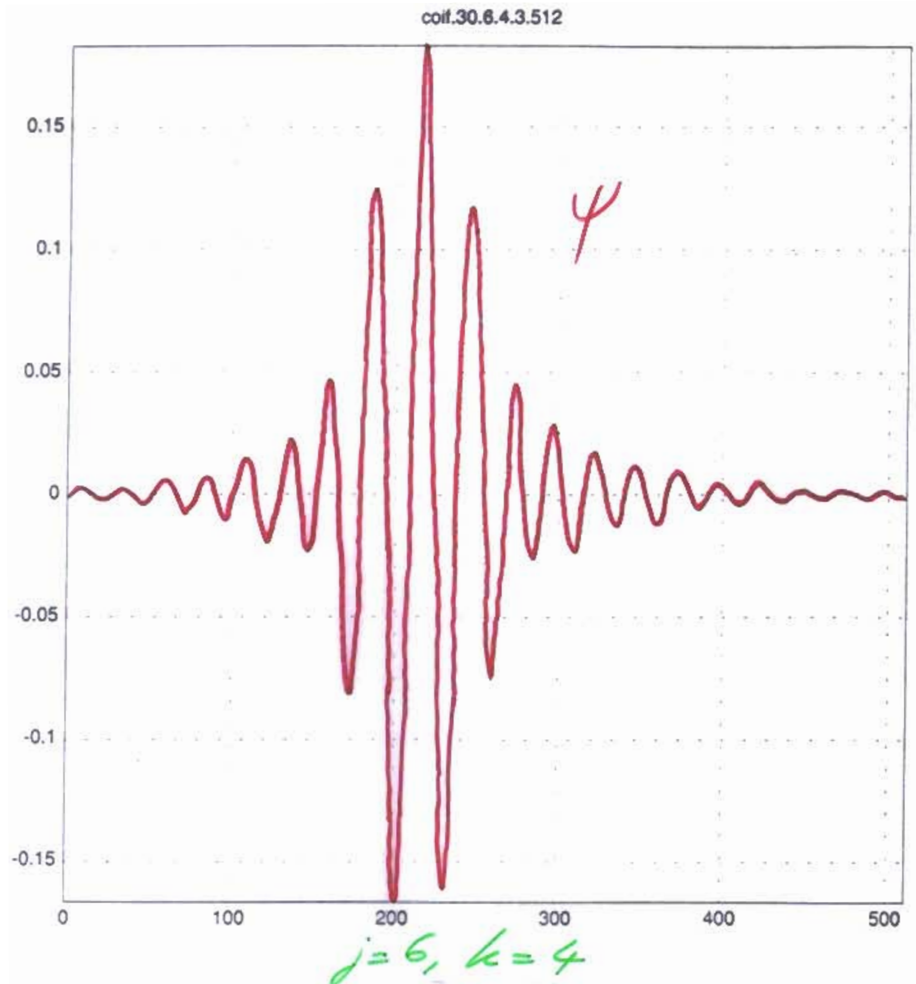
Coifman 6

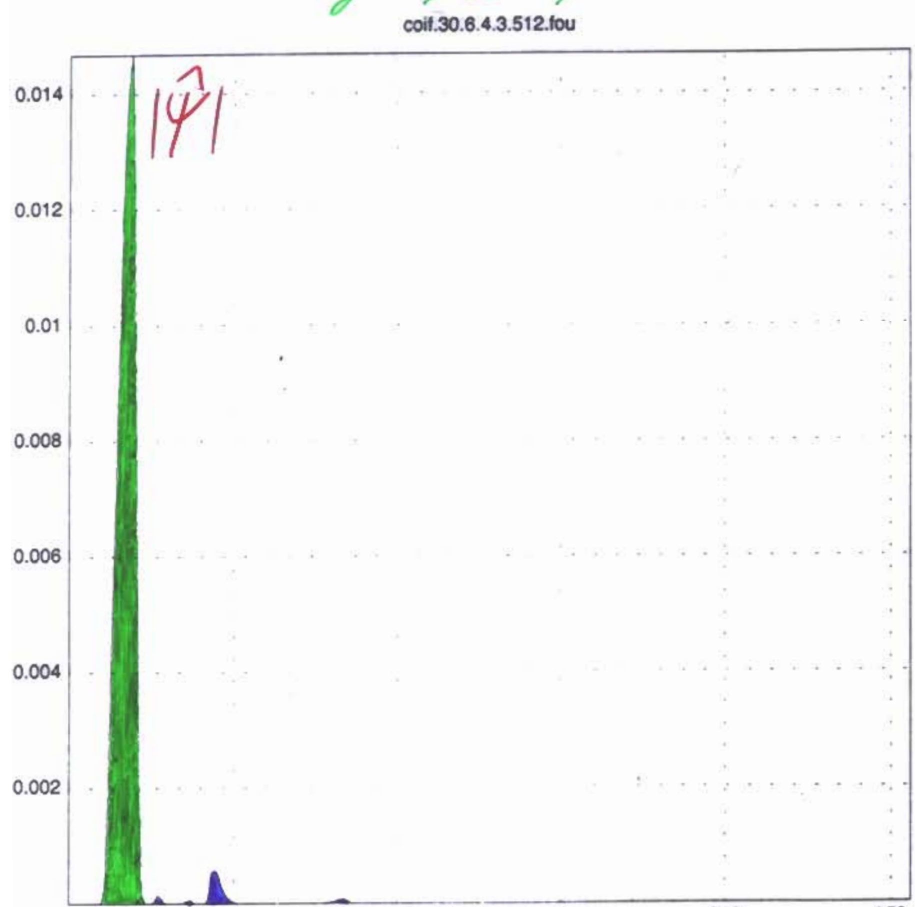


$j=6,\ k=42$



Coifman 30


coif.30.6.4.3.512
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
0
100
200
300
400
500
ψ
j = 6, k = 4


coif.30.6.4.3.512.fou
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
50
100
150
200
250
|ψ̂|

Coifman 30

ORTHOGONAL WAVELET DECOMPOSITION PACKET

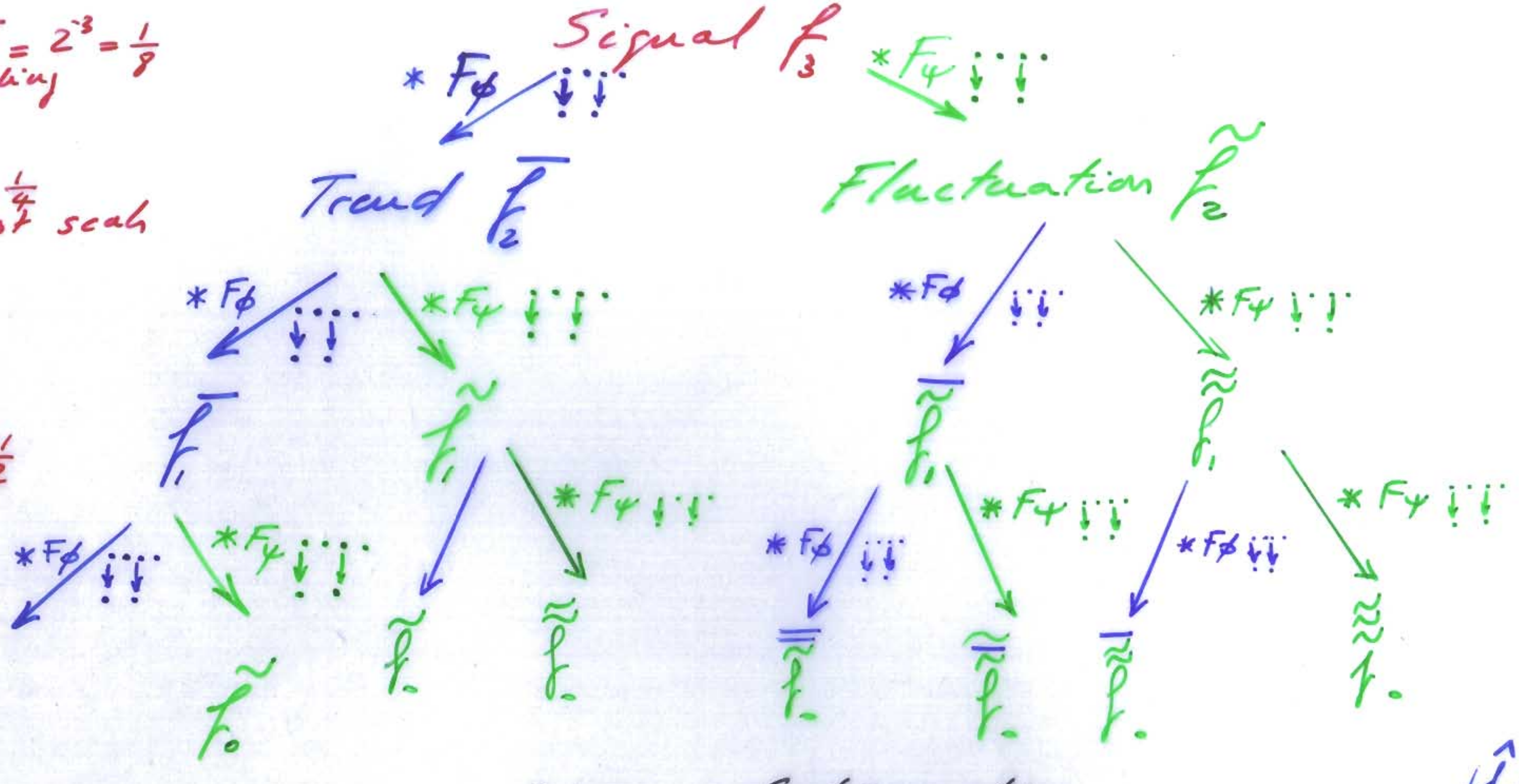
$\ell = 2^{-j}$

$\ell = 2^{-J} = 2^{-3} = \frac{1}{8}$ sampling

$\ell = 2^{-2} = \frac{1}{4}$ smallest scale

$\ell = 2^{-1} = \frac{1}{2}$

$\ell = 2^{0} = 1$

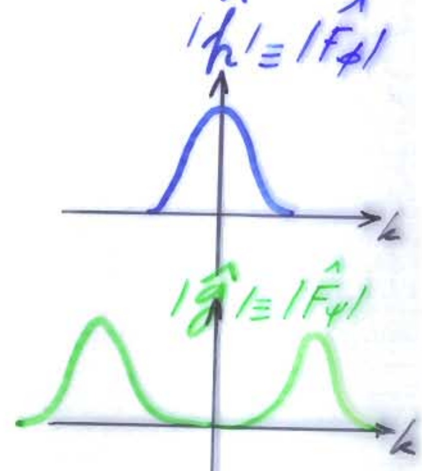


Subsampling

$F_\phi \equiv h$ QMF corresponding to the scaling function ϕ_j

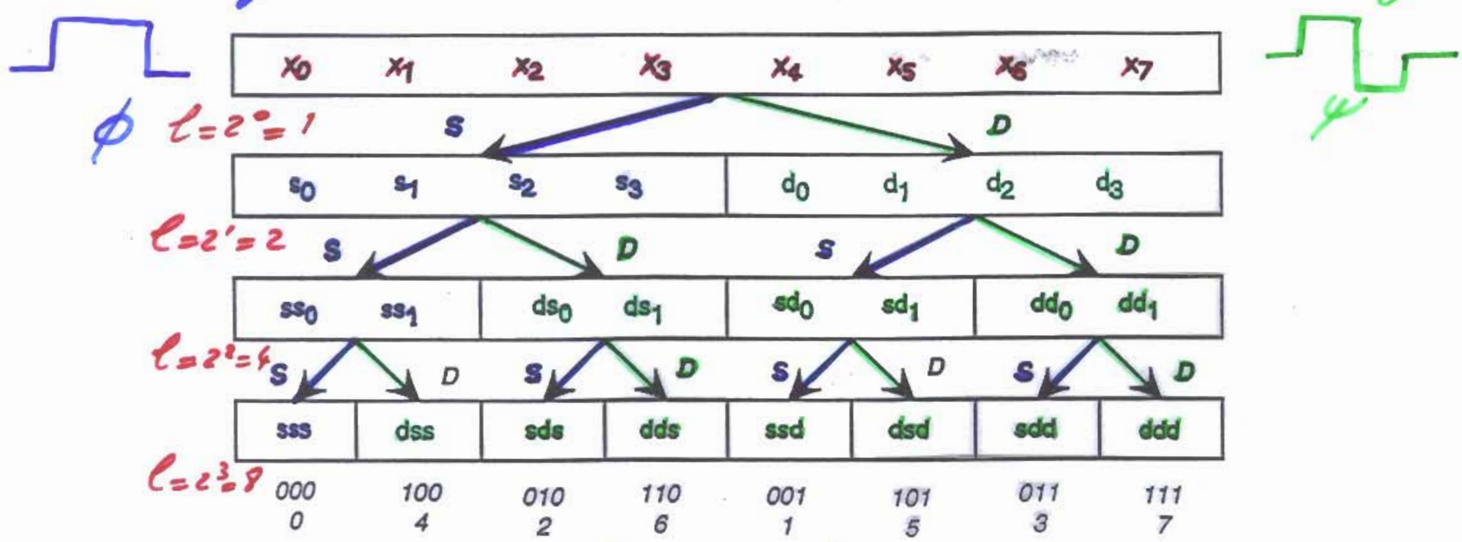
$F_\psi = g$ QMF corresponding to the wavelet ψ_j

Signal $f_0 \equiv \{\bar{f}_0, \tilde{f}_0, \tilde{f}_1, \tilde{f}_2\}$



1D WAVELET PACKET BASES

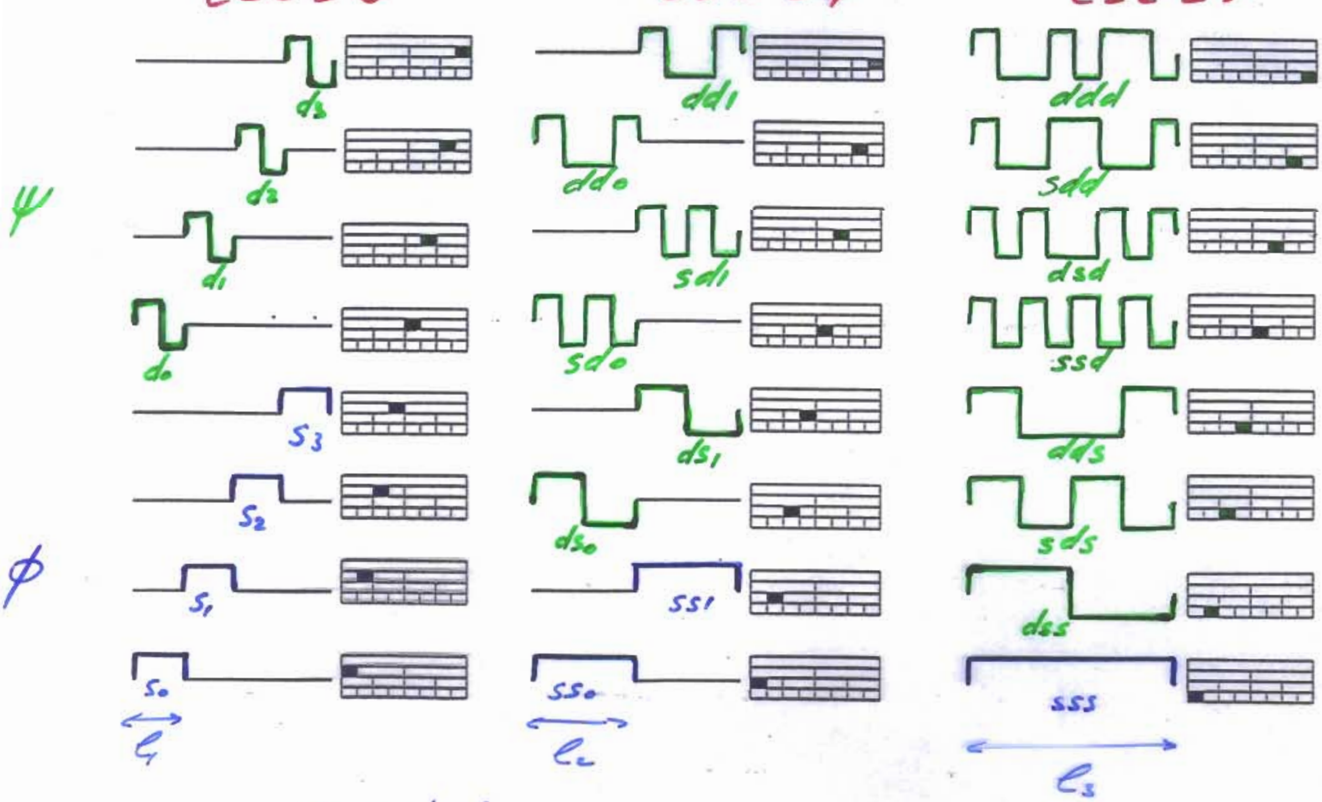
Smoothing S — Naturally ordered wavelet packets on $\mathbb{R}^8$ — Detailing D



Example:
Wavelet packet bases generated from a 8 grid-point sampling using Haar wavelet filter: $h=\left\{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right\}$, $g=\left\{\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right\}$

Haar wavelet packets on $\mathbb{R}^8$:

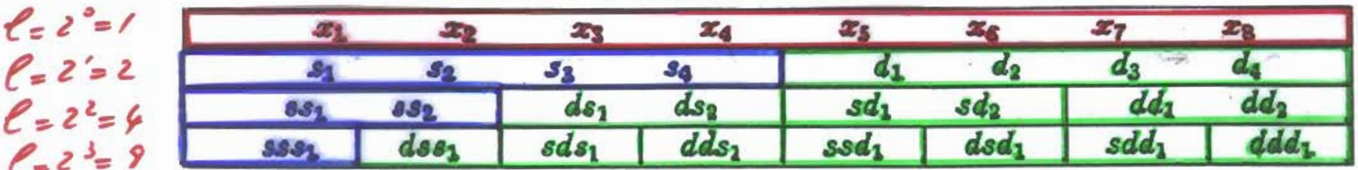
Small	Intermediate	Large
Smallest scale = level 1;	Intermediate scale = level 2;	Largest scale = level 3.
$\ell=2^1=2$	$\ell=2^2=4$	$\ell=2^3=8$



$$ss_0=\frac{1}{\sqrt{2}}(s_0+s_1) \qquad ds_0=\frac{1}{\sqrt{2}}(s_0-s_1)$$

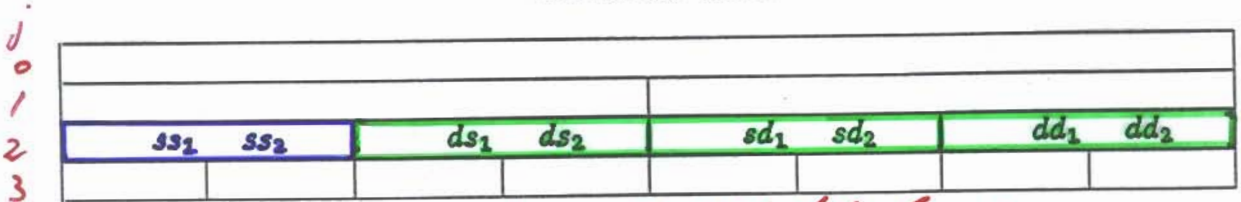
$$ss_1=\frac{1}{\sqrt{2}}(s_2+s_3) \qquad ds_1=\frac{1}{\sqrt{2}}(s_2-s_3)$$

A rectangle of wavelet packet coefficients.



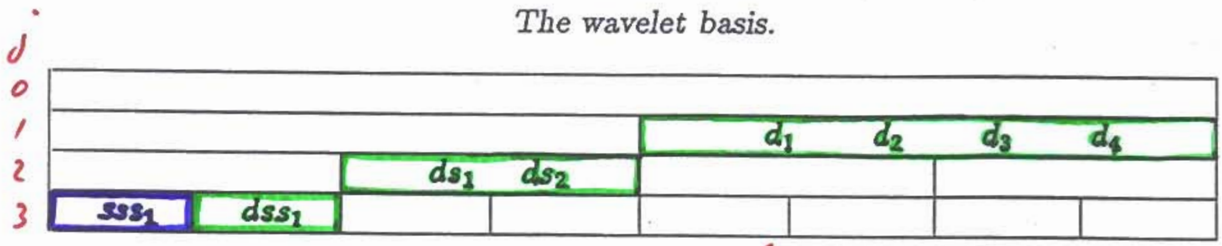
Wavelet packet coefficients

A subband basis.



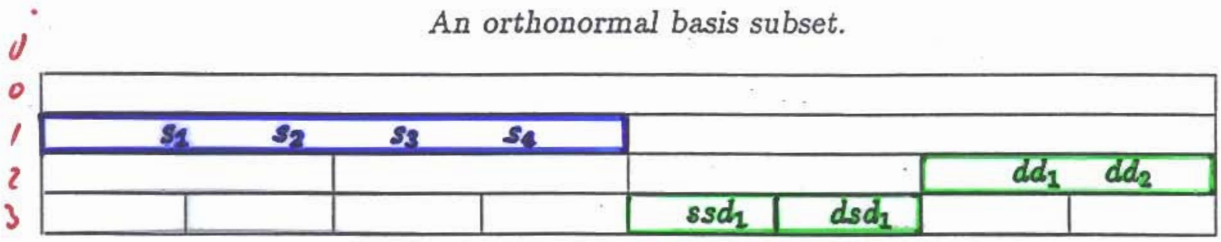
Windowed Walsh basis

The wavelet basis.



Wavelet basis

An orthonormal basis subset.



Other basis

BEST BASIS REPRESENTATION

It is the orthogonal basis made of wavelet packets which gives the smallest Shannon entropy expansion of the vorticity

$$S_B = \min_{\mathcal{B}} S = -\sum |\langle \omega / \psi_{\mathcal{B}} \rangle|^2 \log |\langle \omega / \psi_{\mathcal{B}} \rangle|^2$$

It corresponds to the smallest measure of distance between ω and its WP orthogonal decomposition $\langle \omega / \psi \rangle \Longrightarrow$

Theoretical dimension of ω: $d_B = e^{S_B}$

To compress information we then truncate in B and keep only $\langle \omega / \psi_B \rangle > \varepsilon$.
The threshold ε can be defined a priori if we assume that the incoherent components (those $< \varepsilon$) correspond to iid zero mean Gaussian noise (cf. Donoho and Johnstone's Wavelet Shrinkage method)

BEST BASIS SELECTION

For a given signal f, made of N samples, we project it onto all orthogonal bases B_n $(n = 1, 2^N)$:

$$\tilde{f}_{ni} = \langle f / \psi_{ni} \rangle$$

For each projection n we compute the cost function:

$$c_n = \sum_i \tilde{f}_{ni}^2 \log \tilde{f}_{ni}^2$$

and we select the projection which minimizes c as the best basis.

The total operation cost of the best basis selection is $N \log_2 N$ operations.

INFORMATION COST FUNCTIONS

We define a real-valued functional $\mathcal{M}_m(f)$ on sequences $\{\tilde{f}_m\}$, set of all projections of the signal f onto all possible orthogonal bases ψ_m, describing the cost of representing f in the basis ψ_m.

Then we search for the most economical representation, which gives the best basis for f.

POSSIBLE $\mu_m(f)$

- Number of coefficients $\tilde{f}_m$ above a threshold value ε:

  $$\mu_m(f) = \#\{\tilde{f}_m < \varepsilon\}$$

- ℓ^p norm with $p<2$:

  $$\mu_m(f) = \sum_i |\tilde{f}_{mi}|^p$$

- Shannon-Weaver information entropy:

  $$\mu_m(f) = -\sum_i |\tilde{f}_{mi}|^2 \log |\tilde{f}_{mi}|^2$$

- Logarithm of energy:

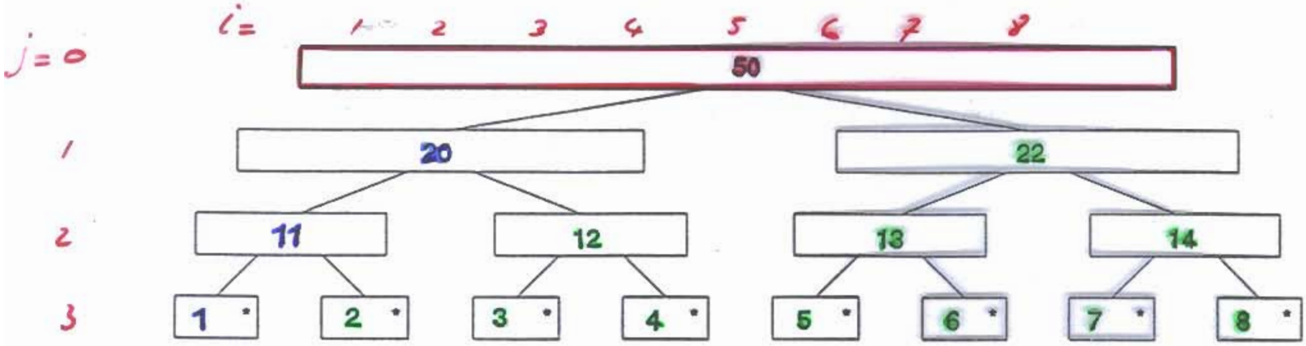
  $$\mu_m(f) = \sum_i \log |\tilde{f}_{mi}|^2$$

- Bit counting:

  $$\mu_m(f) = \sum_i \log_2 \left(1 + \frac{\tilde{f}_{mi}}{\varepsilon}\right)$$

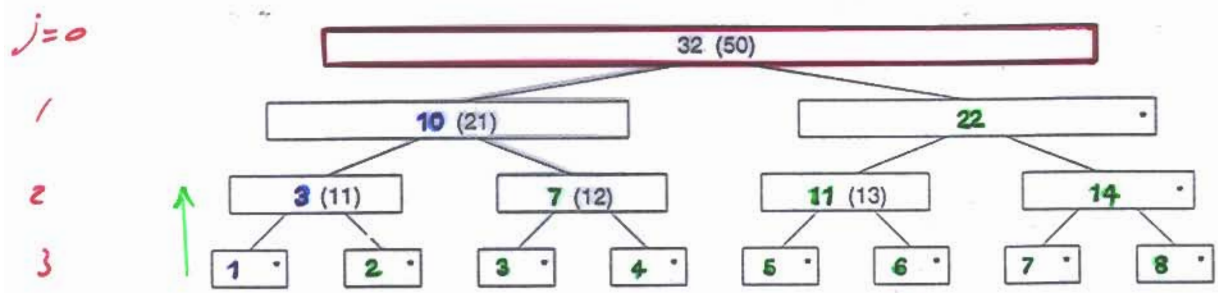
  with ε machine precision.

Step 1 in the best basis search: mark all bottom nodes



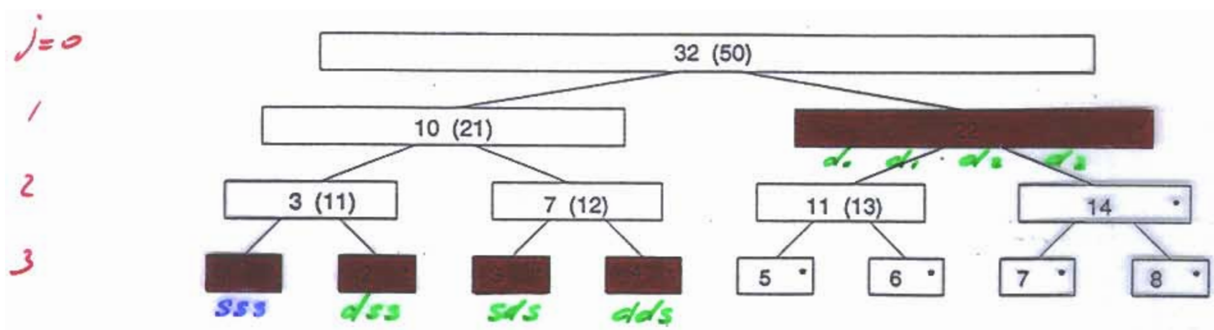
All possible wavelet packets

Step 2 in the best basis search: mark all nodes of lower cost



Lower cost nodes

Step 3 in the best basis search: retain topmost marked nodes



Best Basis

THEORETICAL DIMENSION

The dimension of the representation in basis B_n is:

$$d_n = e^{-C_n}$$

It measures how many non negligible coefficients are present in the representation of f in basis B_n.

This dimension is minimal for the best basis representation of f which gives the theoretical dimension d_{th} of f. It is related to the dimension of the approximate inertial manifold characterizing the dynamical system which has generated f.

$d_{th} \ll N$ low-order dynamics

$d_{th} = N$ white noise

THEORETICAL DIMENSION

We can compare sequences $\{\tilde{f}_m\}$ by their rate of decay, namely the rate at which their elements become negligible if they are rearranged in decreasing order.

The theoretical dimension of $\{\tilde{f}_m\}$ is:

$$d = \exp\left\{-\sum_i |\tilde{f}_{mi}|^2 \log |\tilde{f}_{mi}|^2\right\}$$

$\Longrightarrow$

the most efficient representation of f will correspond to the lower dimension d.

BEST BASIS SELECTION ALGORITHM

The field is defined on $N = 2^J$ grid-points

1. Then we generate the $2^{N/2}$ possible orthogonal bases constructed with $\psi^m_{\ell,j,i,k}$

2. We project ω onto each possible orthogonal basis m

3. We compute the information entropy $\sigma_m = -\sum_i \tilde{\omega}^2_{mi} \log \tilde{\omega}^2_{mi}$ for each projection m

4. We select the projection $\overline{m}$ which maximizes the entropy σ_m

5. We project ω onto the best basis $\overline{m}$ and get N coefficients $\tilde{\omega}_{\overline{m}}$

6. We select the $N' \ll N$ strongest coefficients $\tilde{\omega}_{\overline{m}}$

7. We reconstruct $\omega_>$ from the N' strongest coefficients

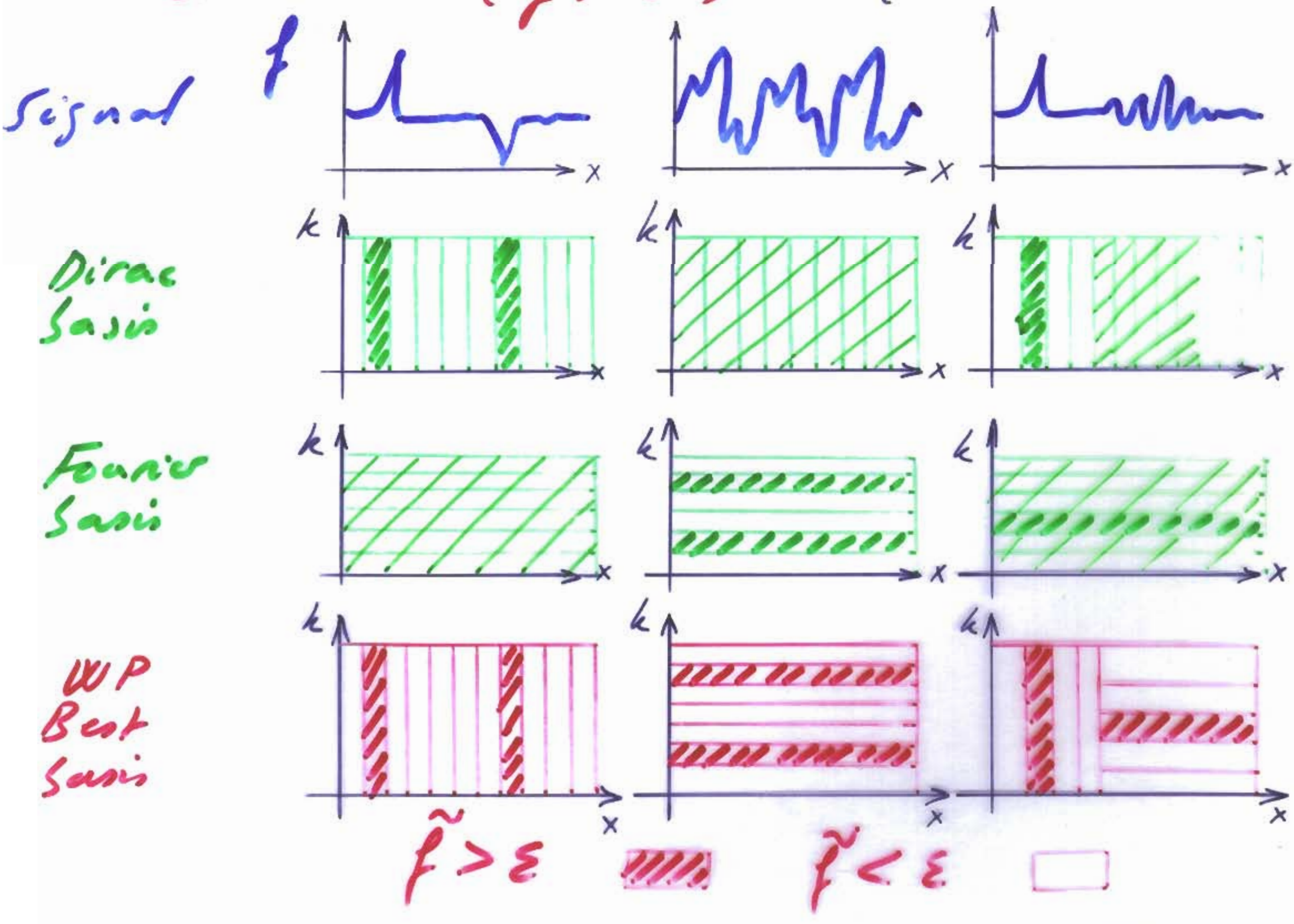
8. We reconstruct $\omega_<$ from the $N - N'$ weakest coefficients

The computation of the whole procedure requires $N \log_2 N$ operations

'BEST BASIS' SELECTION

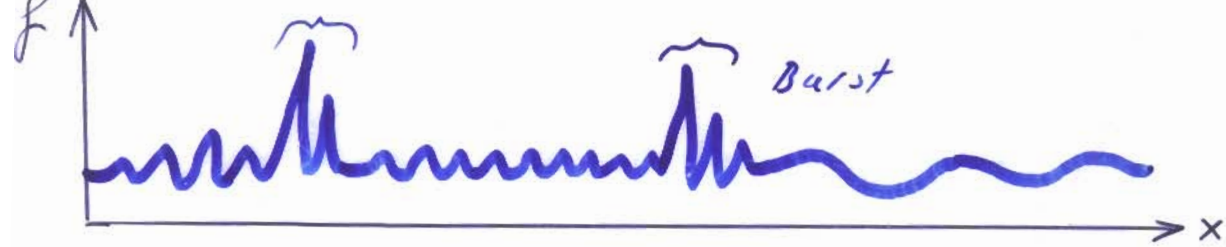
Applied to a signal sampled on N points, the wavelet packet algorithm generates the 2^N possible orthogonal bases and then selects the 'best basis', i.e. the one which minimizes the number of significant coefficients ($\tilde{f} > \varepsilon$).

$$c = +\sum_i |\tilde{f}_i|^2 \log |\tilde{f}_i|^2$$

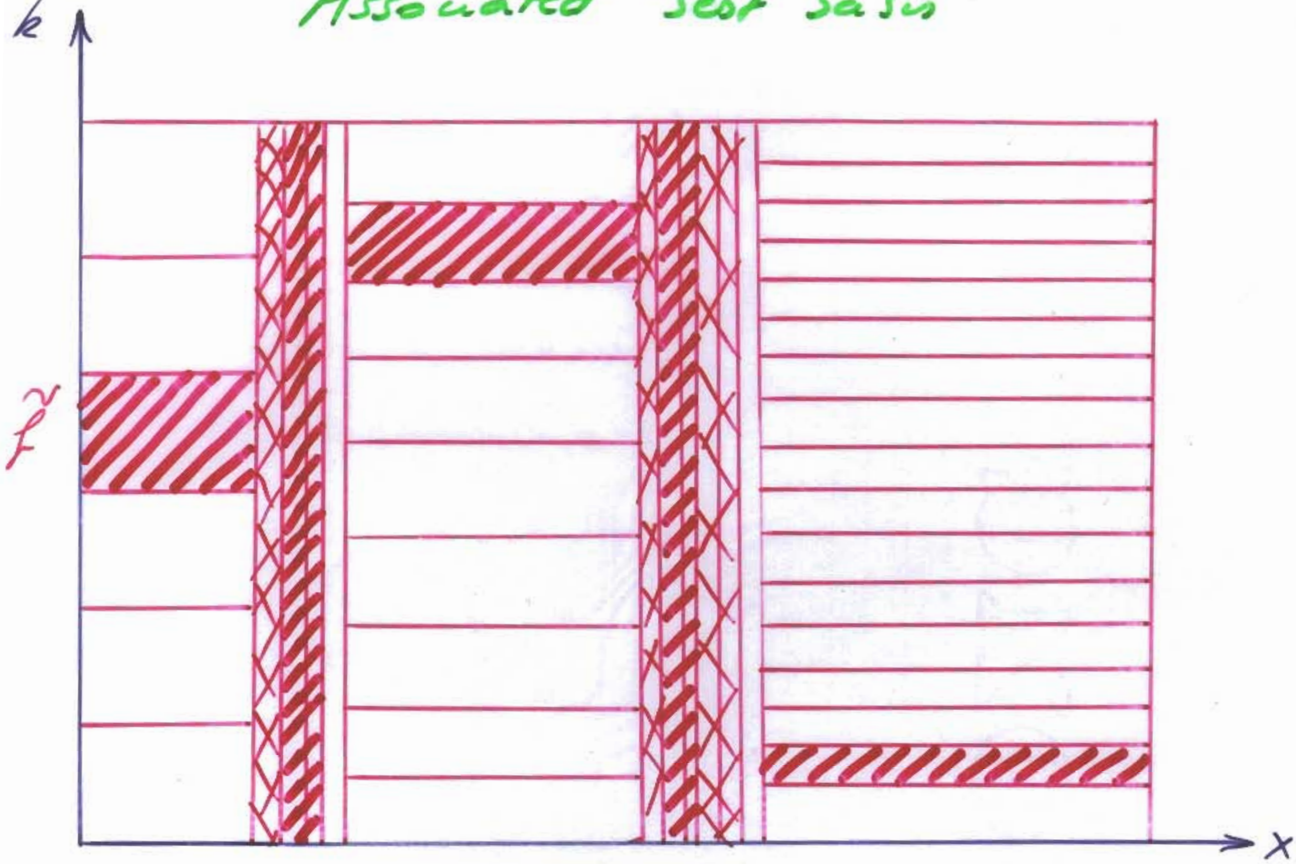


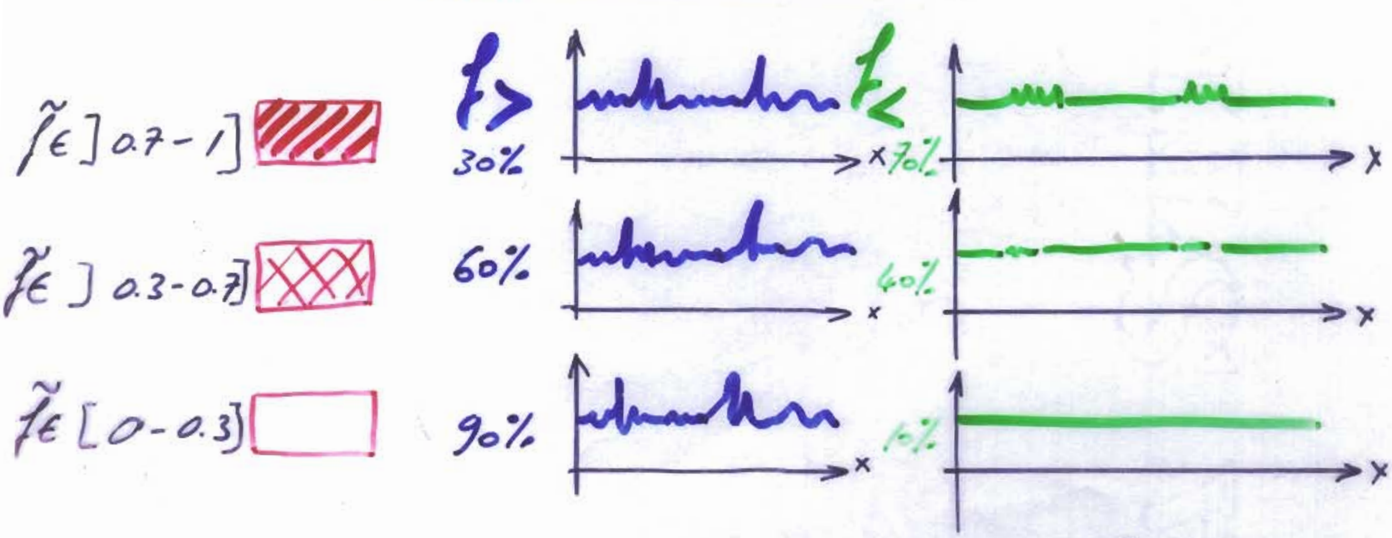
EXAMPLE

1D Intermittent turbulent signal



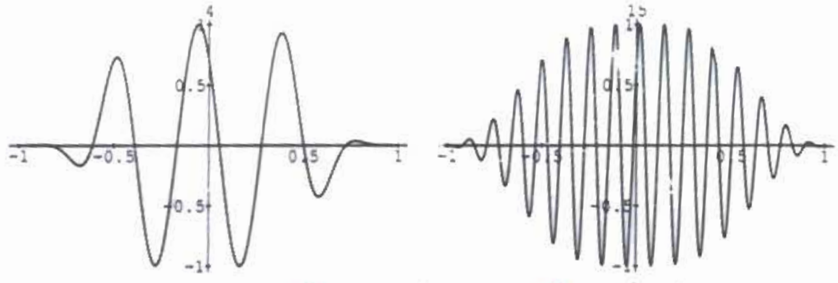
Associated 'best basis'



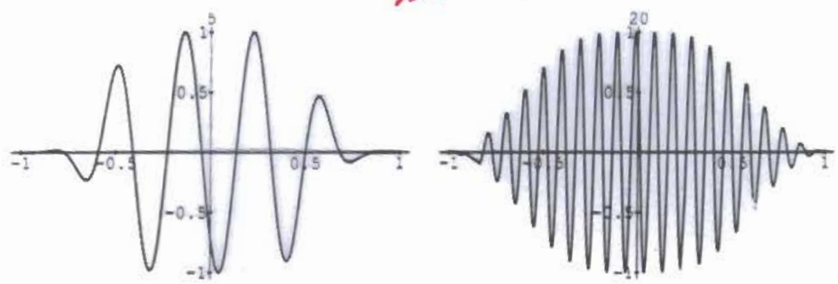


Reconstructions

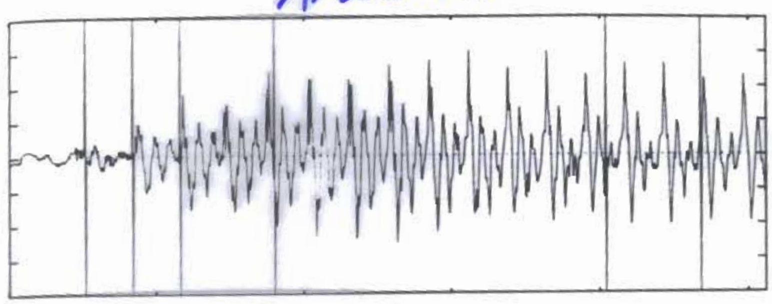
SPEECH SIGNAL COMPRESSION



Local adapted sines



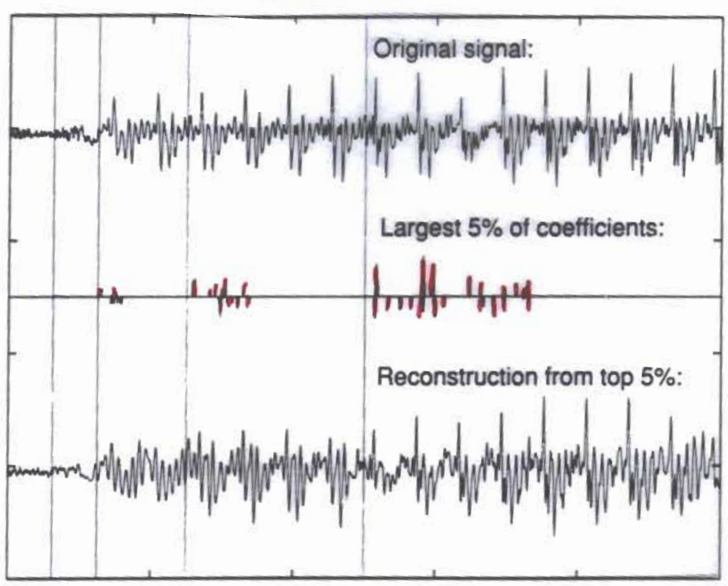
'Armadillo'



Lowest entropy windows ≡ largest efficiency representation

FIGURE 2. Automatic segmentation of a voice recording (armadillo) by using least entropy windowing in the local sine library. The windows are selected to obtain optimal efficiency in representing the signal. We see different patterns fall into distinct windows.

'Armadillo'



Selection of windows of lowest entropy

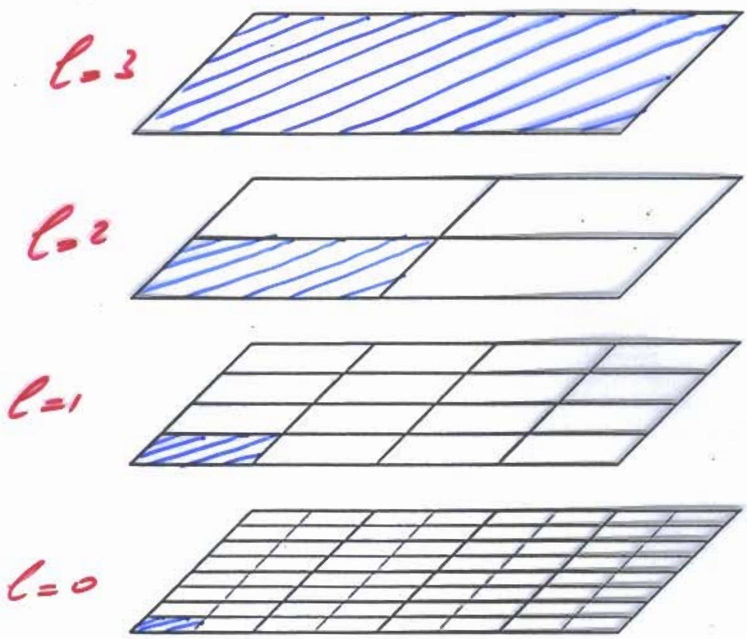
Sorting of the coefficients

Reconstruction from the top 5% coefficients

FIGURE 1. The first 1024 samples ($\frac{1}{8}$ second) of the word armadillo, are plotted on the top part. The library of local sine waveforms is then used to select the combination of windows of highest efficiency (lowest entropy). Expansion coefficients are then ordered by window in decreasing order. The top 5% are plotted in the center and used to reconstruct a compressed form of the signal which is plotted below.

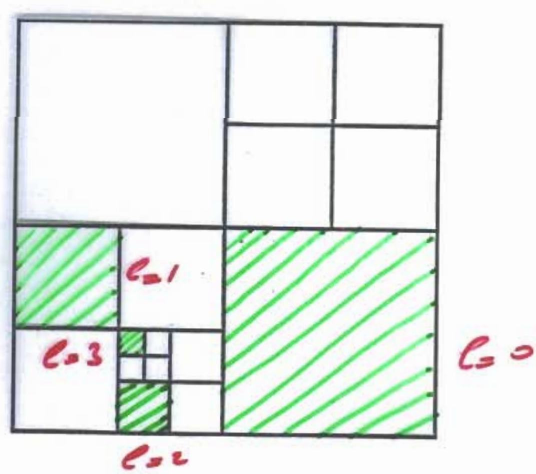
2D WAVELET PACKET BASES

2-dimensional wavelet decomposition to level 3



Example 2-dimensional wavelet packet bases

Wavelet basis

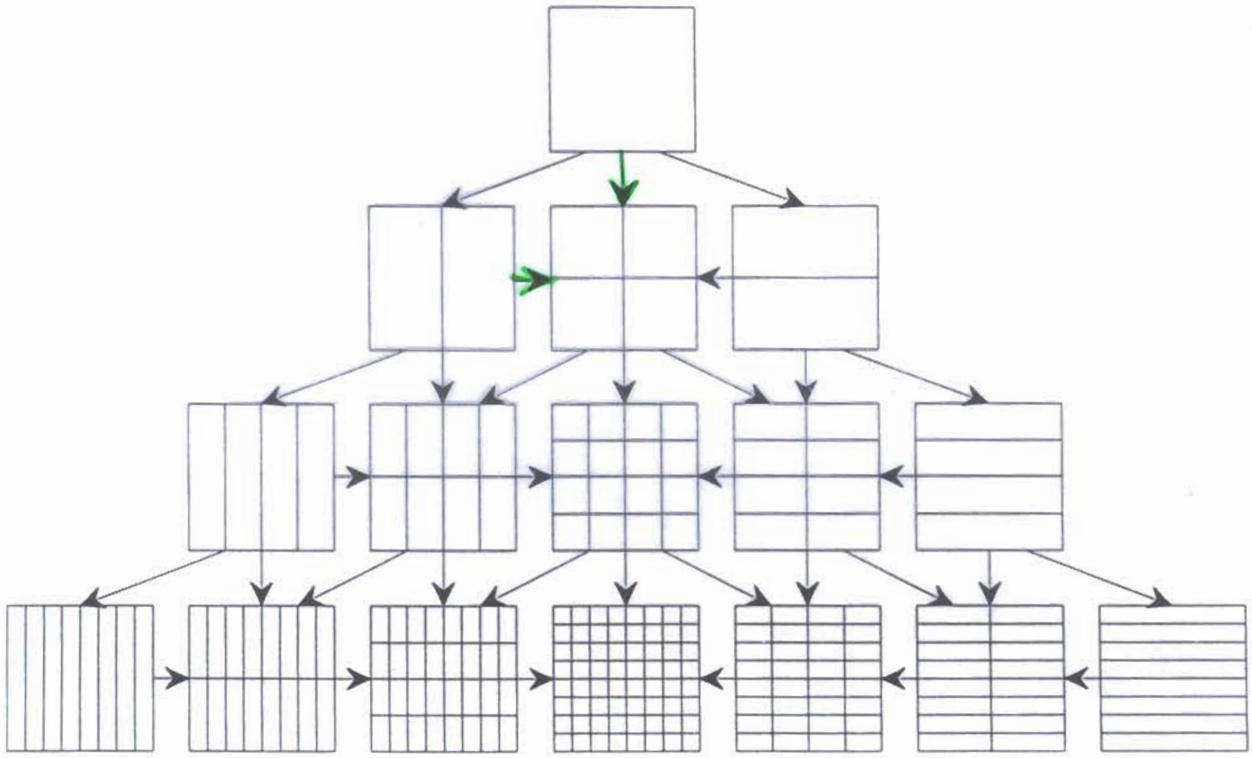


Another basis

2D WAVELET PACKET ALGORITHM

→ Convolution - decimation in the X direction

↓ Convolution - decimation in the Y direction



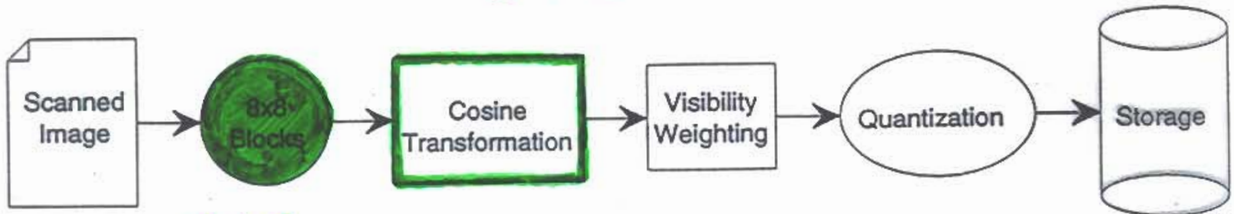
All tensor products of Wavelet Packets in 2D

There are $O[N(\log N)^2]$ coefficients

IMAGE COMPRESSION

JPEG picture compression

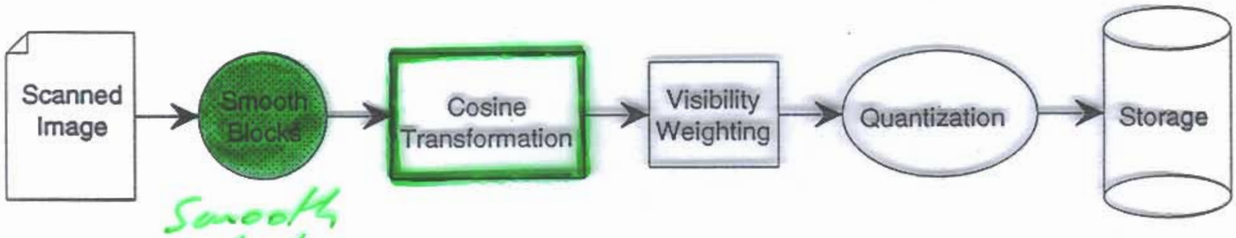
JPEG



8 X 8 Blocks

Local cosine picture compression

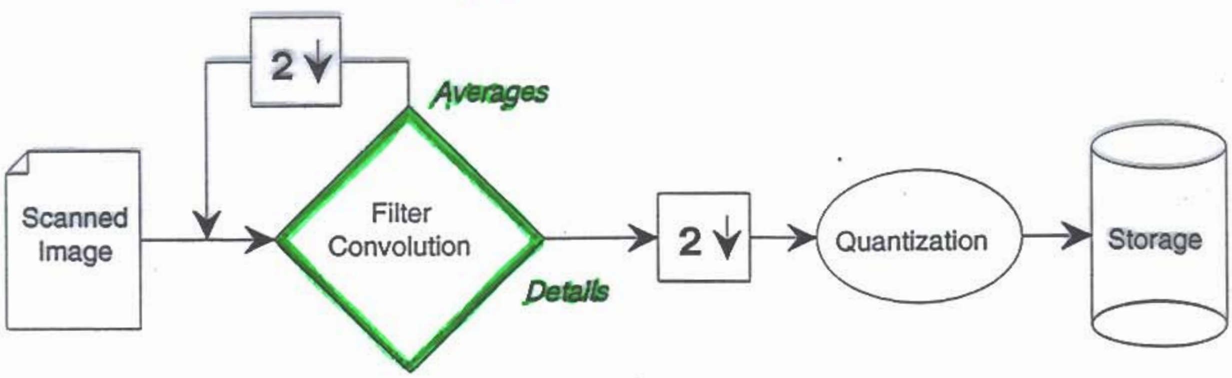
Local Cosines



Smooth Blocks

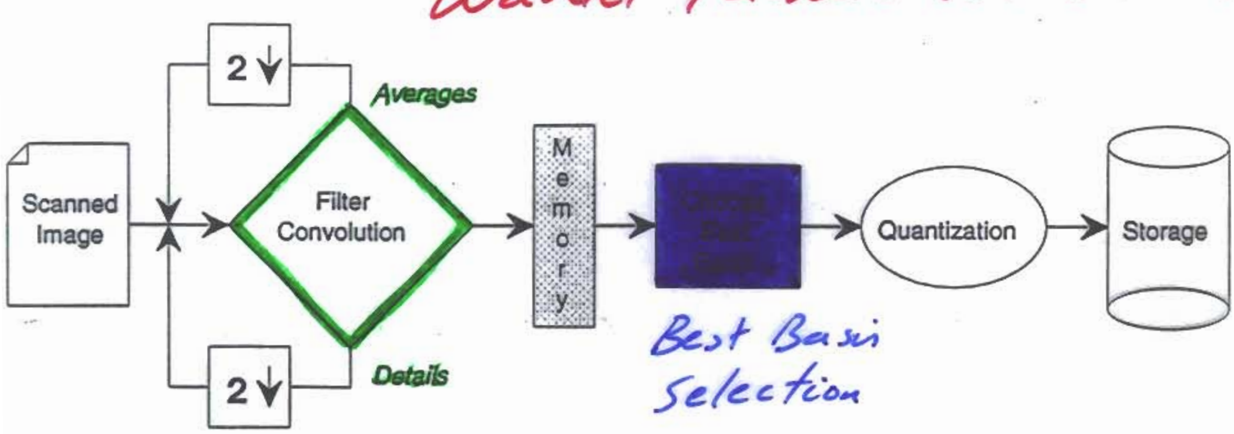
Wavelet or multiresolution picture compression

Wavelets



Wavelet packet best-basis picture compression

Wavelet Packets (JPEG 2000)



Best Basis Selection

TURBULENT FLOW COMPRESSION

We are looking for the most concentrated representation of the vorticity field, in order to have the lowest dimensional system to compute and to store.

$\longrightarrow$

We want to find the orthogonal basis which will avoid unuseful complexity and therefore optimally concentrate information.

Karhunen-Loève (POD or PCA) approach

designed for statistical ensembles

Considering a set of realizations and inverting the correlation matrix, it provides an orthonormal basis which is optimal in a L^2 average sense

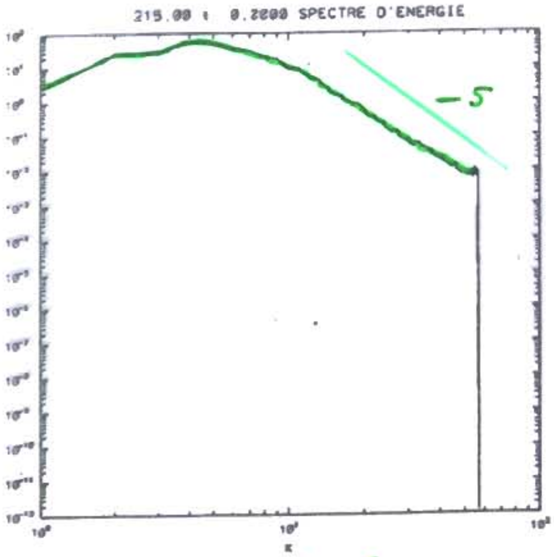
$\Longrightarrow N^3$ operations

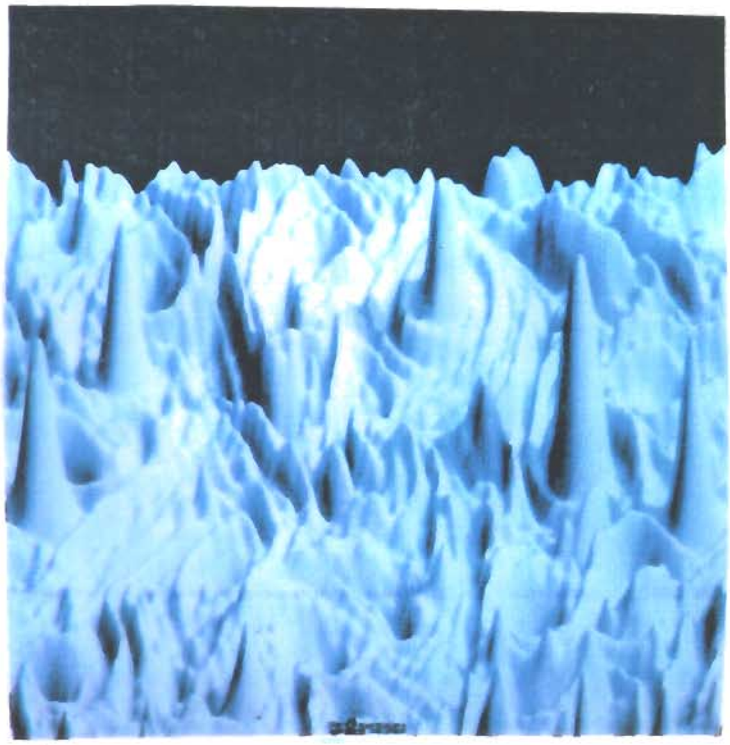
'Best Basis' approach

designed for a given realization

Search the basis which minimizes entropy out of a large library of orthogonal bases

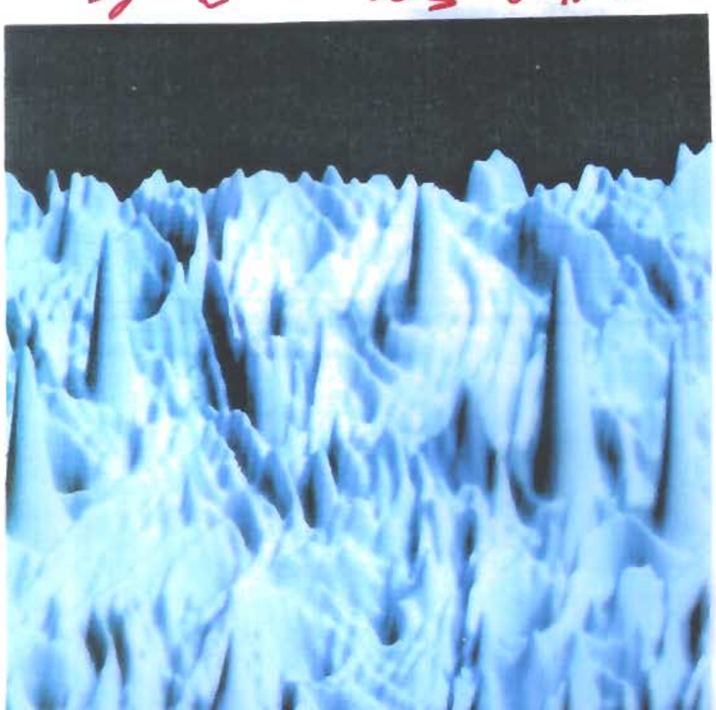
$\Longrightarrow N \log N$ operations



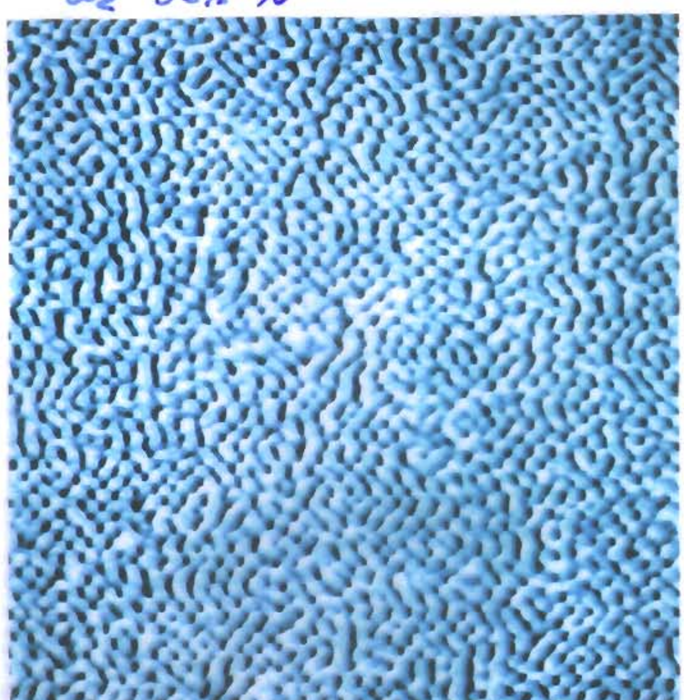


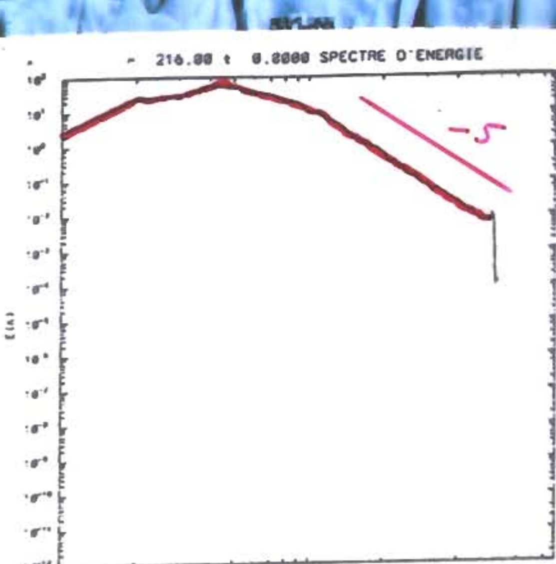
Vorticity field
$N = 256^2$

Compression by 2 $\quad \omega > 50\%\ N$ $\qquad \omega$ $\qquad \omega < 50\%\ N$



+



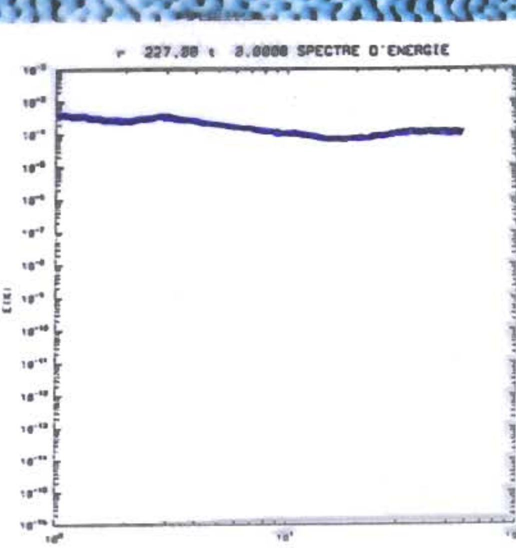


'Active' vortices

Splitting using orthogonal wavelet packets

99.96%

(Farge & al. Fluid Dyn. Res. 10, 229, 1992)



'Passive' background

$N = 262\,144$ grid-points $= 512^2$

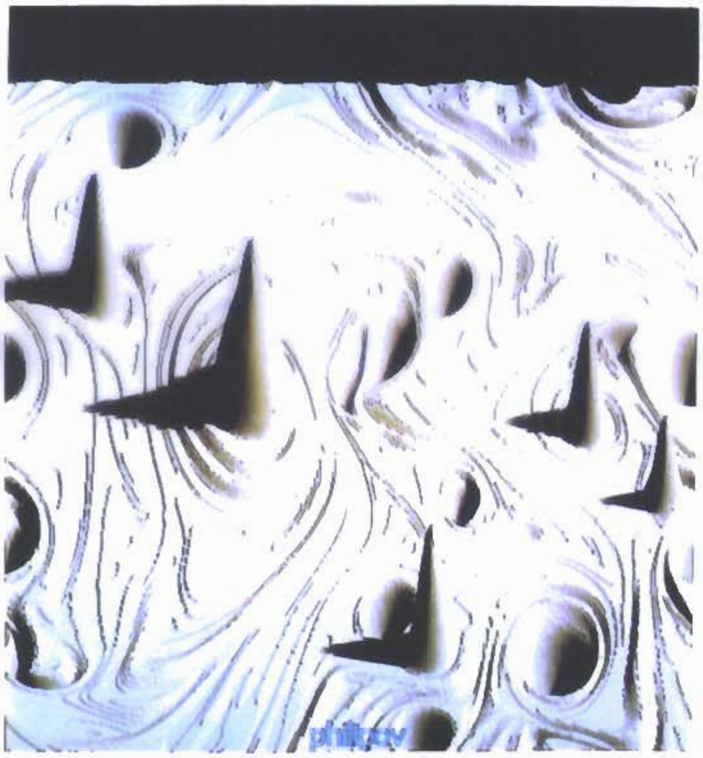
Vorticity field reconstructed from the strongest Wavelet Packets

99.9% Z
3000 WP
1.14 % N

99%
739 WP
0.28% N

90% Z
73 WP
0.028% N

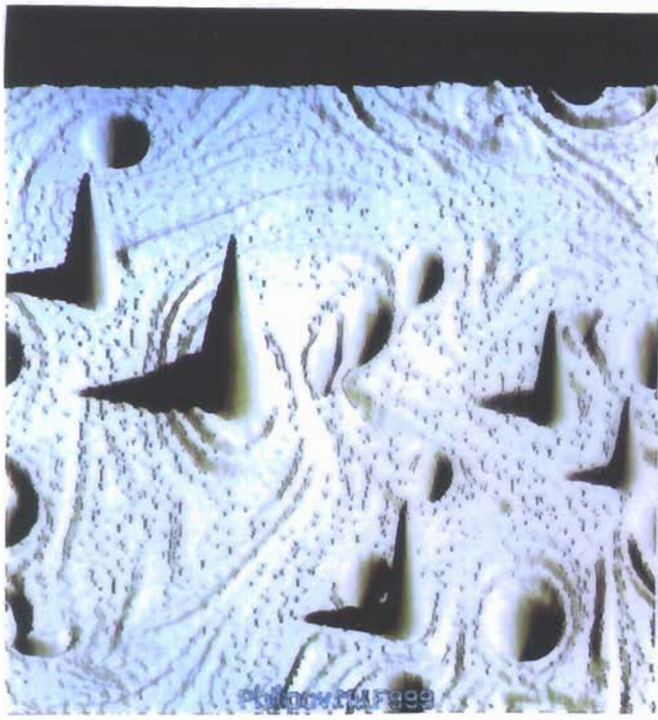
50% Z
14 WP
0.005% N

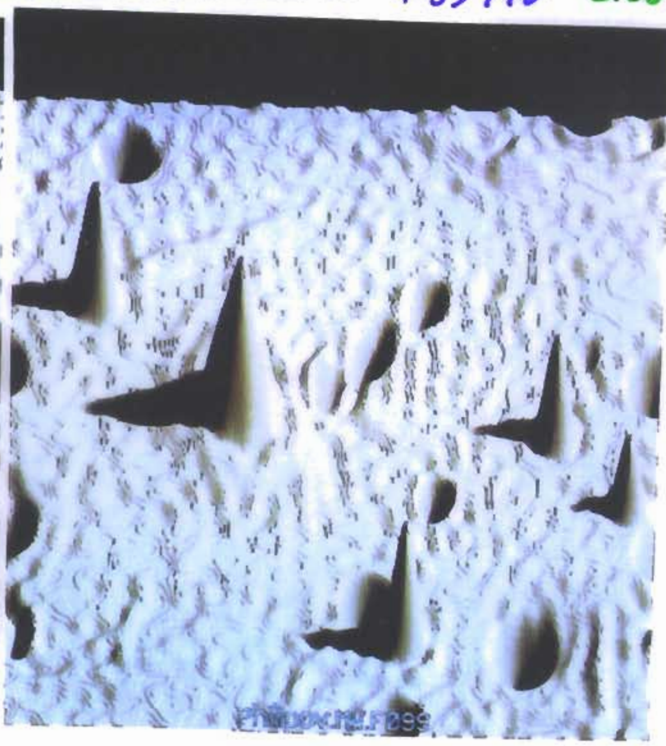


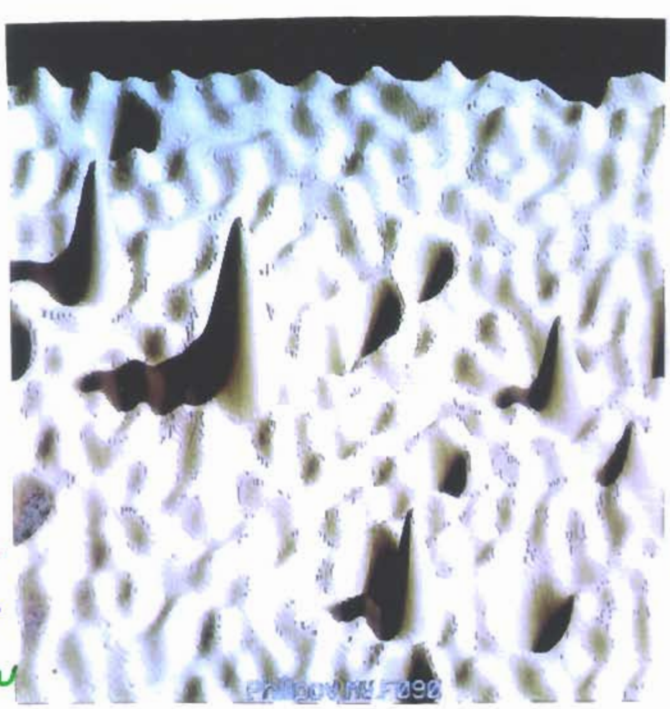
Vorticity field reconstructed from the strongest Malvar wavelets (locally adapted cosines)

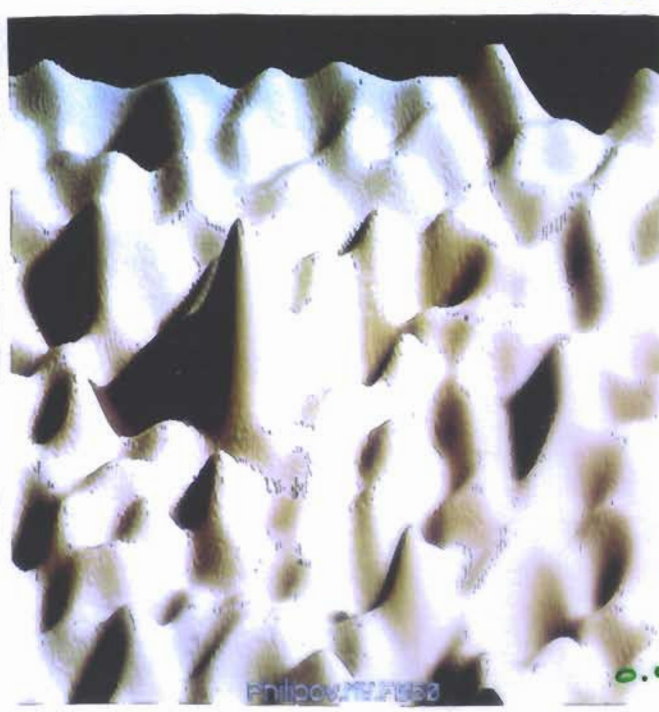
99.3% Z
3000 MW
1.14% N

96% Z
739 MV 0.28% N



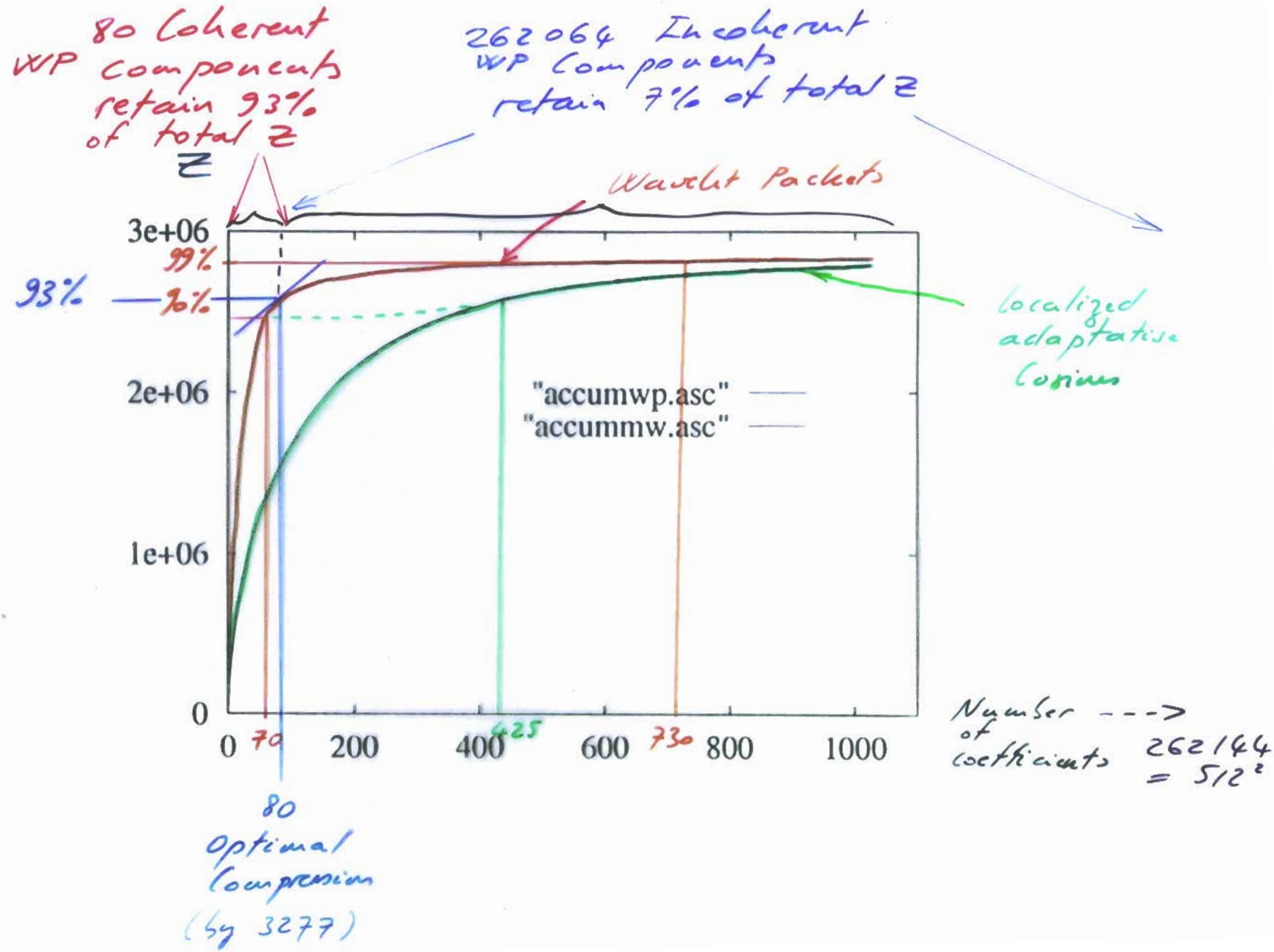






45% Z
73 MV
0.028% N

15% Z
14 MW
0.005% N



80 Coherent WP components retain 93% of total Z
262064 Incoherent WP Components retain 7% of total Z
Z
Wavelet Packets
3e+06
2e+06
1e+06
0
99%
93%
90%
"accumwp.asc"
"accummw.asc"
localized adaptative Cosines
0
70
200
400
425
600
730
800
1000
Number of coefficients --->
262144 = 512²
80 Optimal Compression (by 3277)

Nicholas KEVLAHAN & Marie FARGE

JFM 97

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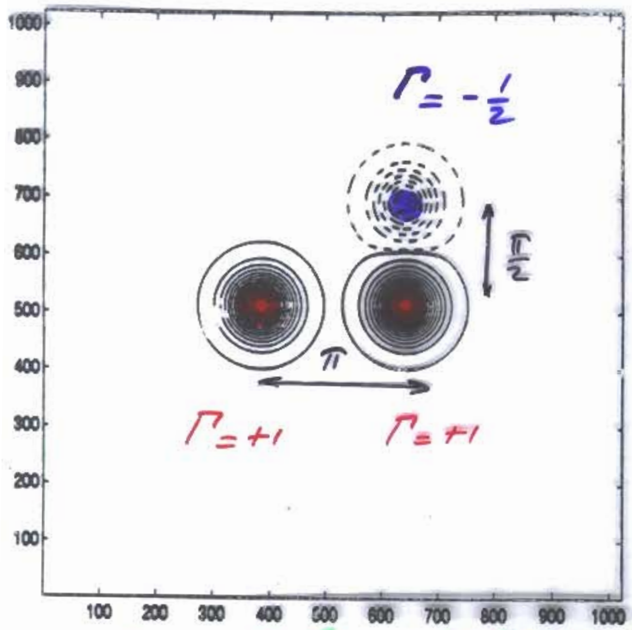
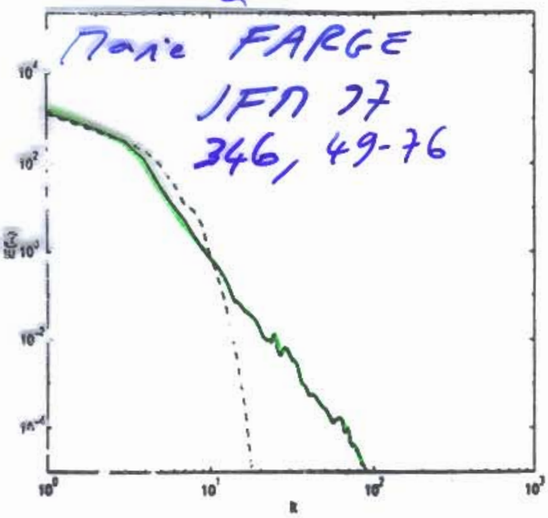


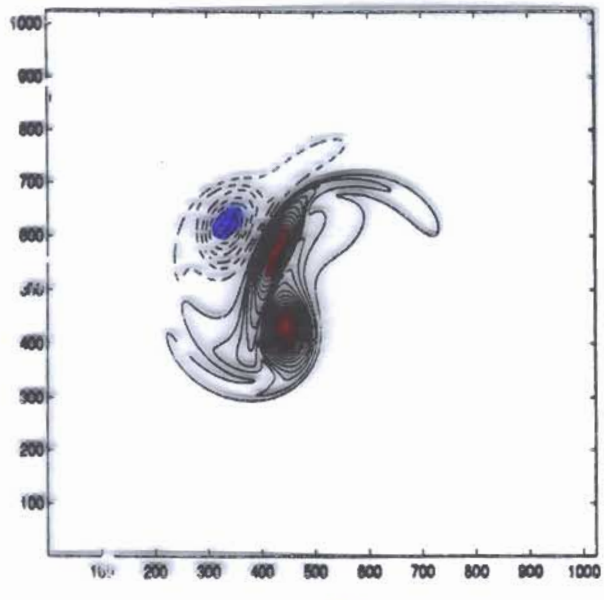


Inspired from Aref & Novikov 3 vortex configuration

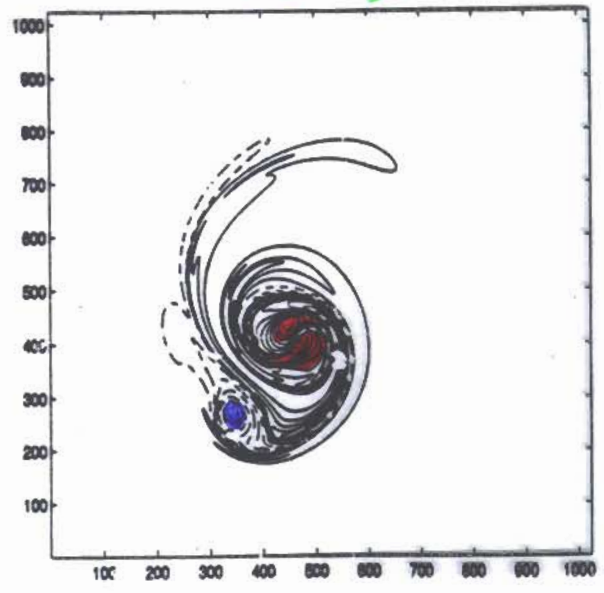
$N = 1024^2$

$D = 5.10^{-5}\,\nabla^2$

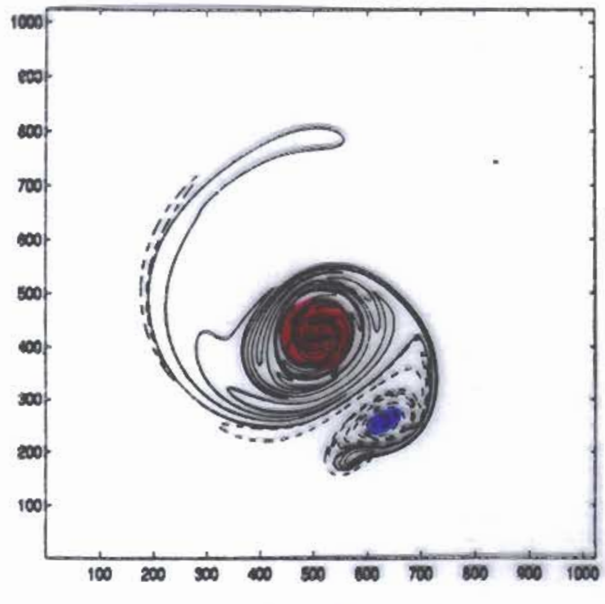
'Re' $= \ldots = 2.10^4$



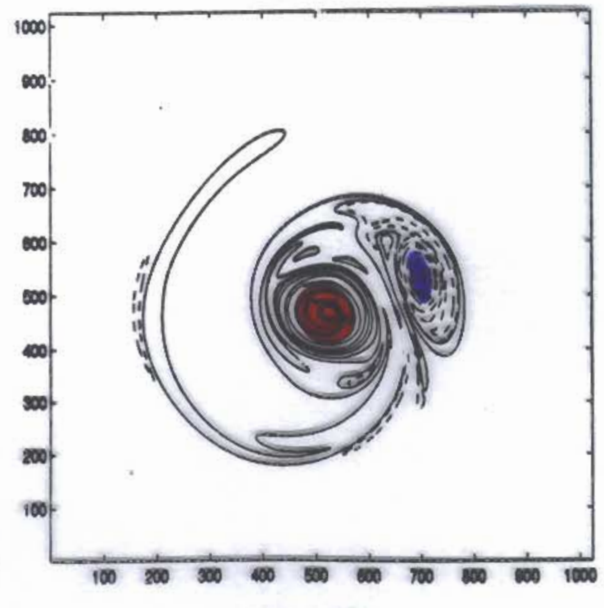
(a) $t = 10$



(b) $t = 20$

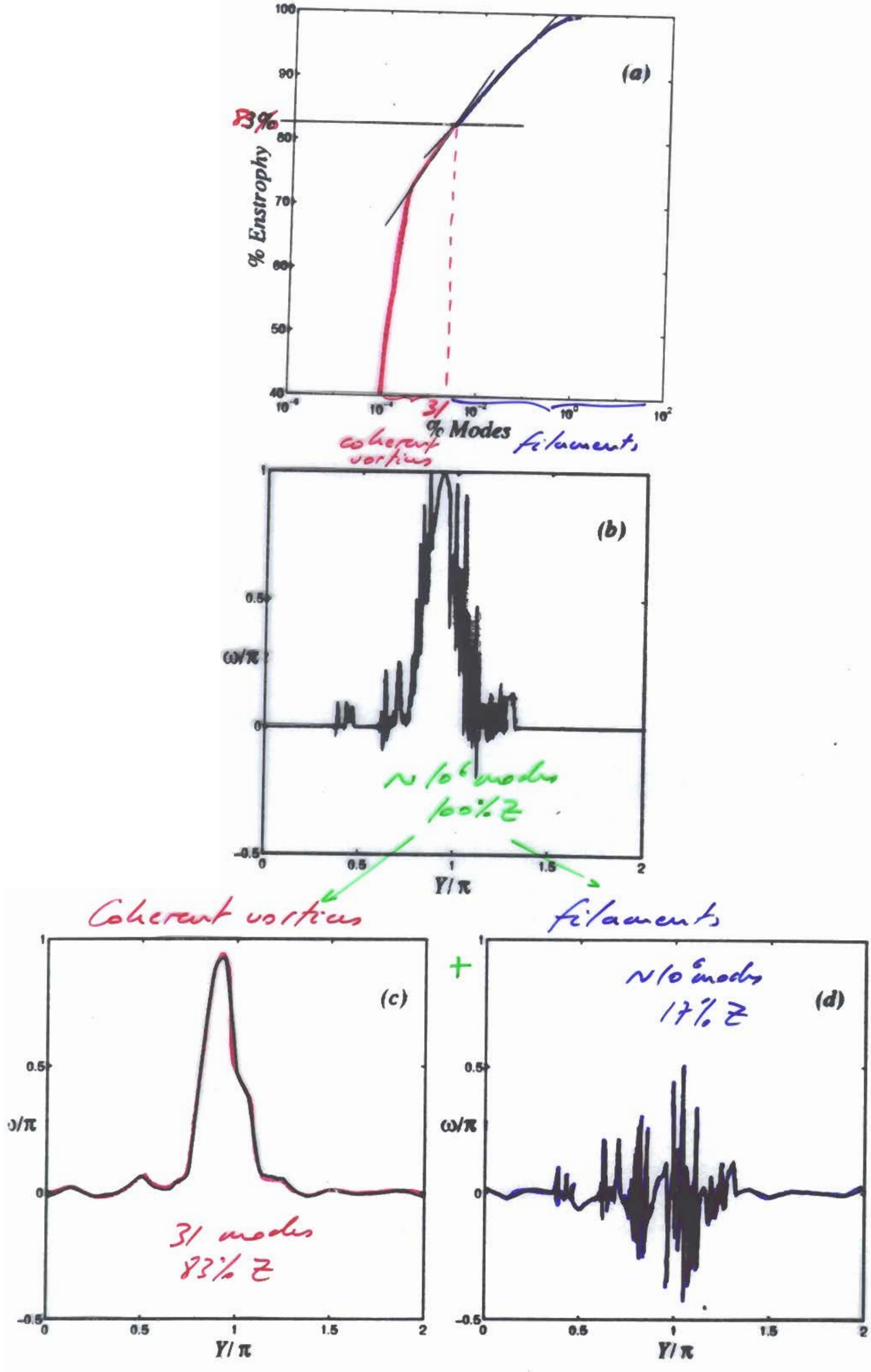


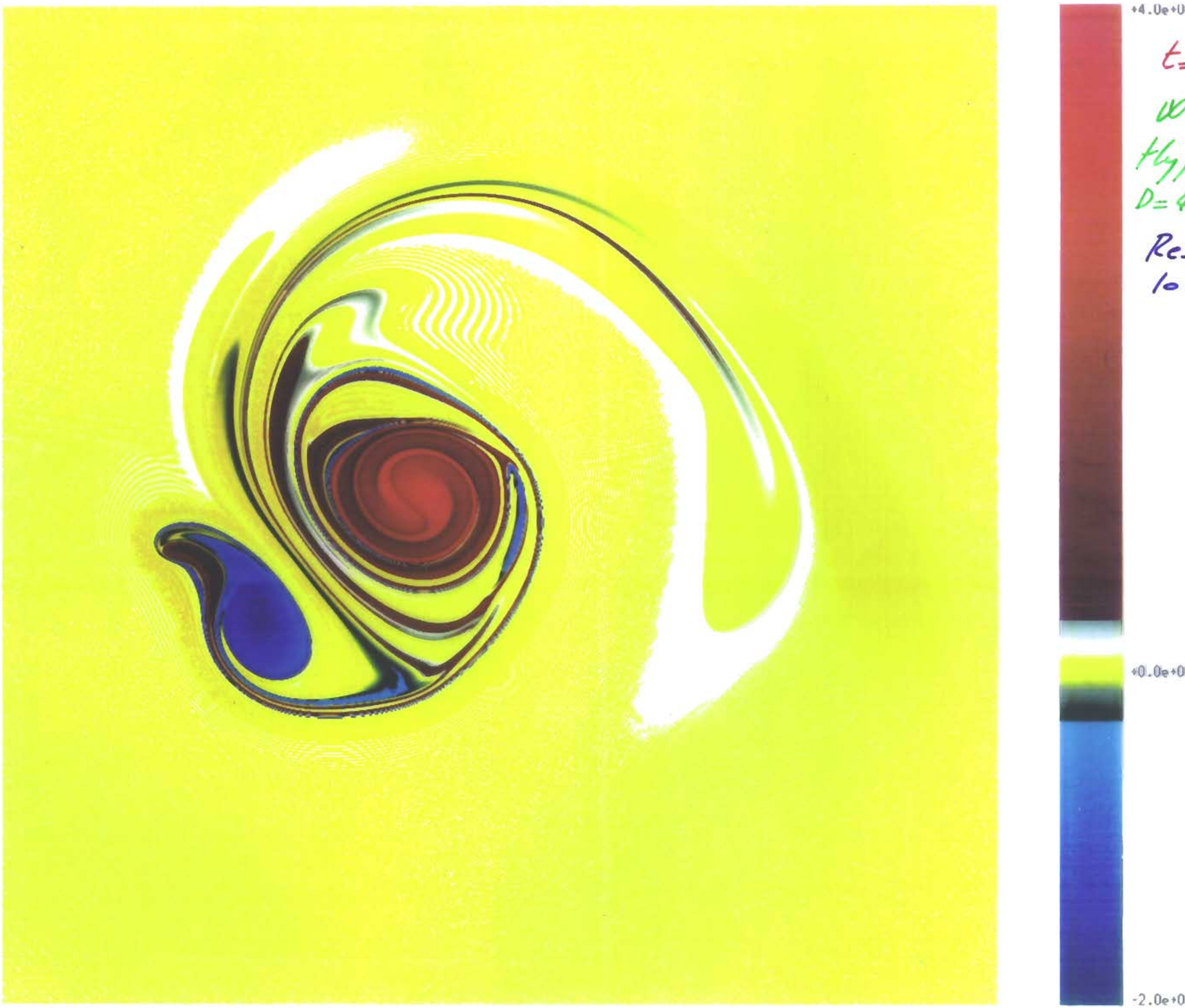
(c) $t = 30$



(d) $t = 40$

Pseudo-Fourier


(a)
% Enstrophy
% Modes
83%
31
coherent vortices
filaments
(b)
ω/π
Y/π
~10⁶ modes
100% Z
Coherent vortices
filaments
+
(c)
ω/π
Y/π
31 modes
83% Z
(d)
ω/π
Y/π
~10⁶ modes
17% Z



$t = 29$

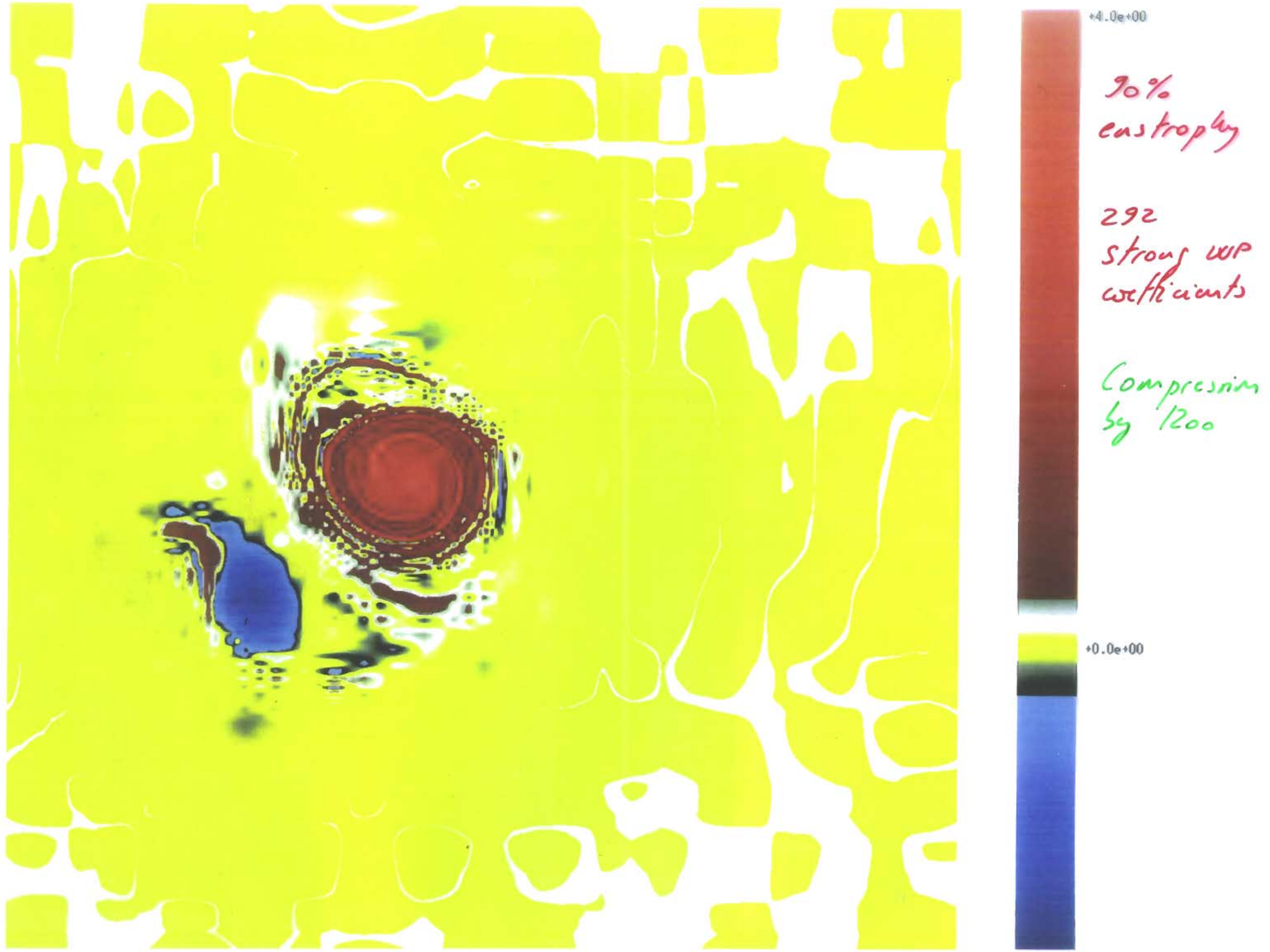
With

Hyperdissipation

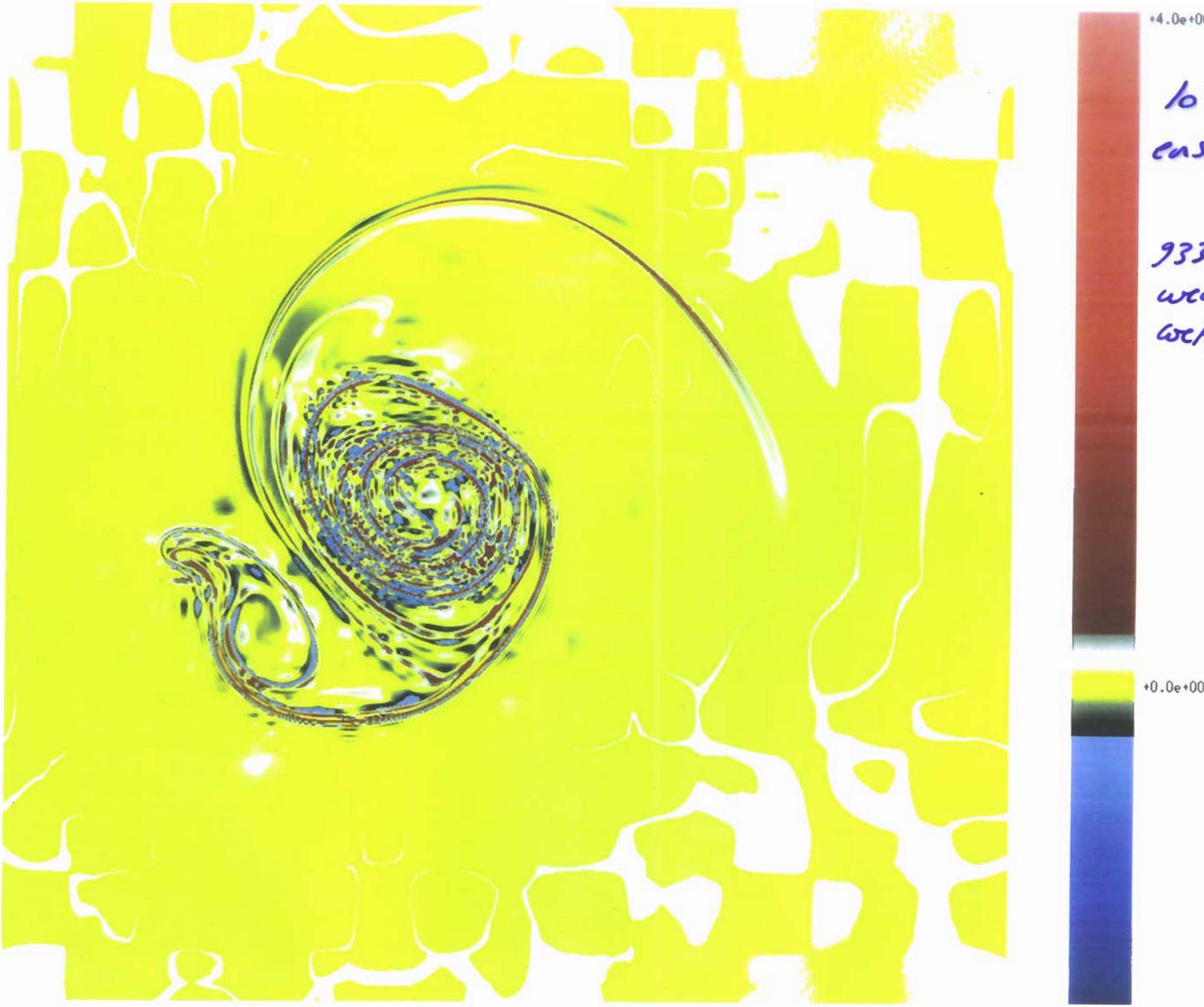
$D = 40(-\nabla^2)^8$

Resolution

1024^2



+4.0e+00
90% enstrophy
292 strong wp coefficients
Compression by 1200
+0.0e+00



10%
enstrophy

933 209
weak WP
coefficients

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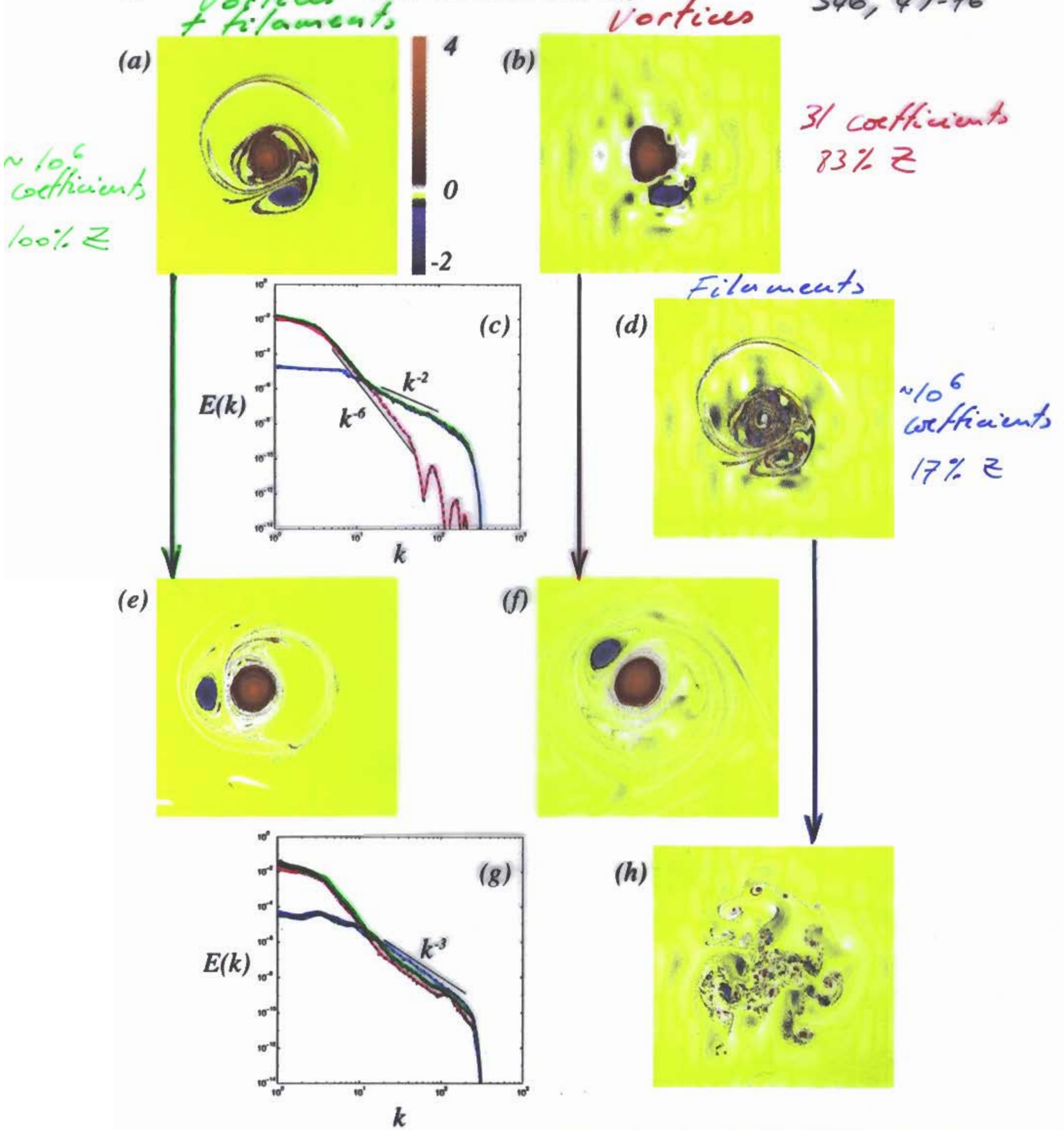

CONJECTURE

We have observed that the strain imposed by the coherent vortices inhibits the nonlinear instabilities to grow in the background

$\Longrightarrow$

Wavelet Gap

The background flow is slaved to the coherent vortices and there is no backscatter from the background to the coherent vortices

$\Longrightarrow$

We propose to model the background flow by a Gaussian (with a k^{-1} scaling) stochastic process.