EXTRACTION OF VORTICES IN 2D AND 3D TURBULENT FLOWS

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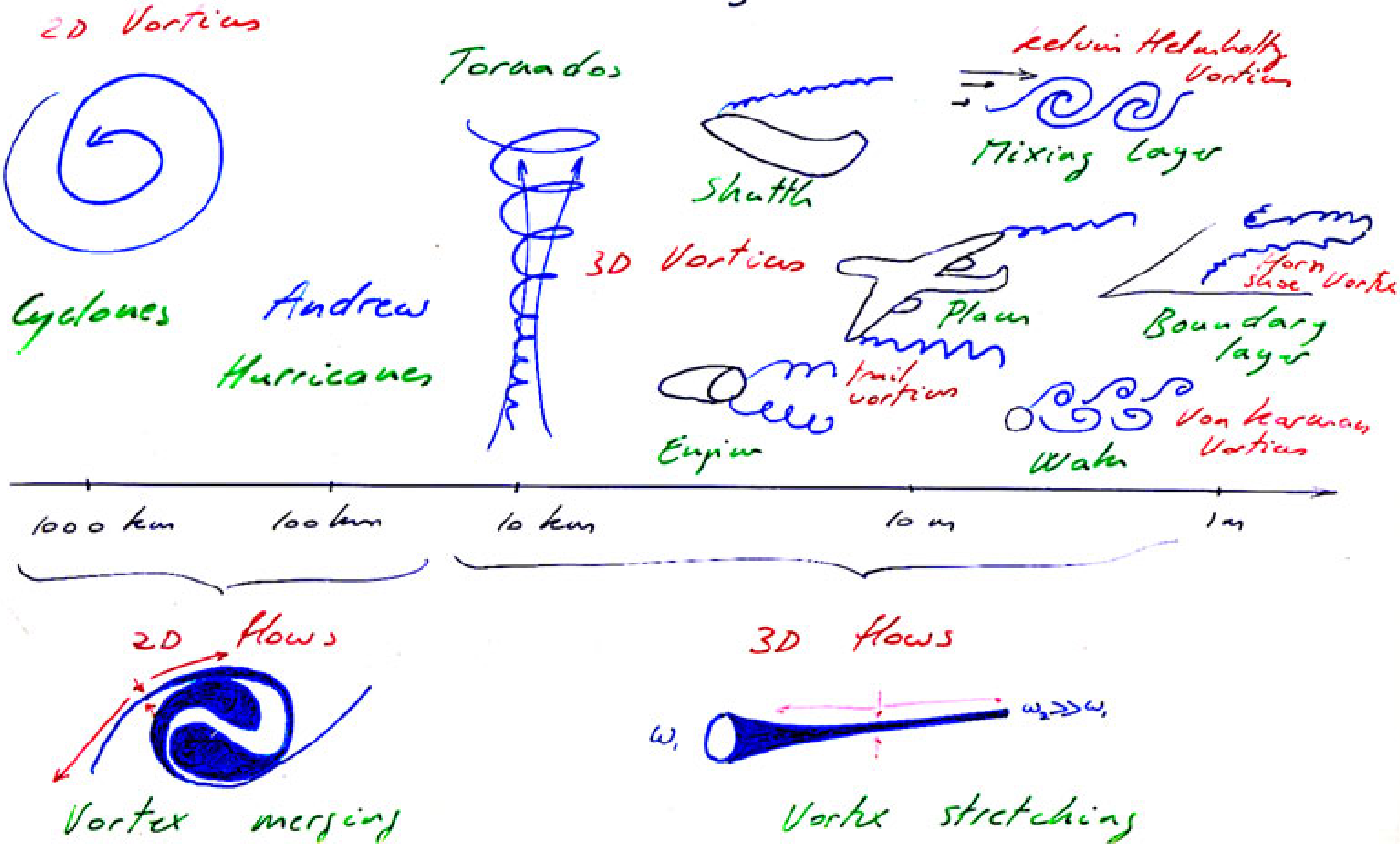
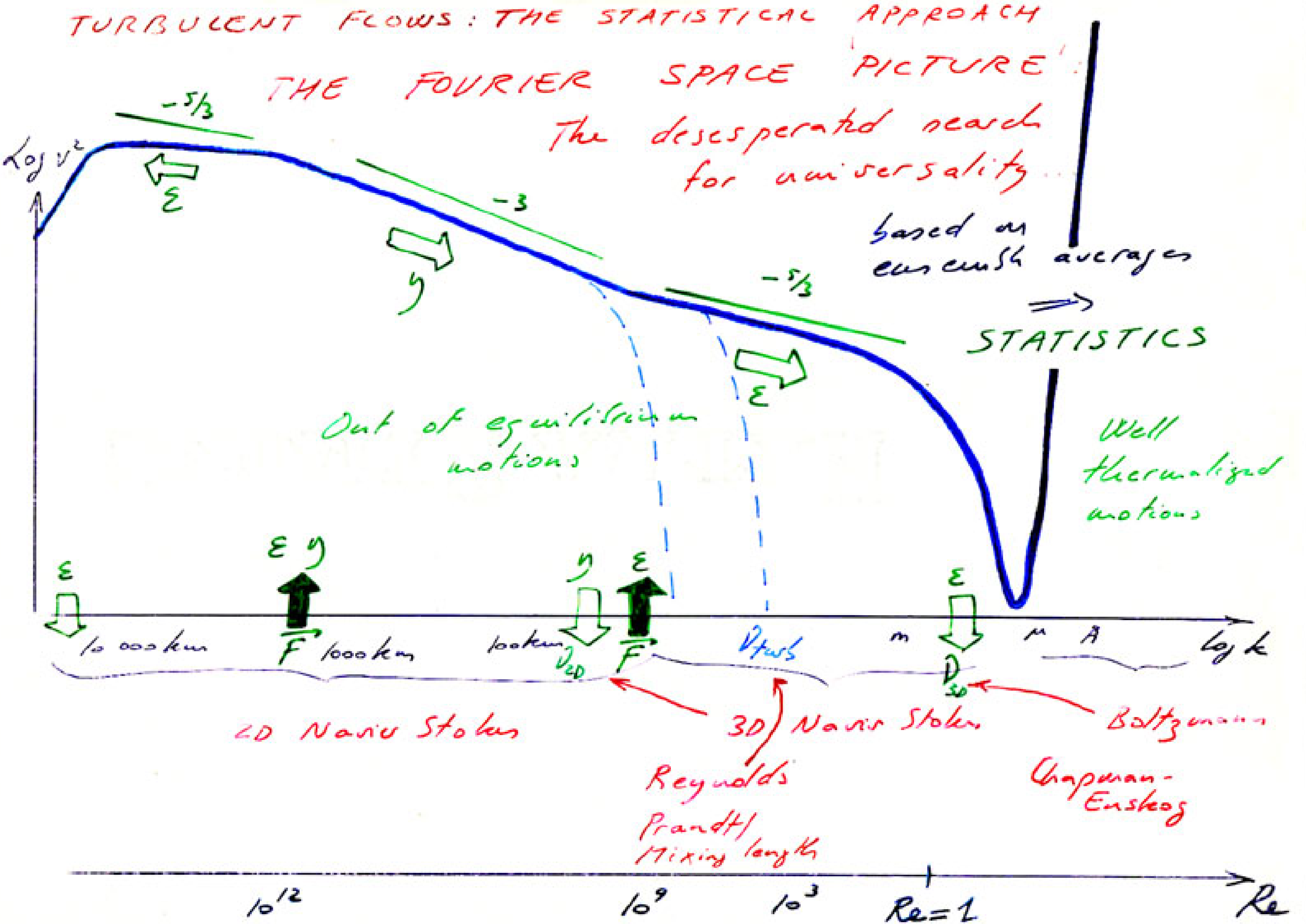
TURBULENT FLOWS: THE DETERMINISTIC APPROACH

THE PHYSICAL SPACE 'PICTURE'.

'The Zoo'

based on each flow realization $\Rightarrow$ DYNAMICS




TURBULENT FLOWS: THE STATISTICAL APPROACH
THE FOURIER SPACE 'PICTURE':
The desesperated search for universality...
based on ensemble averages
⇒ STATISTICS
Log v²
-5/3
-3
-5/3
ε
η
ε
Out of equilibrium motions
Well thermalized motions
ε
ε η
η
ε
ε
10 000 km
F
1000 km
100 km
D_{2D}
F
D_{turb}
m
D_{3D}
μ
Å
Log k
2D Navier Stokes
3D Navier Stokes
Boltzmann
Reynolds
Prandtl/
Mixing length
Chapman-
Enskog
10^{12}
10^{9}
10^{3}
Re=1
Re

Hugh DRYDEN
Adv. Applied Mech., 1, 1948

'At the 5th International Congress
of Applied Mechanics in 1938
Tollmien and Prandtl suggested that
the turbulent fluctuations might
consist of two components, a
diffusive and a non diffusive
component [...] Considerable masses
of fluid move as more or less
coherent units. The process cannot
be smoothed by averaging over a
small volume because it is not
possible to choose dimensions small
compared with a single element [...].
Shall the flow be regarded as
a mean flow that merely transports
and distorts large eddies super-
posed on the flow, the eddies
being of varying size and intensity.
[...] The ideas of Tollmien and Prandtl
that measured fluctuations include
both random and non random elements
are correct, but as yet there is no
known procedure, either experimental
or theoretical, for separating them.'

Robert KRAICHNAN
'On Kolmogorov's inertial range theories' J.F.M. 62, 1974

The terms 'scale of motion' or 'eddy of size ℓ' appear repeatedly in treatments of the inertial range. One gets an impression of little, randomly distributed whirls in the fluid, with the cascade process consisting of the fission of the whirls into smaller ones, after the fashion of Richardson's poem. This picture seems drastically in conflict with what can be inferred about the qualitative structure of high-Reynolds number turbulence from laboratory visualization techniques and from plausible application of Kelvin circulation theorem.'

Ya ZELDOVICH ON TURBULENCE

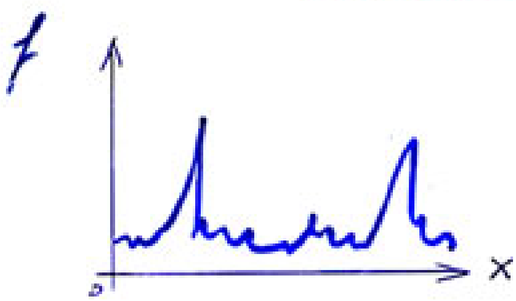
The randomness in turbulent flows comes from a statistically prescribed background against which the evolution proceeds.

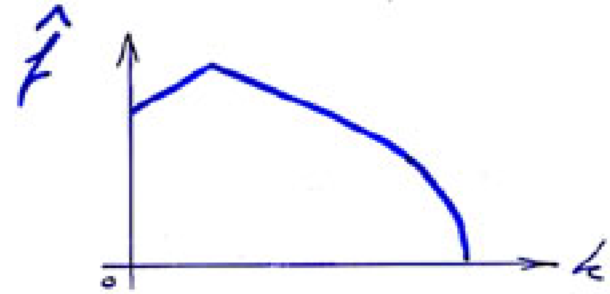
Stagnation points, as well as unstable periodic orbits, probably play a decisive role for the existence of chaotic regions. In 2D turbulence, stagnation points are such that nearby trajectories converge in one direction and diverge in the other. It is important to separate the stable and unstable regions of turbulent flows.

(The Almighty Chance, 1990)

TURBULENT FLOW ANALYSIS

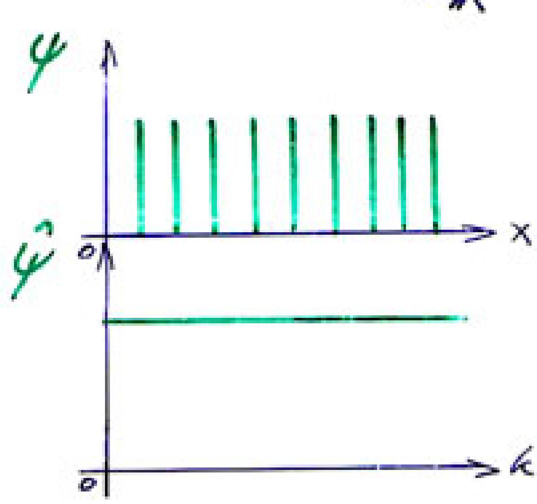
Turbulent flows are described by continuous fields f:

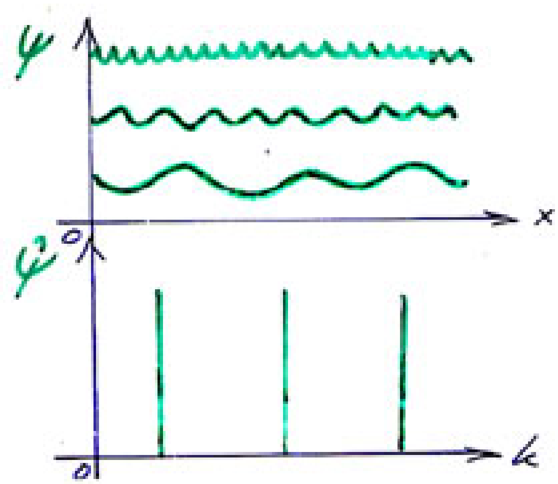




To measure and analyse them, we project them onto orthogonal bases described by analyzing functions ψ_μ:

$$\tilde{f}(\mu) = \int_{\mathbb{R}^n} f(x)\, \psi_\mu(x)\, dx$$





Grid-points or Dirac basis

$$\psi_{x_0}(x) = \delta(x_0)$$

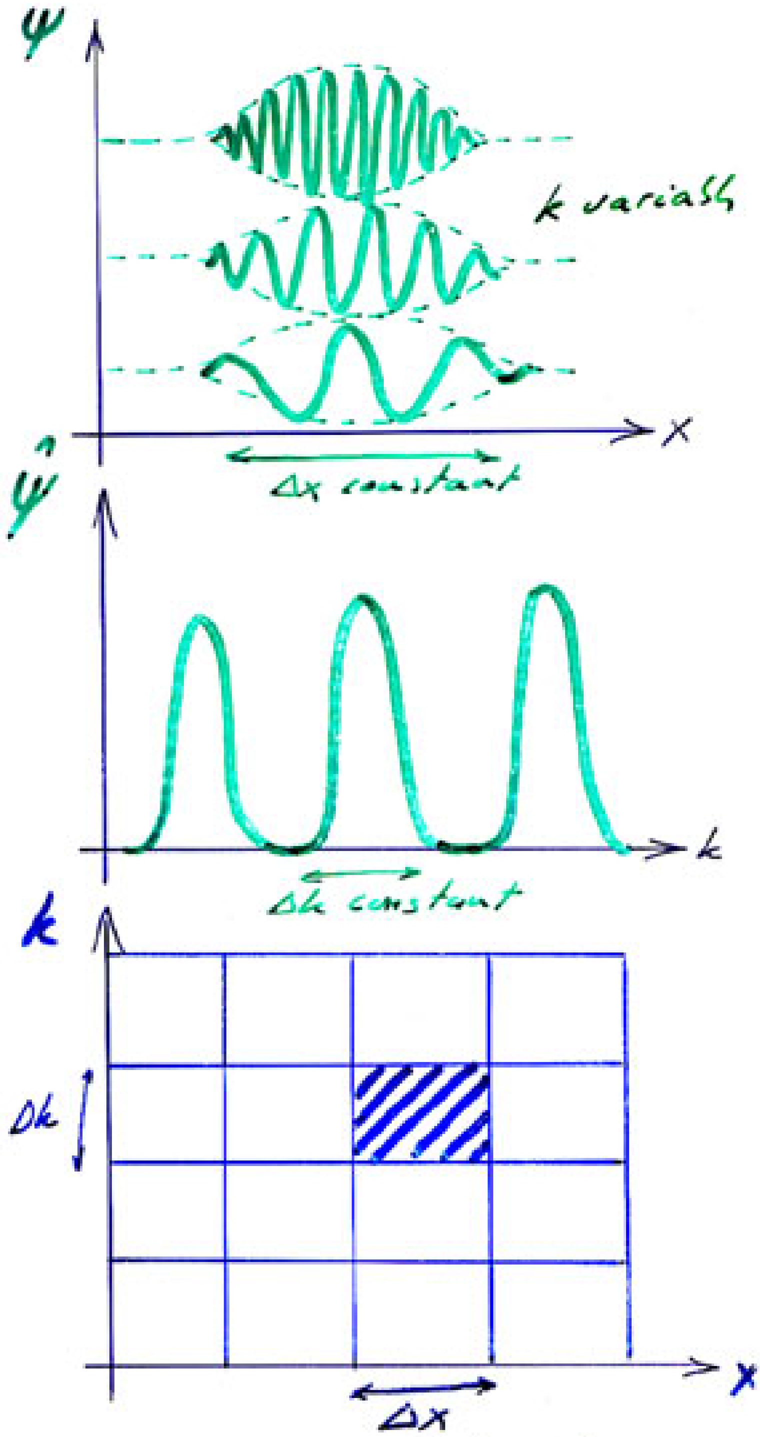
Wavenumbers or Fourier basis

$$\psi_k(x) = e^{-2i\pi k \cdot x}$$

OTHER POSSIBLE BASES

Windowed Fourier

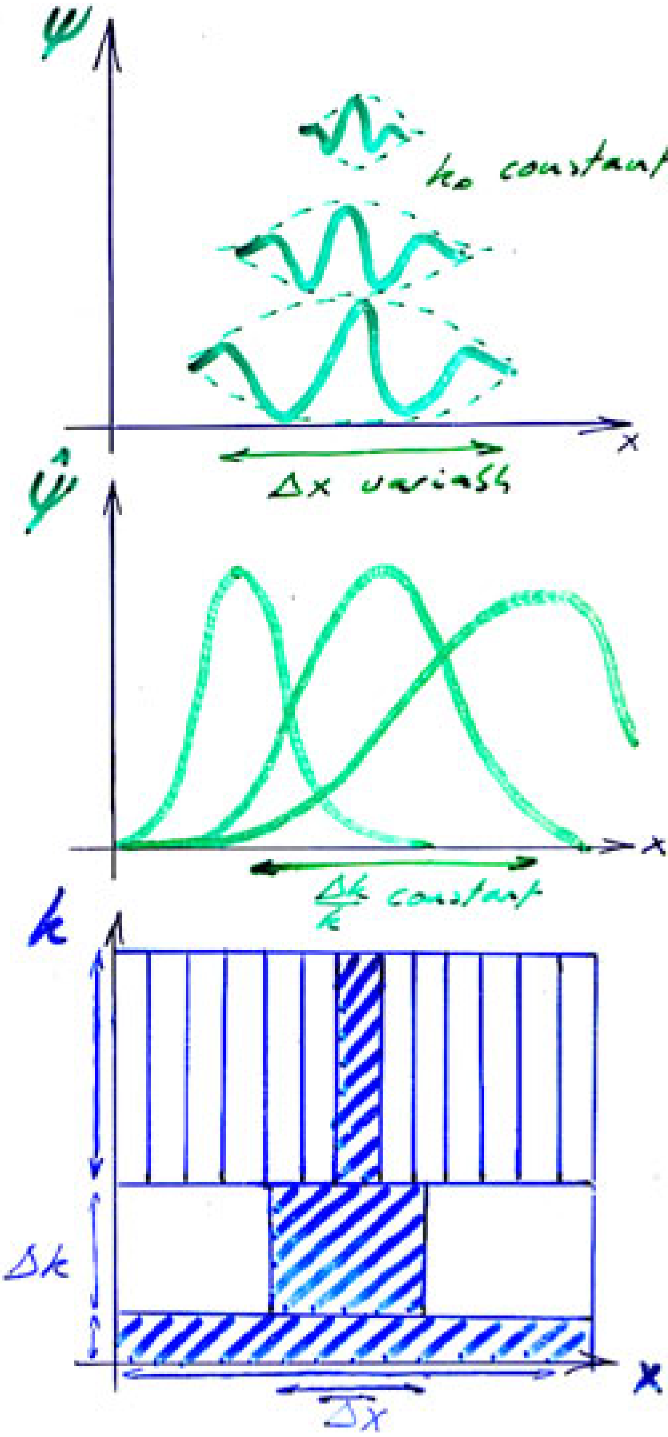
Space-wavenumber decomposition



Balian's obstruction

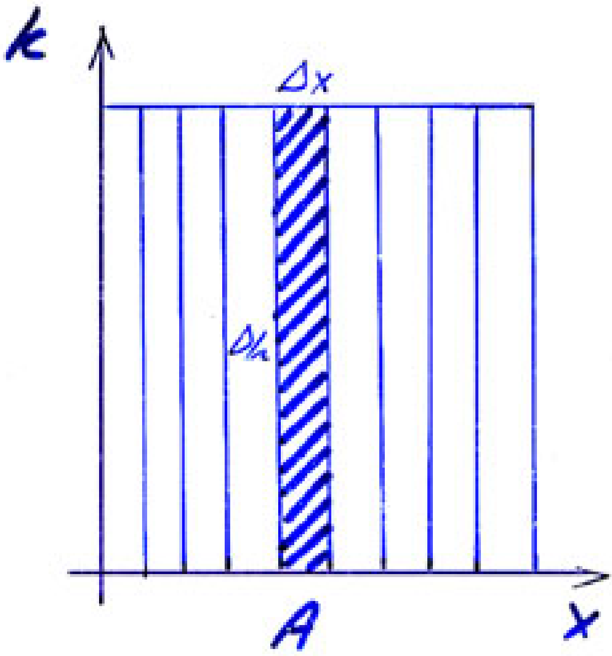
Wavelets

Space-scale decomposition

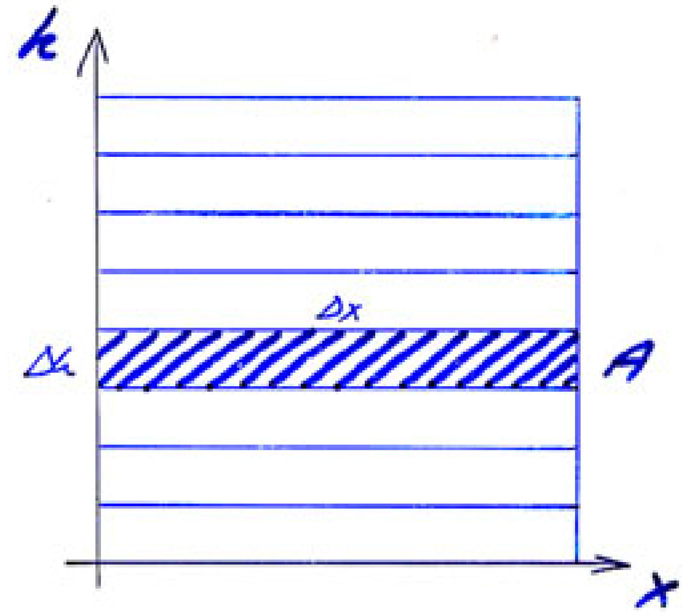


PHASE-SPACE REPRESENTATION
or
INFORMATION PLANE
(Léon Brillouin)

Dirac basis



Fourier basis



Heisenberg's
Uncertainty Principle:
$\Delta x . \Delta k \geqslant A$

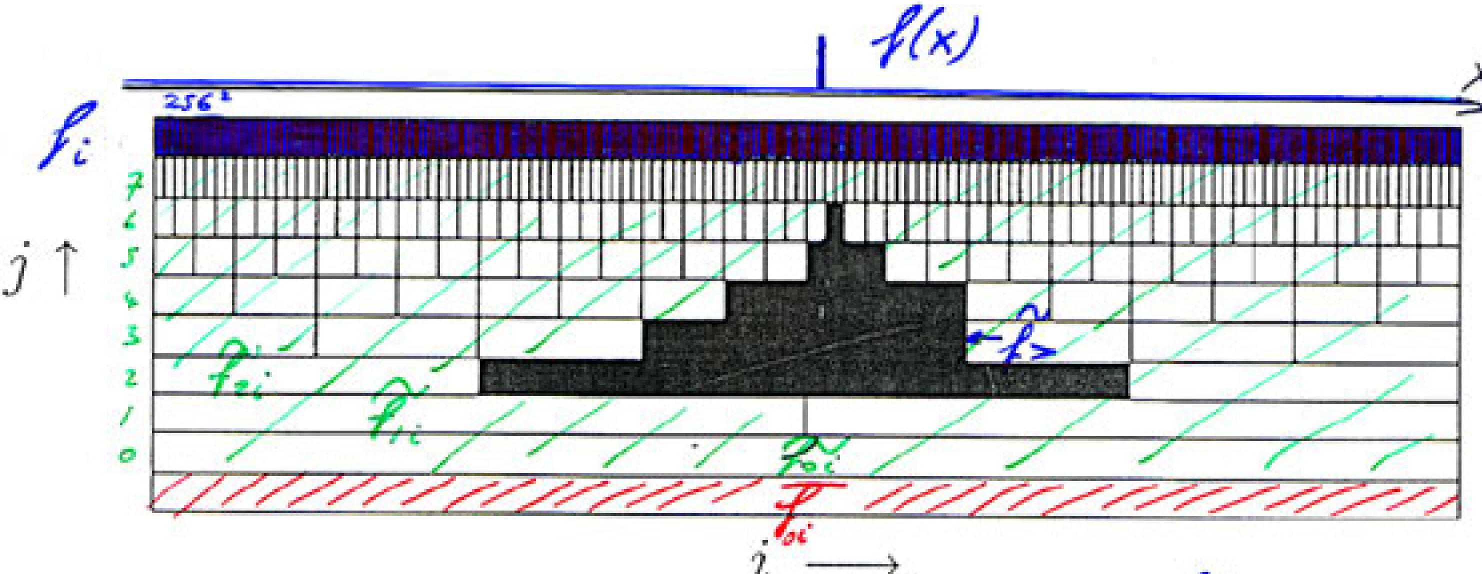
A elementary area
of the information cells
or phase-space 'atoms'
or Heisenberg boxes

ORTHOGONAL WAVELET DECOMPOSITION

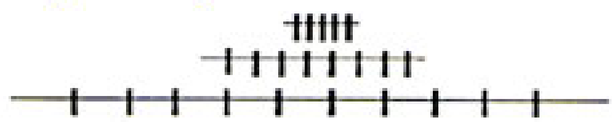
cf. Farge, Ann. Rev. Fluid Mech., 24, 1992

$$f(x) = \sum_{i=-\infty}^{+\infty} \underbrace{\bar{f}_{oi}}_{\text{Scaling coefficients at the largest scale } \ell_{max} = 2^0 = 1} \phi_{oi}(x)$$

$$+ \sum_{j=0}^{+\infty} \sum_{i=-\infty}^{+\infty} \underbrace{\tilde{f}_{ji}}_{\text{wavelet coefficients at scale } \ell = 2^{-j}} \psi_{ji}(x)$$



Adaptive grid associated to $\tilde{f}_>$



Scaling coefficients $\bar{f}_{oi} = \int_{-\infty}^{+\infty} f(x)\,\phi(x-i)\,dx$

Wavelet coefficients $\tilde{f}_{ji} = \int_{-\infty}^{+\infty} f(x)\,\psi(x-2^{-j}i)\,dx$

2D ⊥ WAVELET DECOMPOSITION OF THE VORTICITY

$$\omega(\vec{x}) = [\overline{\omega}_{0,0,0}]\,\phi_{0,0,0}(\vec{x}) + \sum_{j=0}^{J-1}\sum_{i_x=0}^{2^j-1}\sum_{i_y=0}^{2^j-1}\sum_{\mu=1}^{3} [\tilde{\omega}^{\mu}_{j,i_x,i_y}]\,\psi^{\mu}_{j,i_x,i_y}(\vec{x})$$

$\vec{x} \in \mathbb{R}^2$

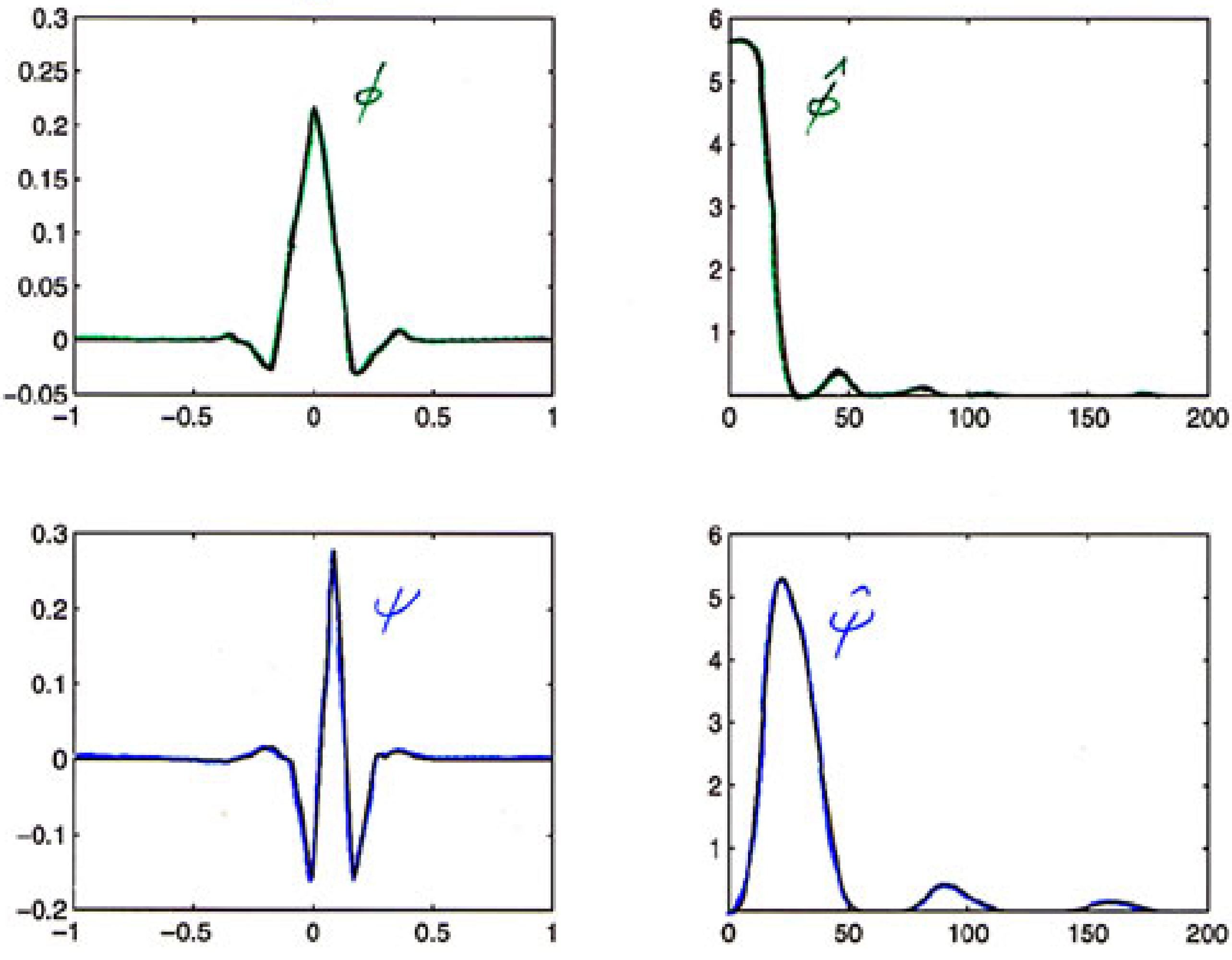
j scale

i position

μ direction

$N = 2^J$

J # octaves



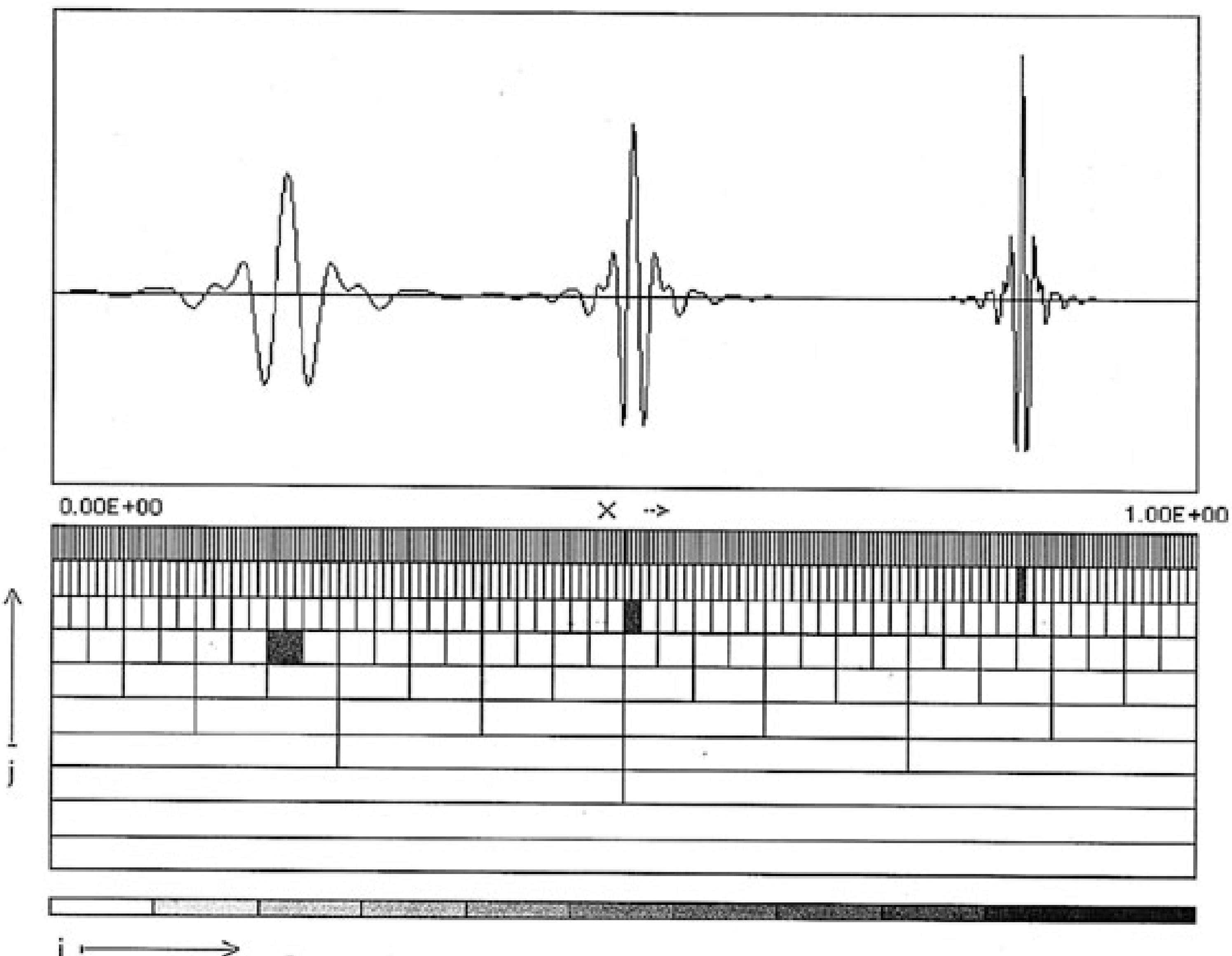
with the scaling coefficients $[\overline{\omega}_{0,0,0}] = \langle \omega, \phi_{0,0,0} \rangle$

and the wavelet coefficients

$$[\tilde{\omega}^{\mu}_{j,i_x,i_y}] = \langle \omega, \psi^{\mu}_{j,i_x,i_y} \rangle$$

$O(N)$ operations

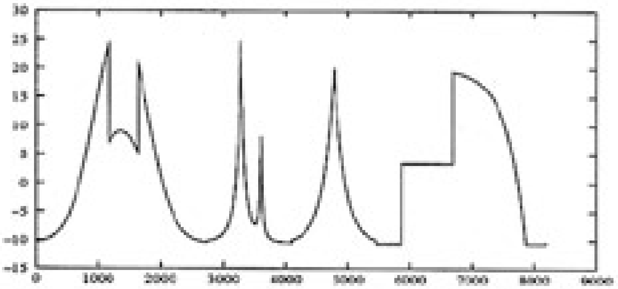
Wavelet family



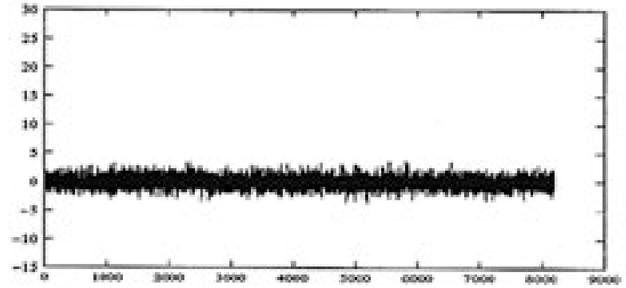
Scale (j) space (i) representation

Numerical application

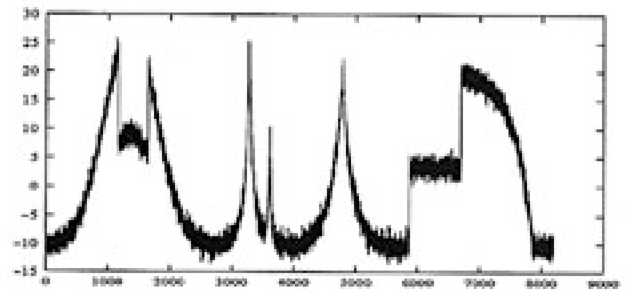
$\underline{f}$



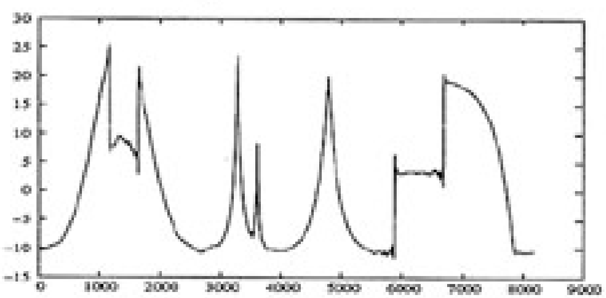
$\underline{W}$



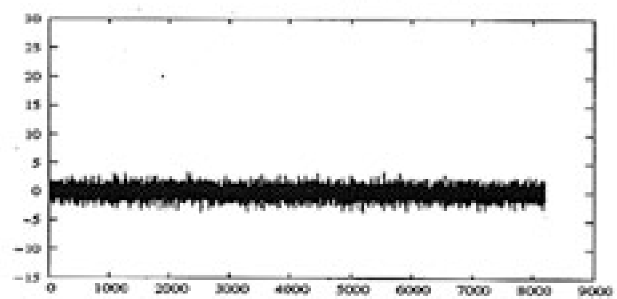
$\underline{X}$



$F_{T_\ell}(\underline{X})$



$F^c_{T_\ell}(\underline{X})$



RESEARCH PROGRAM

The description of turbulent flows f in terms of a mean $\bar{f}$ plus random fluctuations f', i.e. $f = \bar{f} + f'$, is eroded, since turbulent flows are governed by deterministic coherent vortices moving in a random background flow resulting from their nonlinear interactions

$$f = f_C + f_I \begin{cases} f_C & \text{coherent, deterministic} \\ f_I & \text{incoherent \& random} \end{cases}$$

Classical averages based on L^2-norm or 2 point correlations are not appropriate to take into account coherent vortices due to their translation motions.

Coherent vortices are correlated in space and time (phase locking)
$\Rightarrow$ we need wavelet representation

They are intermittent events $\Rightarrow$ we need conditional averages, i.e. each realization is splitted into f_C and f_I before averaging.

COHERENT VORTEX SIMULATION (CVS)

We want to compute the evolution of unsteady turbulent flows at the lowest computational cost.

We consider turbulent flows as made of coherent vortices moving in a random background flow produced by their non linear interactions

$$\vec{f}(\vec{x},t) = \vec{f}_C(\vec{x},t) + \vec{f}_I(\vec{x},t)$$

Coherent deterministic non-diffusive

Incoherent random diffusive

PRINCIPLE OF CVS
(COHERENT VORTEX SIMULATION)

Separation of each flow realization into:

Nonlinear interaction { – Coherent vortices out of equilibrium,

Turbulent dissipation { – Well mixed background flow in 'thermal' equilibrium.

No scale separation { Both components are multiscale but present different statistical behaviours ⟹ CVS:

– the coherent vortices are deterministically computed in an adaptive wavelet basis,

– the effect of the background flow is statistically modelled.

cf. Phys. Fluids, 11 (8), 2187 (1999)

GOAL

We have introduced a new method, called Coherent Vortex Simulation (CVS) cf. Phys. Fluids, 11, 8, 2187 (1999), which can be seen as a multiscale LES method with dynamical grid refinement.

CVS presently works for 2D flows and we want to extend it to 3D flows.

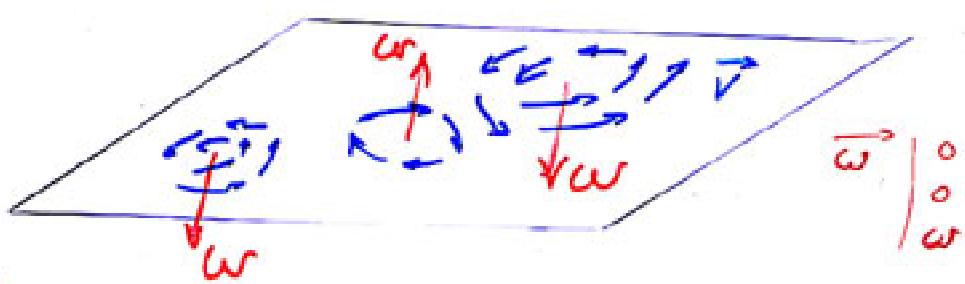
As a first step, we propose to compare CVS and LES filtering in terms of vortex tube extraction and number of d.o.f. compression for 3D flows.

DECAYING 2D TURBULENT FLOWS

Direct Numerical Simulation

Navier-Stokes
$$\begin{cases} \dfrac{\partial \vec{v}}{\partial t} + \overbrace{\vec{v}\cdot\vec{\nabla}\vec{v}}^{\text{nonlinear advection}} + \dfrac{1}{\rho}\vec{\nabla}P = \overbrace{\nu\nabla^2\vec{v}}^{\text{linear dissipation}} \\ \vec{\nabla}\cdot\vec{v} = 0 \end{cases} \quad + \underset{\text{periodic}}{\text{B.C}} \;\&\; \underset{\text{random}}{\text{I.C.}}$$

$$\begin{cases} \omega = \vec{\nabla}\times\vec{v} = \nabla^2\psi \\ \dfrac{\partial \omega}{\partial t} + \vec{v}\cdot\vec{\nabla}\omega = \nu\nabla^2\omega \end{cases} \quad + \text{B.C} \;\&\; \text{I.C.}$$



Laminar flows
$Re \sim 1$

Turbulent flows
$Re \to \infty$

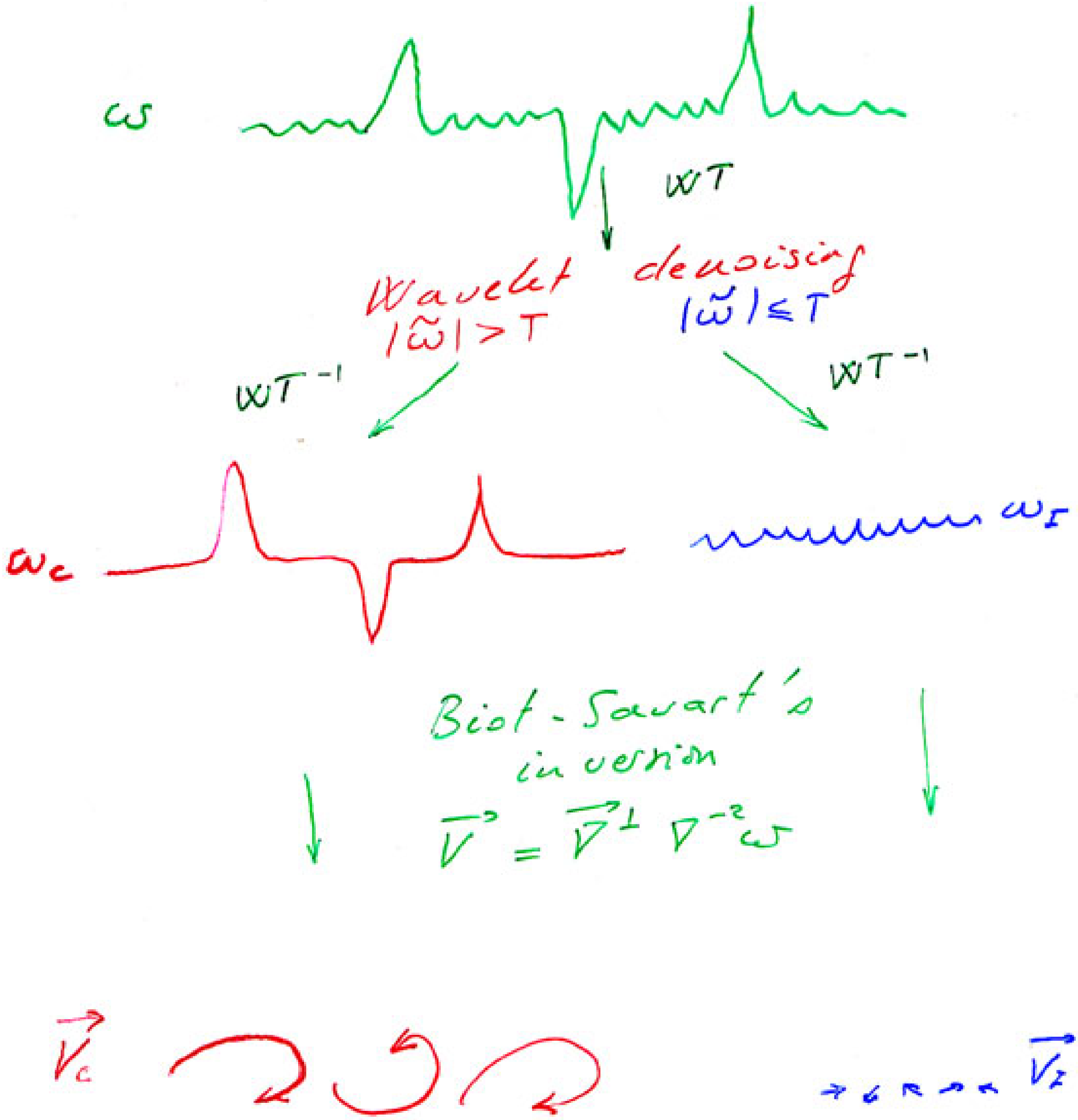
Reynolds Number

$$Re = \frac{\text{non linear advection}}{\text{linear dissipation}}$$

N number grid points $\propto Re$

Pseudo-spectral method

CVS DECOMPOSITION



$$\omega = \omega_c + \omega_I$$
$$\vec{V} = \vec{V_c} + \vec{V_I}$$

CVS

WAVELET NONLINEAR FILTER

$\vec{\omega}$

Vorticity sampled on N grid points

Orthogonal wavelet transform

$\tilde{\omega} = \langle \omega / \psi \rangle$

N wavelet coefficients

Nonlinear threshold

$|\tilde{\omega}| :: \tilde{\omega}_T = \left(2 \langle \vec{\omega}^2 \rangle \log N\right)^{1/2}$

$>$ $\qquad$ $\leq$

N_C strong wavelet coefficients $\tilde{\omega}_C$

N_I weak wavelet coefficients $\tilde{\omega}_I$

Adapted Inverse wavelet transform

$\vec{\omega}_C$ Coherent Vorticity

$\vec{\omega}_I$ Incoherent Vorticity

Biot-Savart kernel

$\vec{V}_C = \nabla \times \nabla^{-2} \vec{\omega}_C$

Coherent Velocity

$\vec{V}_I = \nabla \times \nabla^{-2} \vec{\omega}_I$

Incoherent Velocity

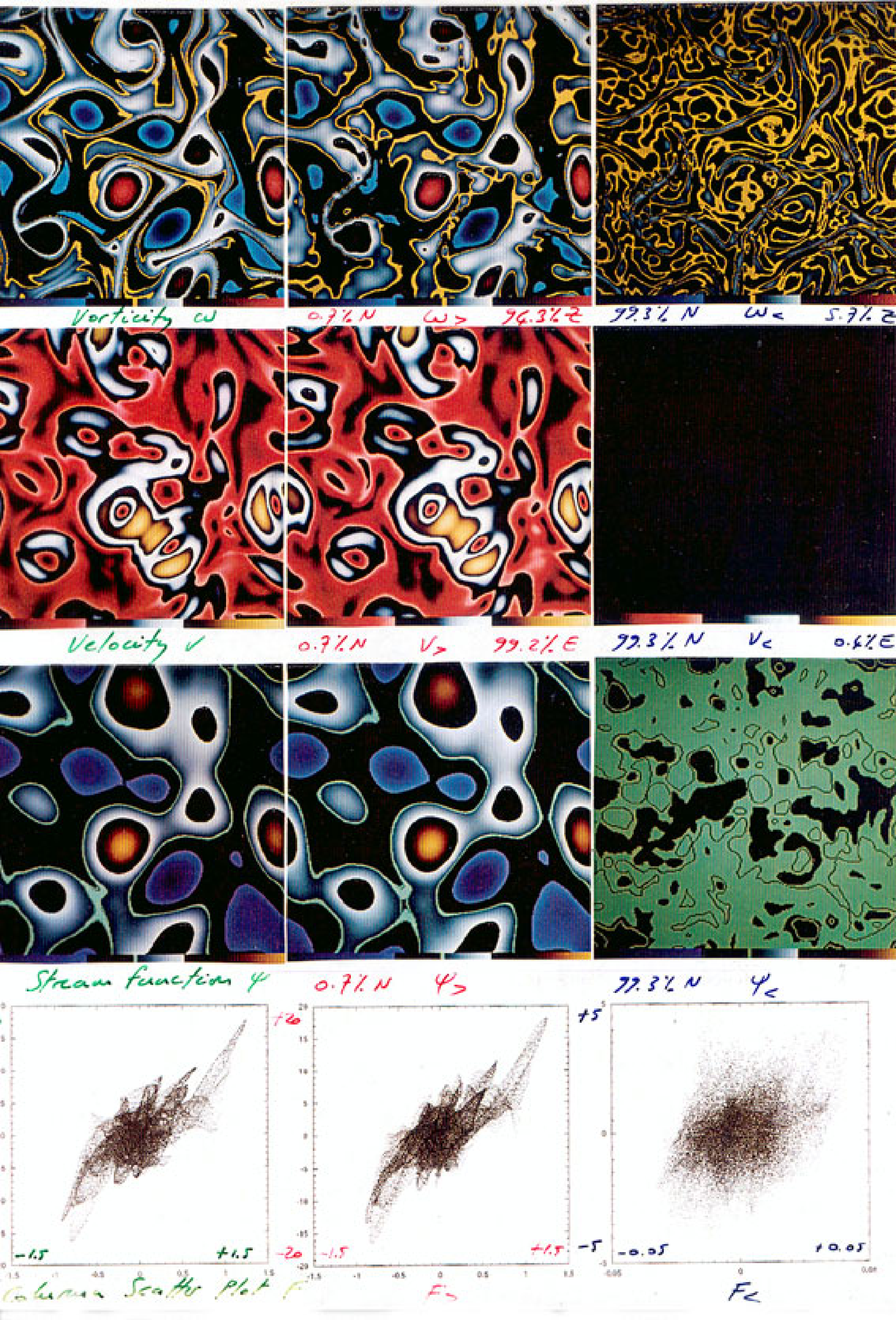
$\vec{\omega} = \vec{\omega}_C + \vec{\omega}_I$

$\vec{V} = \vec{V}_C + \vec{V}_I$

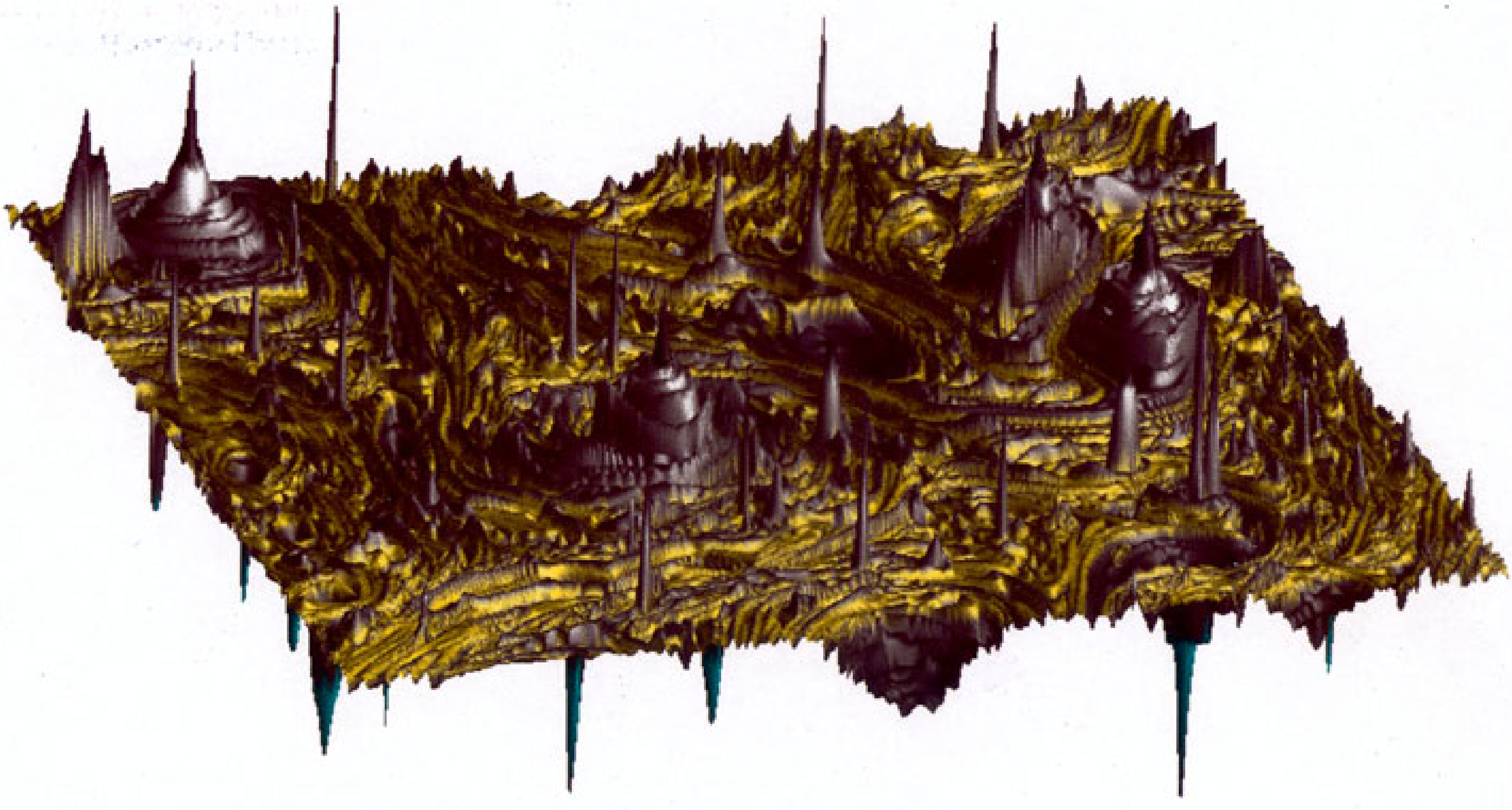
cf. Farge, Schneider, Kevlahan, Physics of Fluids Vol. 11, n°8, 2187 August 1999

Schneider, Kevlahan, Farge, Theor. Comput. Fluid Dynam. Vol. 9, 191, 1997

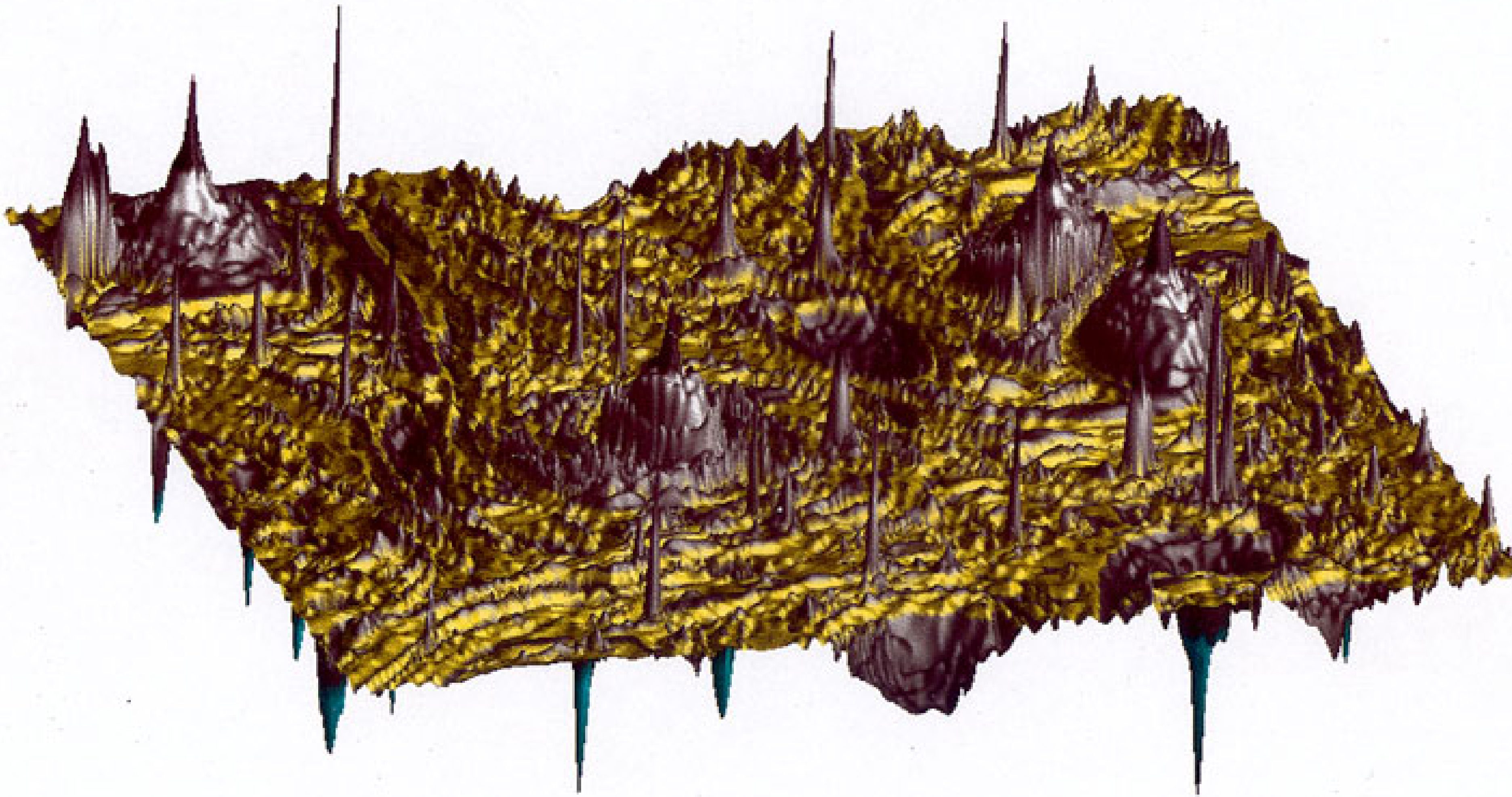
Farge, Goirand, Meyer, Pascal, Wickerhauser, Fluid Dynam. Research Vol. 10, 229 1992



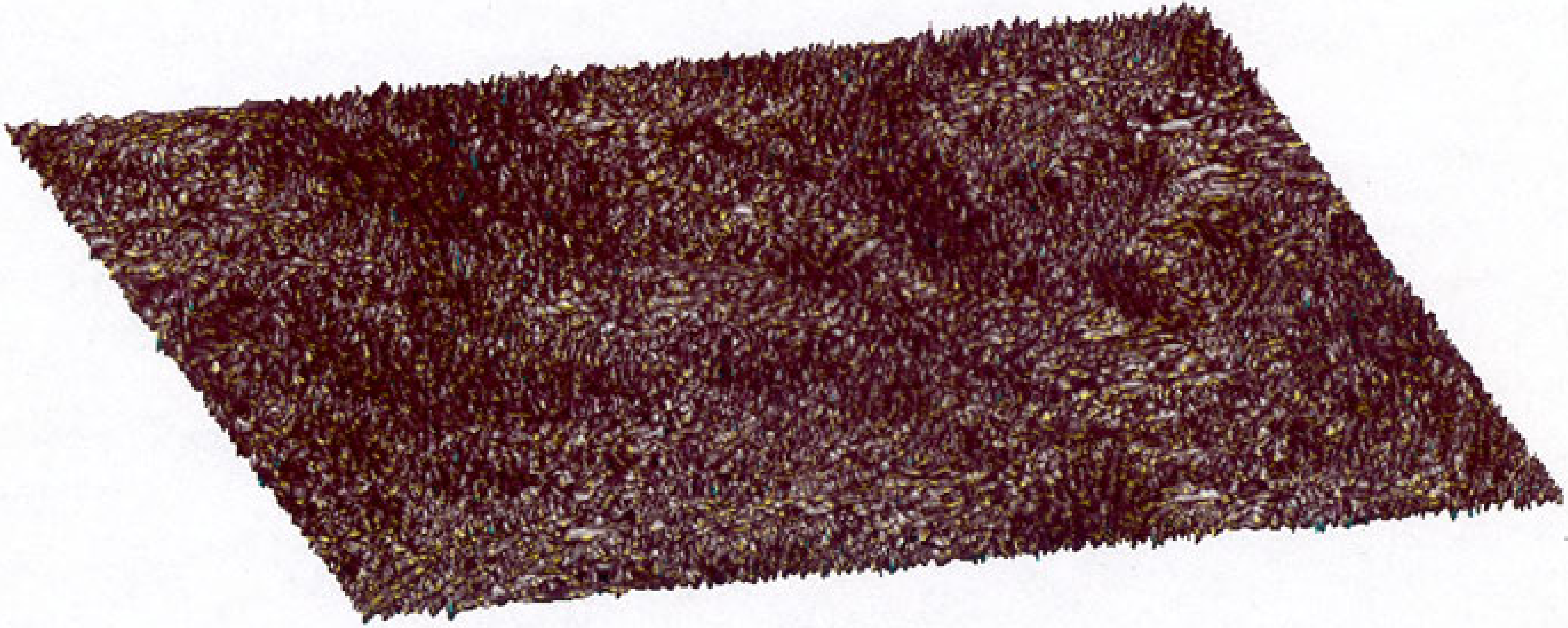
Vorticity ω
0.7% N ω> 94.3% Z
99.3% N ω< 5.7% Z
Velocity v
0.7% N v> 99.2% E
99.3% N v< 0.6% E
Stream function ψ
0.7% N ψ>
99.3% N ψ<
+20
-20
-1.5
+1.5
+5
-5
-0.05
+0.05
Column Scatter Plot F
F>
F<



Total Vorticity field
$N = 512^2$

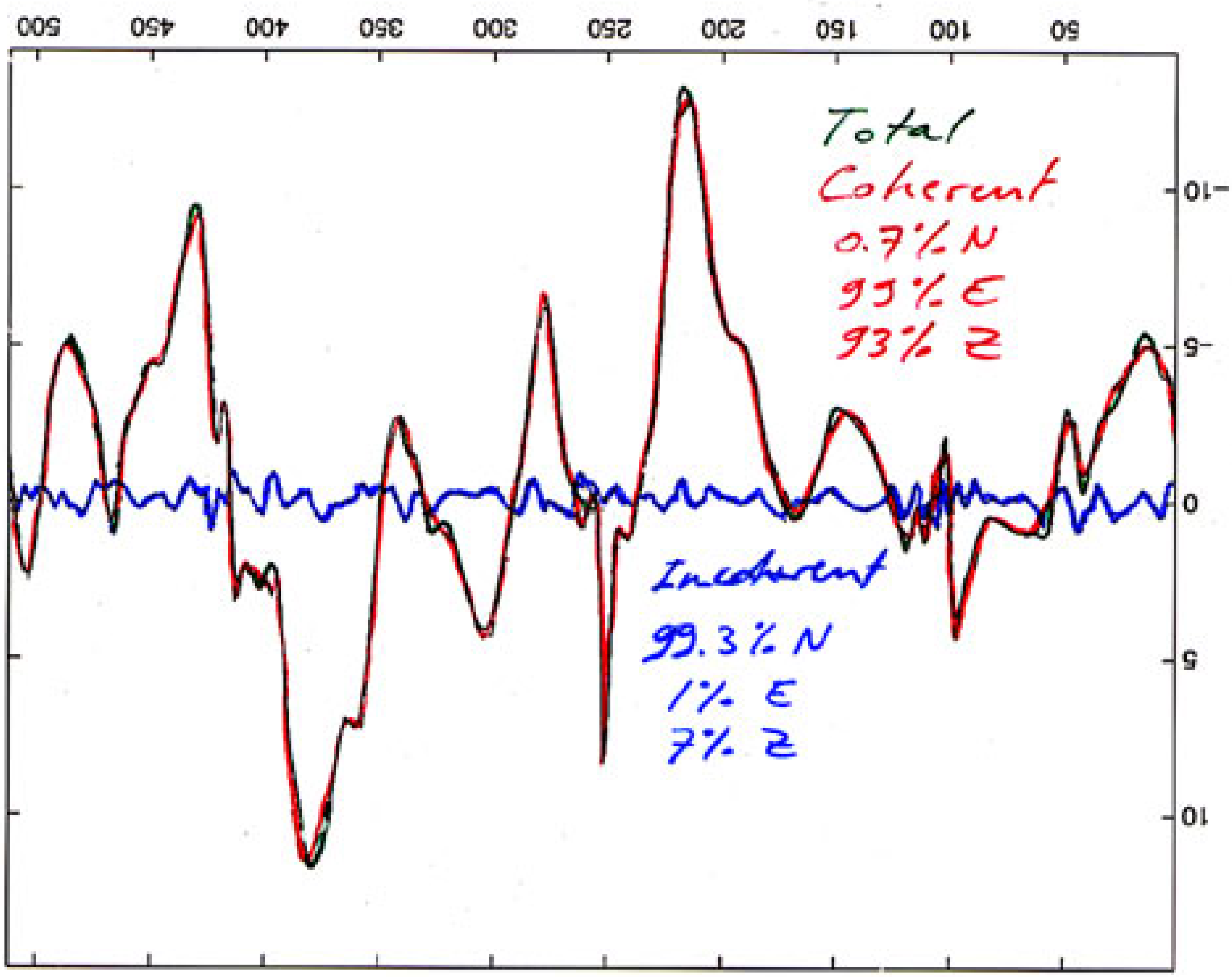


Coherent Vorticity field
0.7% N 99% E 93% Z

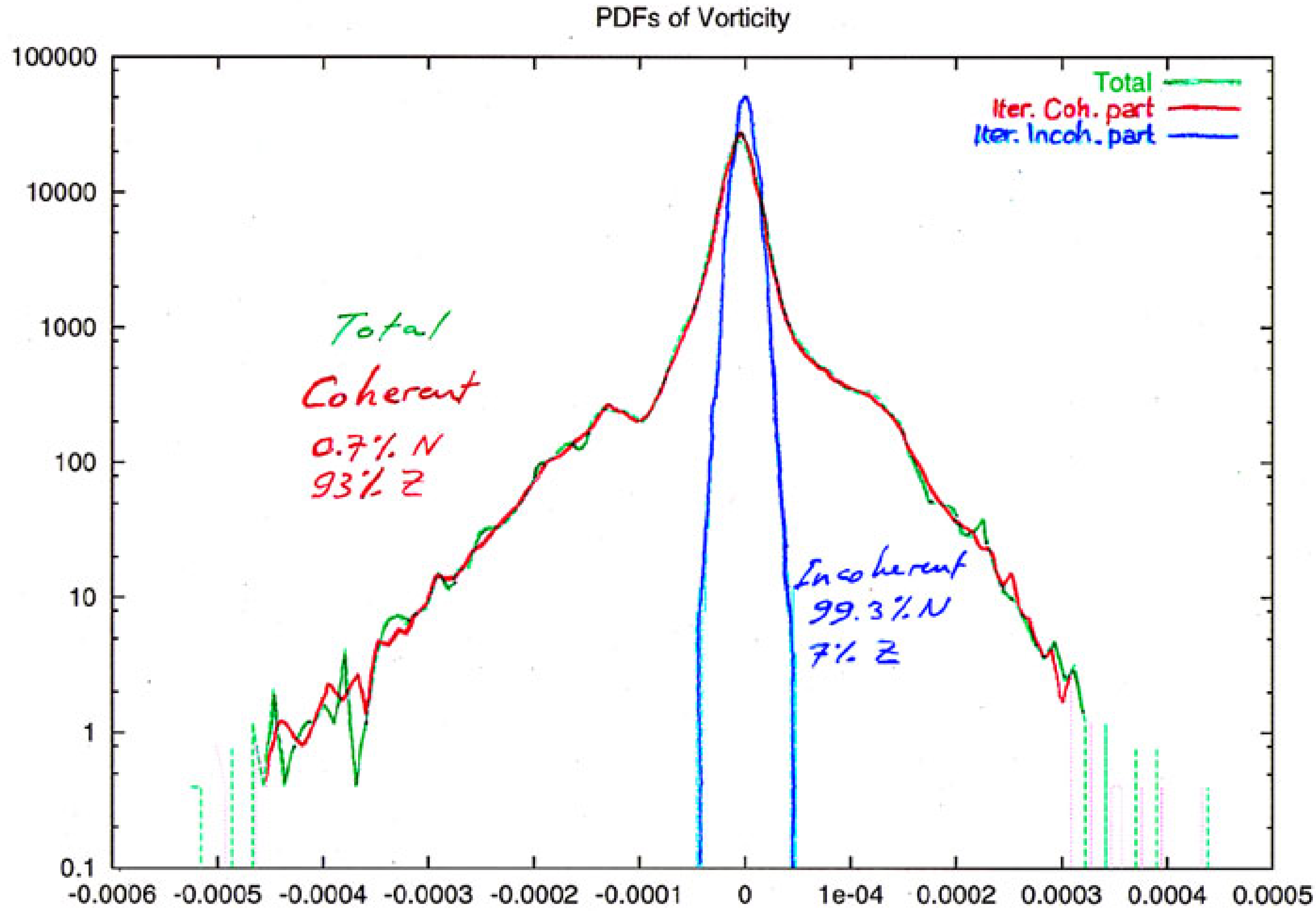


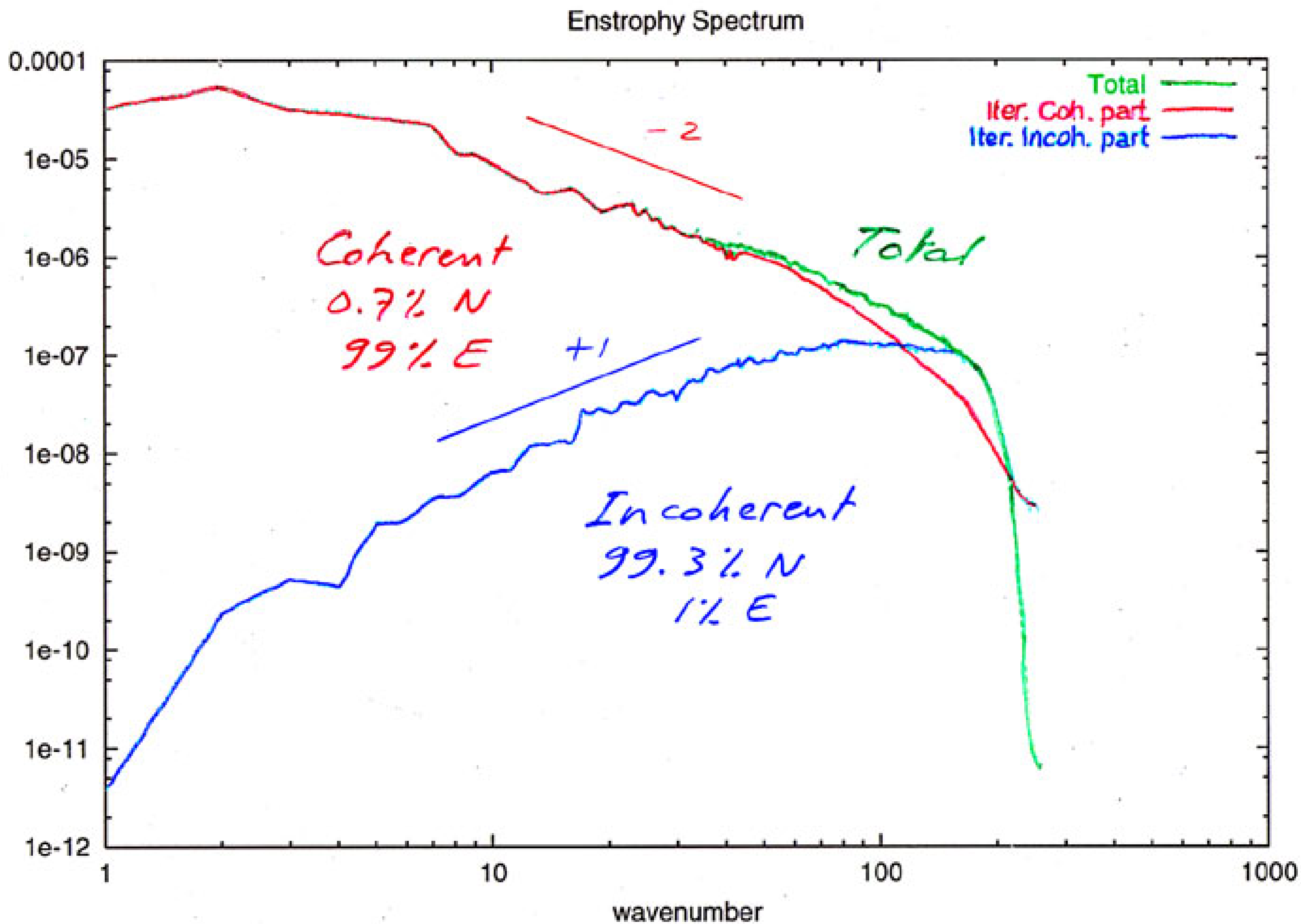
Incoherent Vorticity field

99.3% N 1% E 7% Z



10 Cut of Vorticity


PDFs of Vorticity
Total
Iter. Coh. part
Iter. Incoh. part
Total
Coherent
0.7% N
93% Z
Incoherent
99.3% N
7% Z
100000
10000
1000
100
10
1
0.1
-0.0006
-0.0005
-0.0004
-0.0003
-0.0002
-0.0001
0
1e-04
0.0002
0.0003
0.0004
0.0005

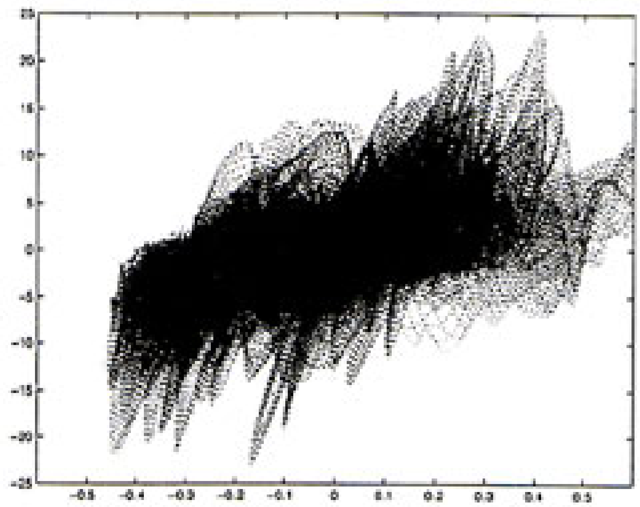


Enstrophy Spectrum
0.0001
1e-05
1e-06
1e-07
1e-08
1e-09
1e-10
1e-11
1e-12
1
10
100
1000
wavenumber
Total
Iter. Coh. part
Iter. Incoh. part
-2
+1
Coherent
0.7% N
99% E
Total
Incoherent
99.3% N
1% E

COHERENCE SCATTERPLOT BETWEEN VORTICITY ω AND STREAMFUNCTION ψ

$\omega = F(\psi)$

ω

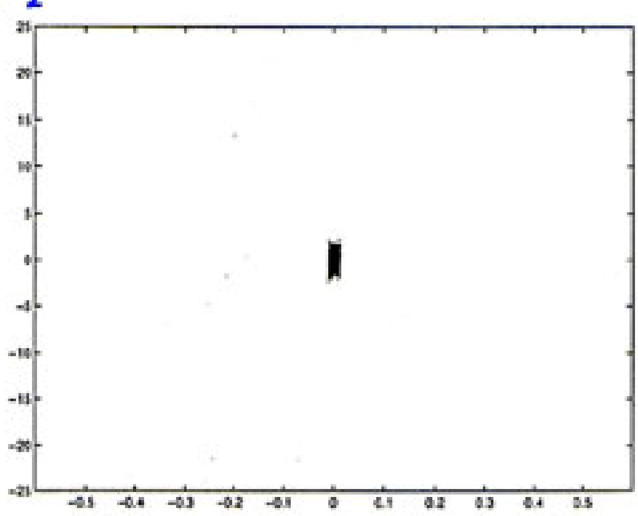


ψ

Total

$\omega_I = F_I(\psi_I)$

ω_I



ψ_I

Incoherent

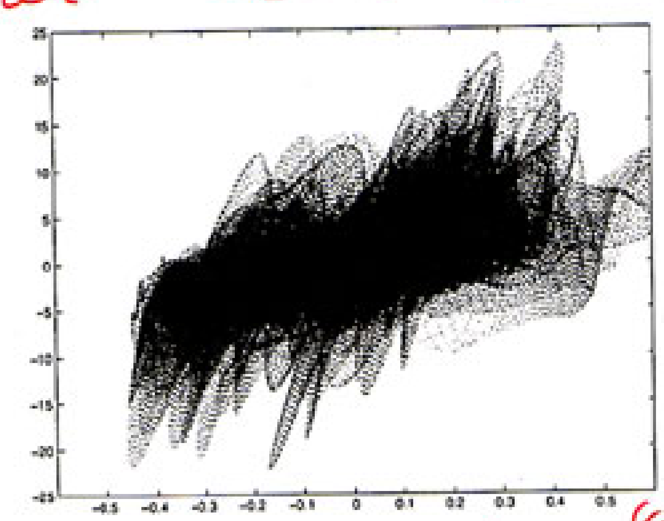
99.3% N

1% E

7% Z

$\omega_c = F_c(\psi_c)$

ω_c



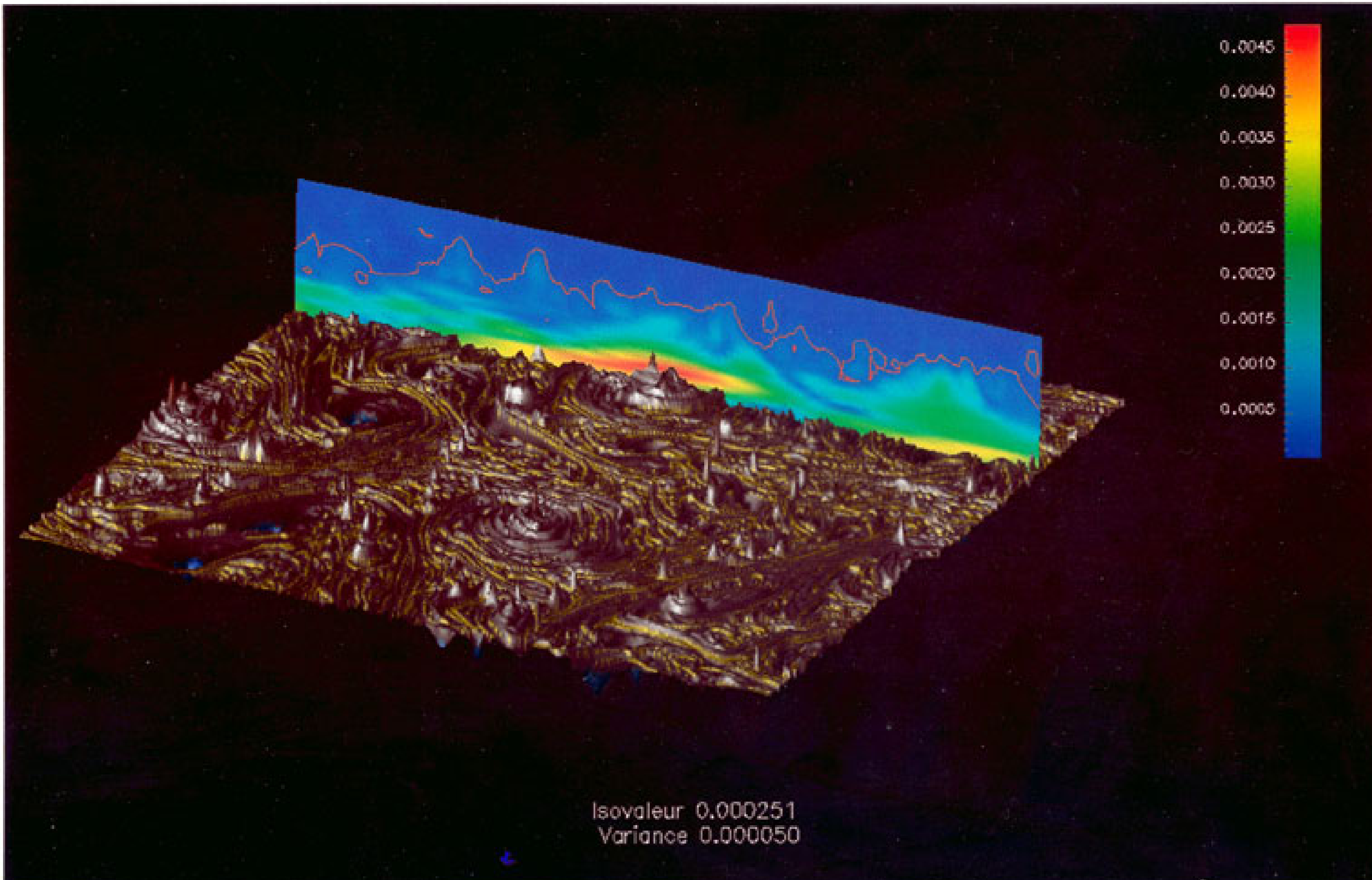
ψ_c

Coherent

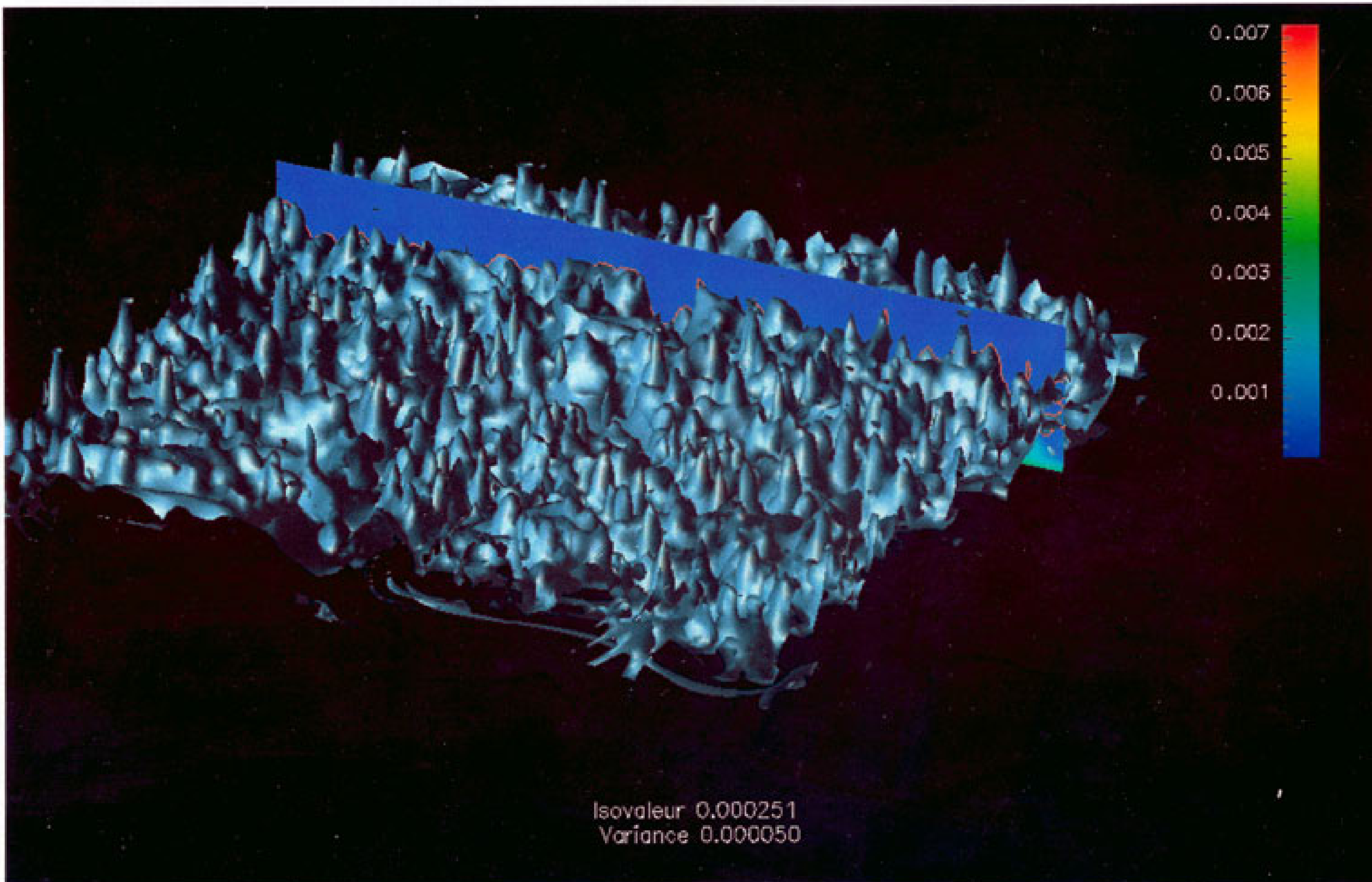
0.7% N

99% E

93% Z



Wavelet coefficients   Isolignes $\tilde{\omega}_T(\vec{x}, \ell)$ Azzalini & Farge 2001



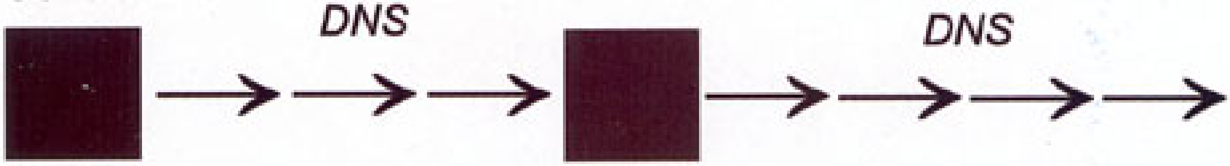
Wavelet coefficients Isosurface $\tilde{\omega}_7(\vec{x},\ell)$ Azzalini & Farge 2001

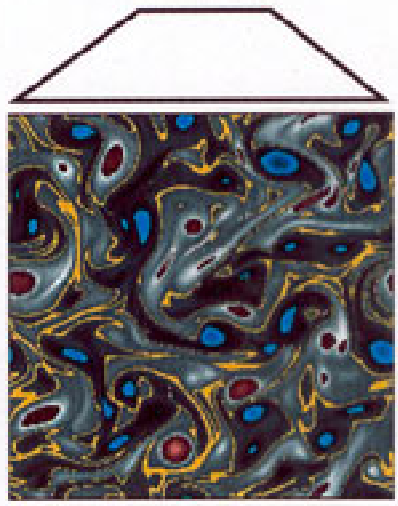
Filtering of DNS data

Nonlinear wavelet filtering (CVS)

$N = 256^2$

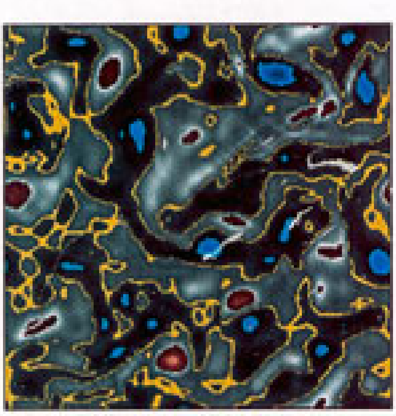
Initial Condition





Coherent vortices
0.7% N
99% energy
93% enstrophy

Incoherent background
99.3% N
0.7% energy
7% enstrophy

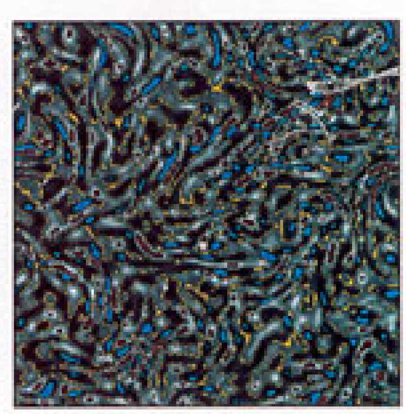


Color scale:

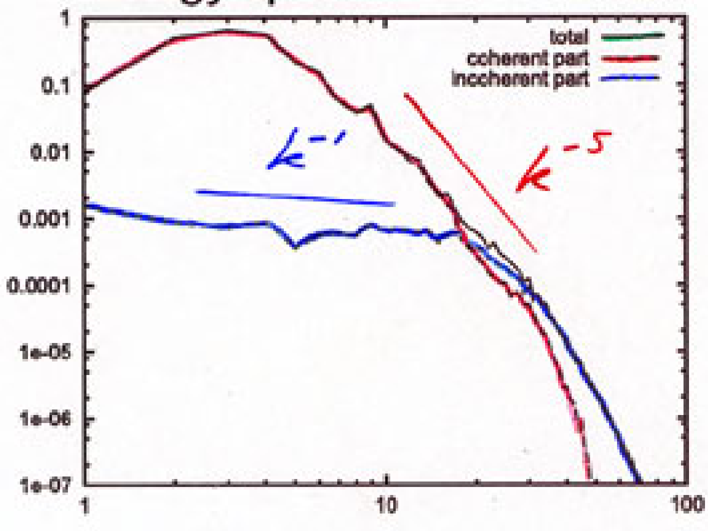
$-\omega_{min}$ to $-0.3\,\omega_{min}$

$-0.3\,\omega_{min}$ to $+0.3\,\omega_{max}$ with $\omega = 0$ yellow

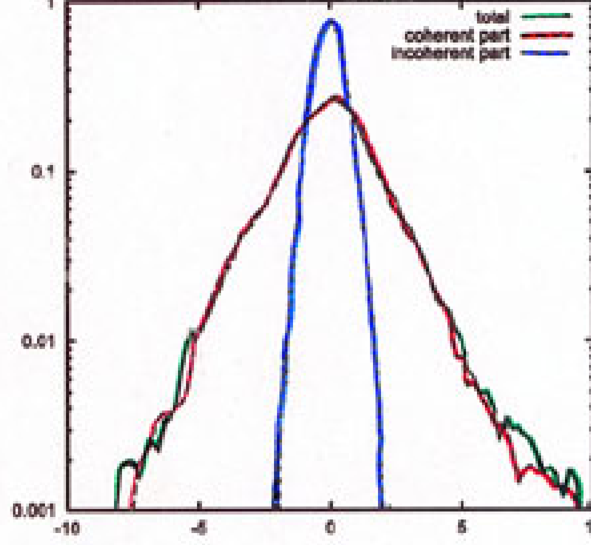
$+0.3\,\omega_{max}$ to ω_{max}



Energy spectrum



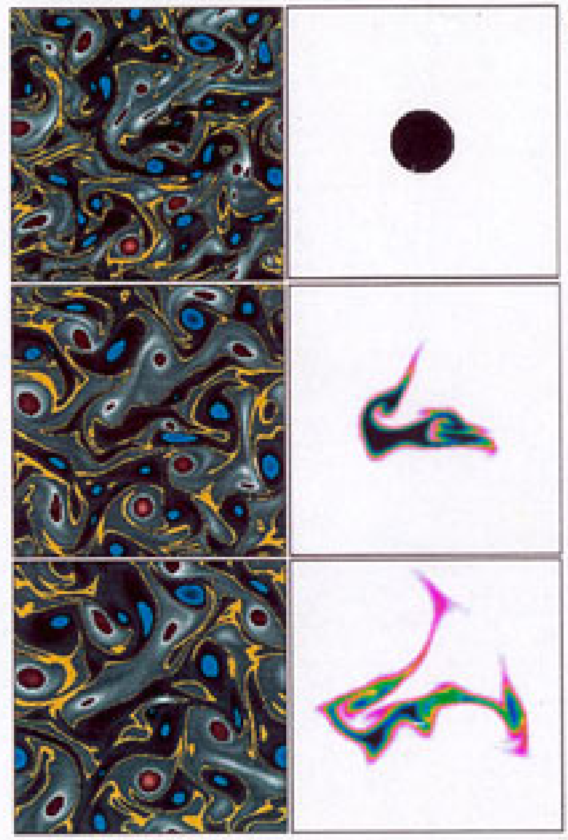
PDF of vorticity



Coherent Vortex Simulation (CVS) for the mixing of a passive scalar

$S_c = \frac{\nu}{D} = 1$

Total field $N = 256^2$



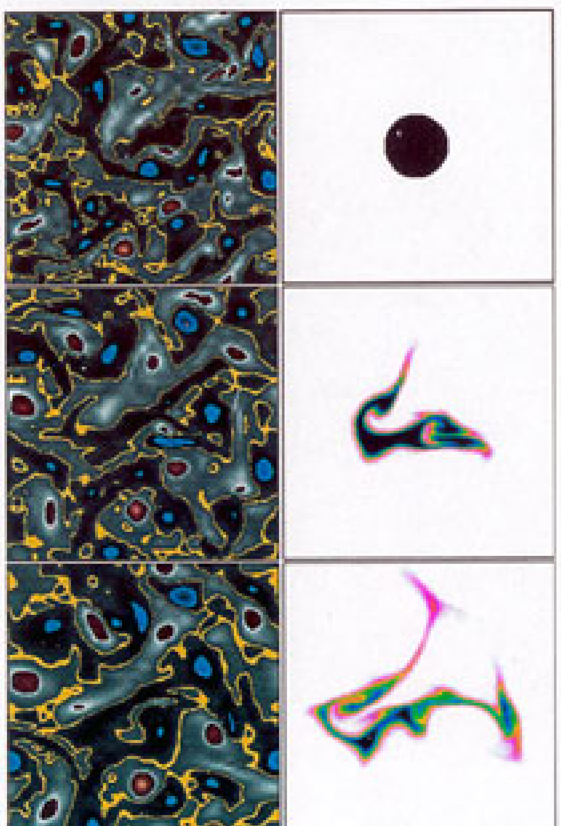
t=0 eddy turnover times

t=7

Coherent part

0.7% N
99% energy
93% enstrophy

t=0

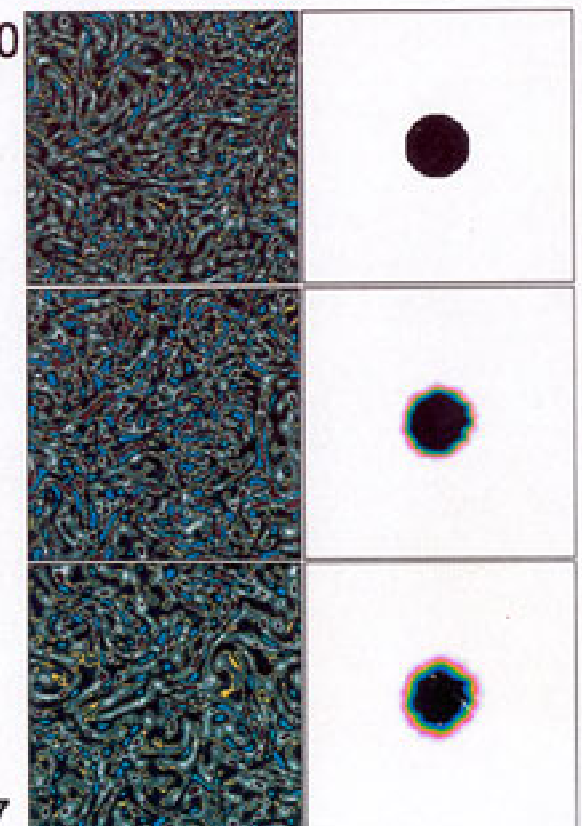


t=7

Incoherent part

99.3% N
0.7% energy
7% enstrophy

t=0



t=7

cf. Beta, Schneider, Farge
Chem. Eng. Sci., 58, 2003

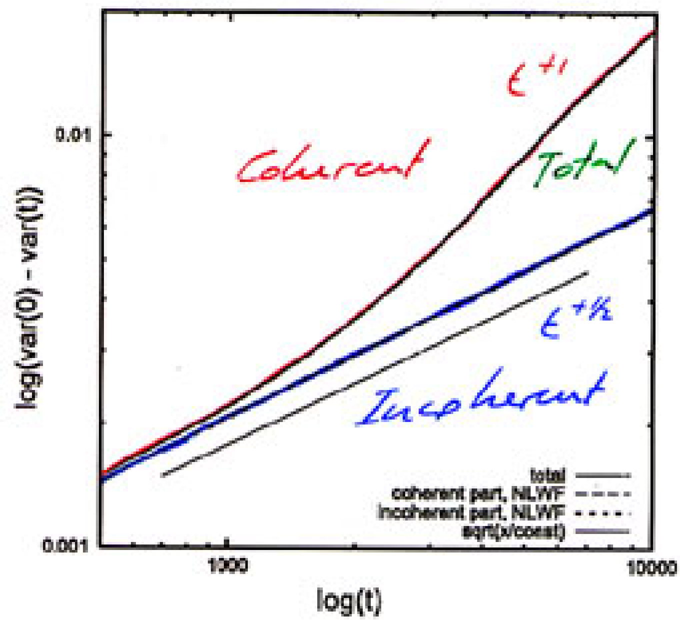
Measuring the mixing process

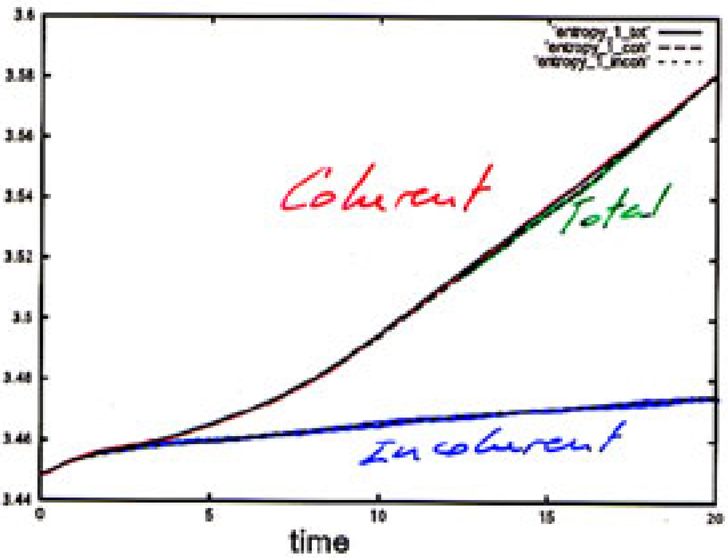
Evolution of quantities,
related to the mixing process

CVS

Difference of
Variance of
Concentration

$$D(t) = \sigma^2(t=0) - \sigma^2(t)$$
with
$$\sigma^2 = \int (\theta - \bar{\theta})^2 \, d\vec{x}$$



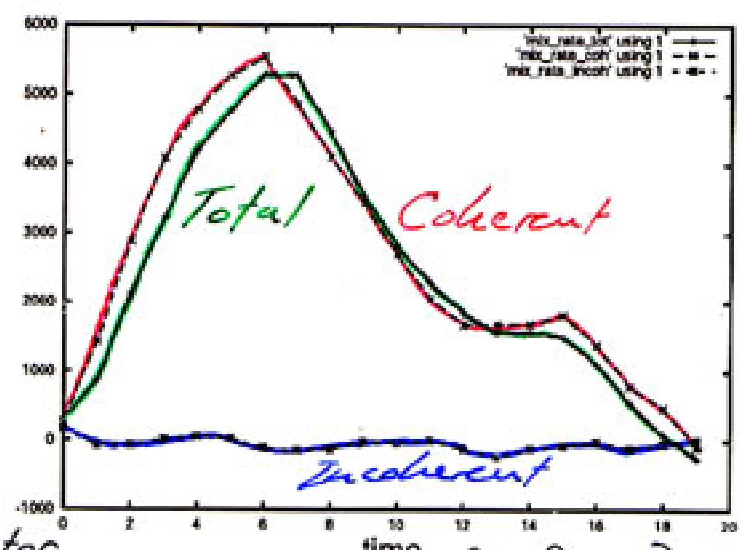


Entropy

$$S(t) = \sum_i p_i(t) \log|p_i(t)|$$
with
$$p_i = \frac{\theta_i}{(\|\theta_i\|_2)^{1/2}}$$

Relative
production rate
of scalar gradient

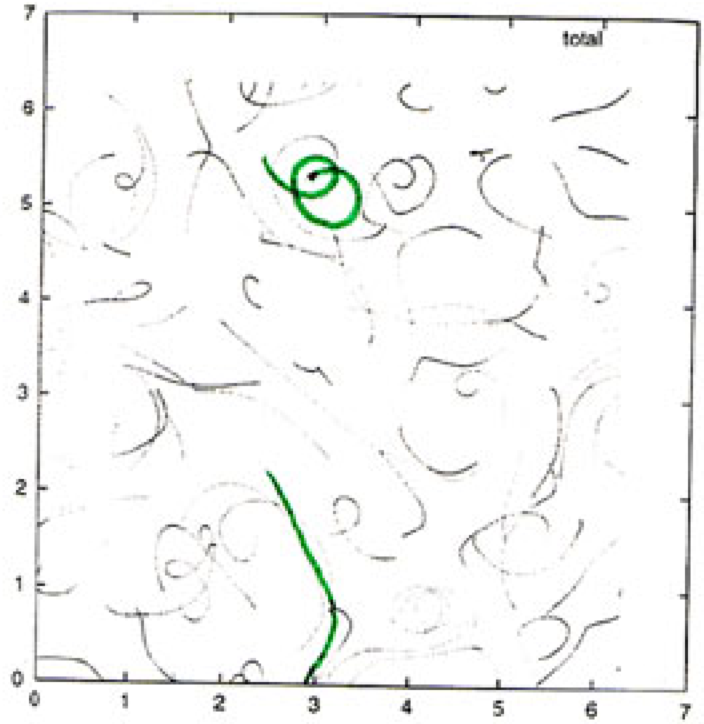
$$G(t) = \int |\vec{\nabla}\theta| \, s \cos 2\alpha \, d\vec{x}$$
with
$$\alpha = \{\vec{\nabla}\theta, \vec{l}_2\}$$
$\vec{l}_2$ compressing eigenvector of strain tensor $\bar{\bar{S}}$



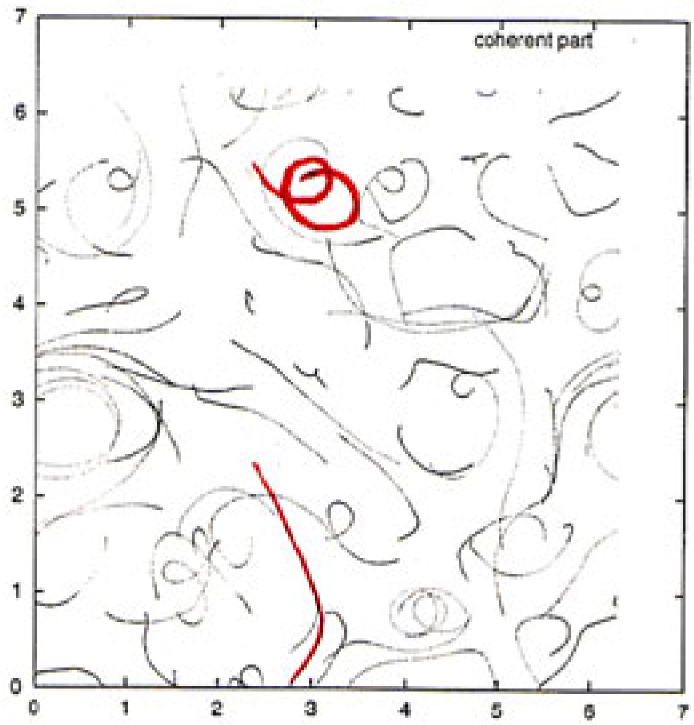
$$s = \sqrt{s_{11}^2 + s_{12}^2} \qquad \begin{cases} s_{11} = \partial_x u - \partial_y v \\ s_{12} = \partial_x v + \partial_y u \end{cases}$$

Total flow 100% N

$$S_c = \frac{\nu}{D} = \infty$$

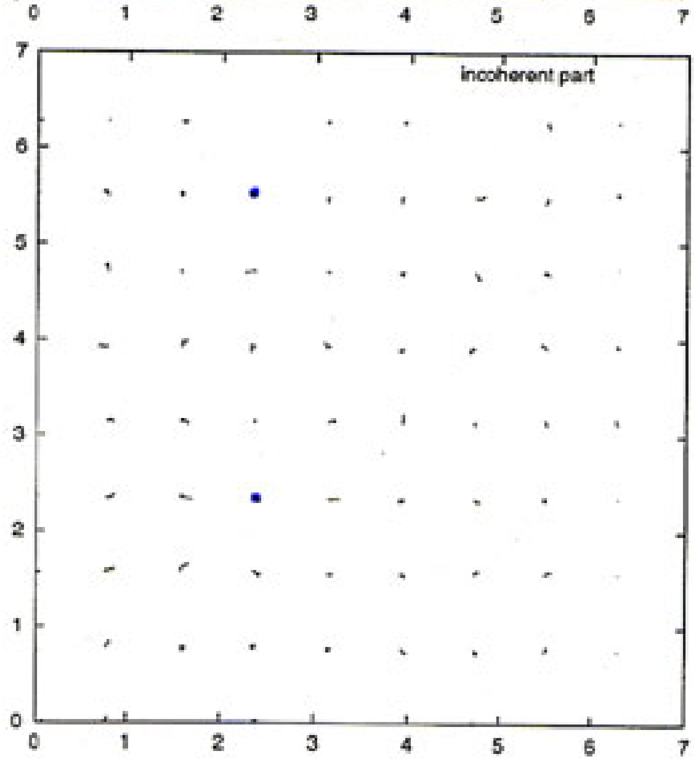


Coherent flow 0.7% N

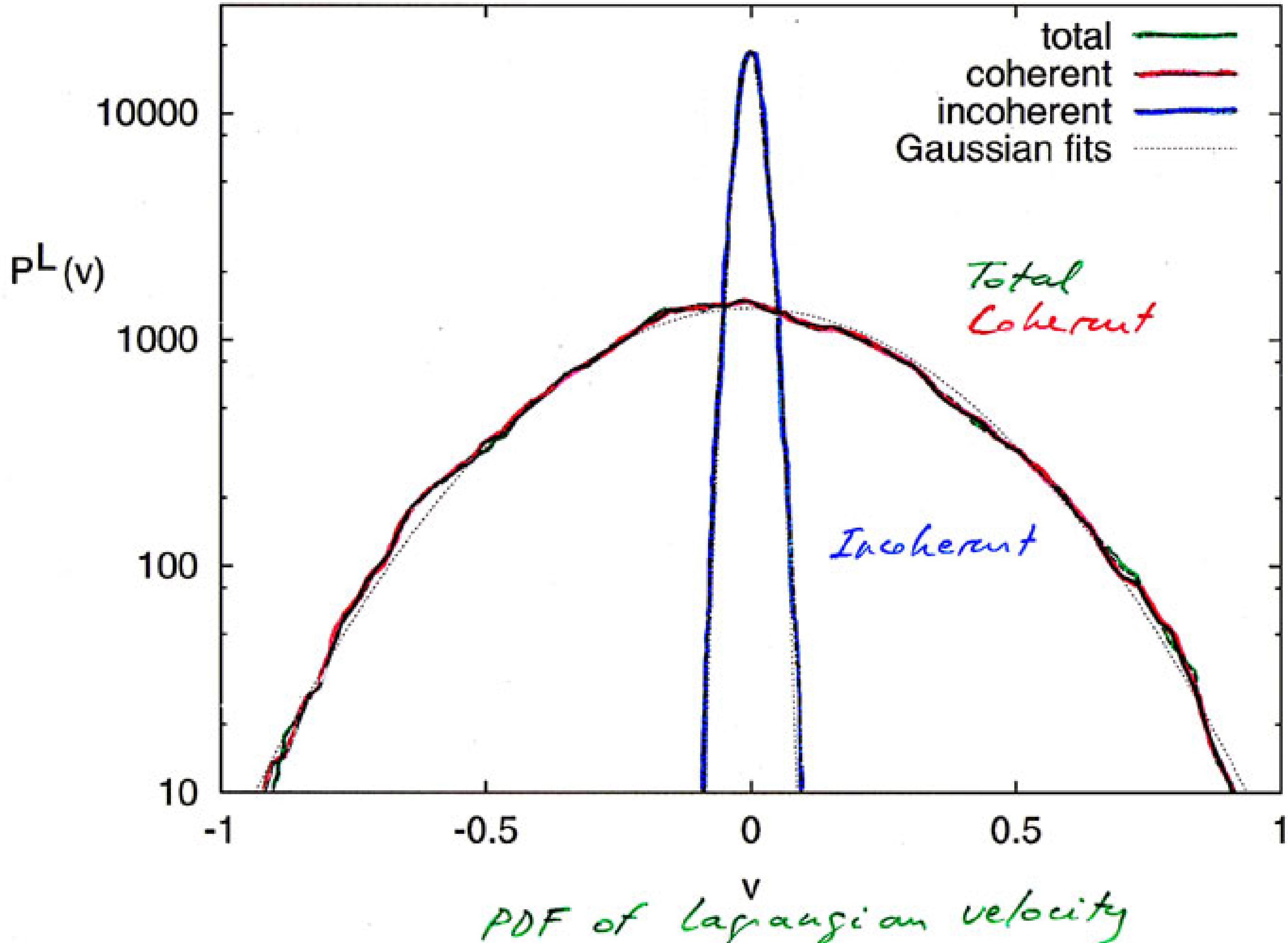


Transport by the coherent vortices velocity field

Incoherent flow 99.3% N



Diffusion by Brownian-like motion of the incoherent velocity field



total
coherent
incoherent
Gaussian fits
10000
1000
100
10
P^L(v)
Total
Coherent
Incoherent
-1
-0.5
0
0.5
1
v
PDF of Lagrangian velocity

We have shown that the CVS filtering disentangles two different dynamical behaviours for scalars and particles:

1. Transport by the coherent vortices → turbulent NL cascade with long time correlation → strong mixing and fast production of entropy,

2. Diffusion by the incoherent background flow → Brownian motion with short time correlation → weak mixing and slow production of entropy, which effect can be modelled by a Langevin equation or discarded as turbulent dissipation.

LABORATORY EXPERIMENT

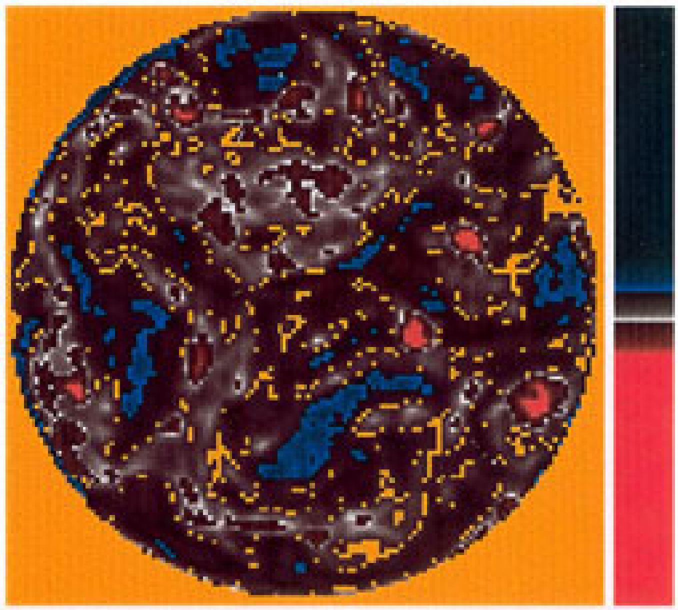
$N = 128^2$

Total Vorticity

Rossby number

$$R_o = \frac{\|\omega\|}{f}$$

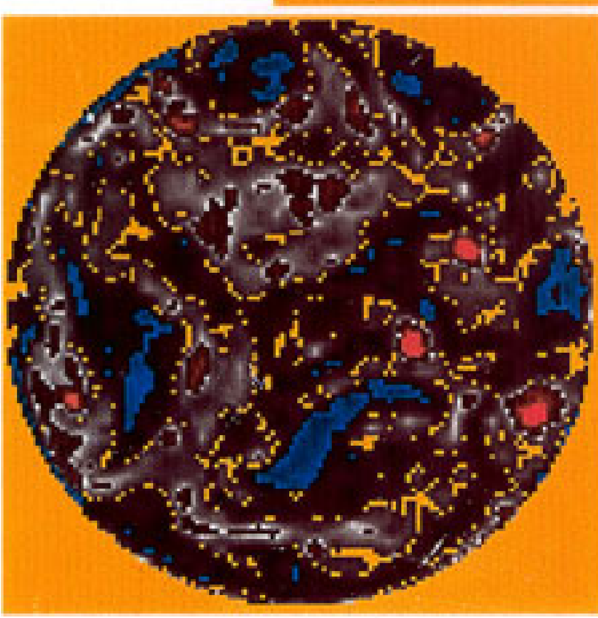
$= 0.5$



Reynolds number

$$Re = \frac{\|u\| D}{\nu}$$

$= 5 \cdot 10^4$



Coherent Vorticity

9% N

98% Z

99.9% E

Incoherent Vorticity

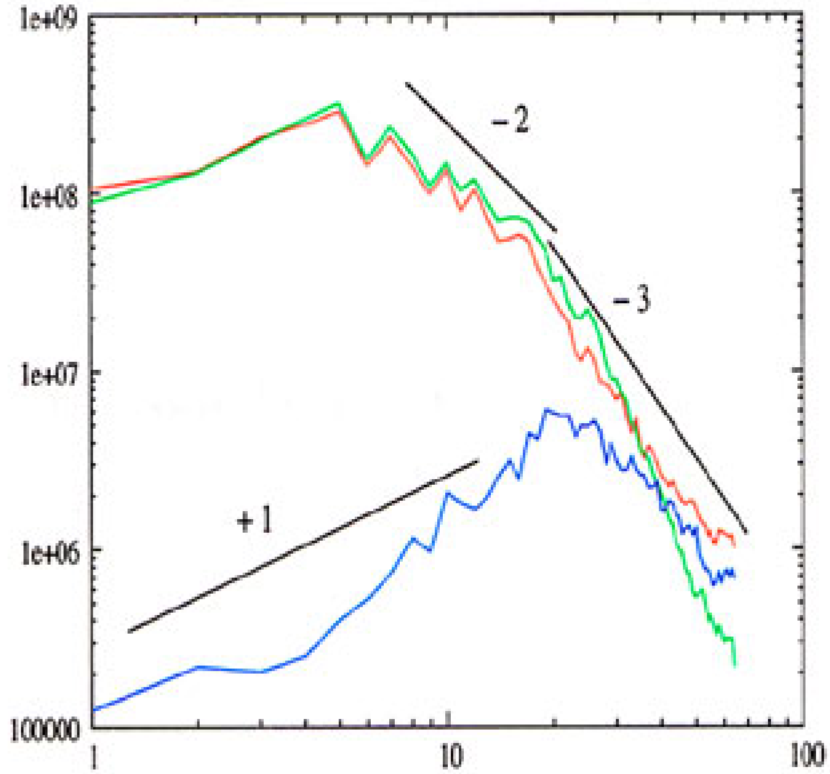
91% N

2% Z

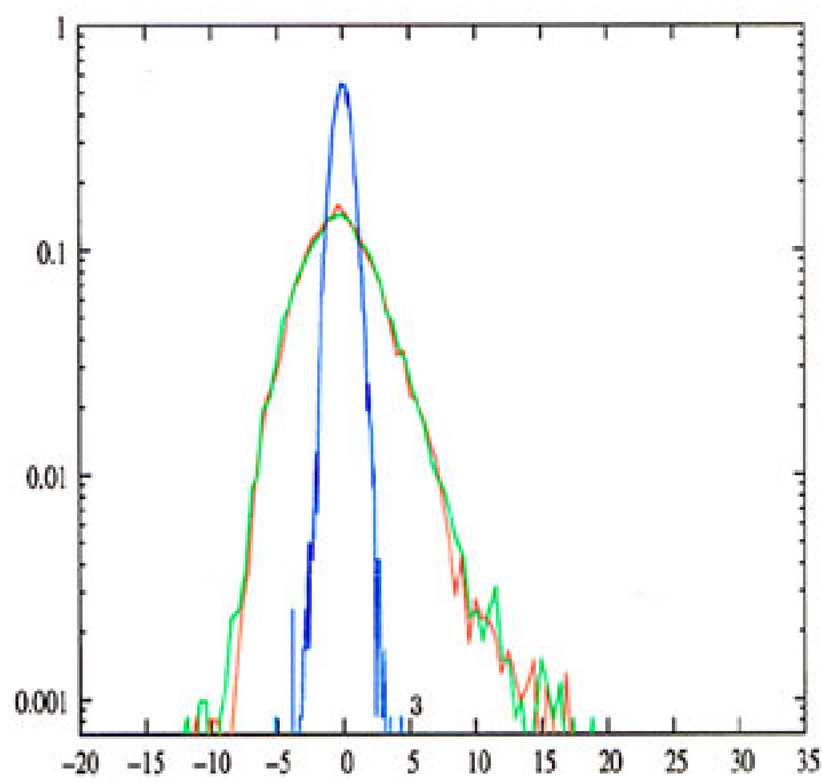
0.1% E

LABORATORY EXPERIMENT

$Ro = 0.5$ $Re = 5 \cdot 10^4$



Eustrophy spectrum



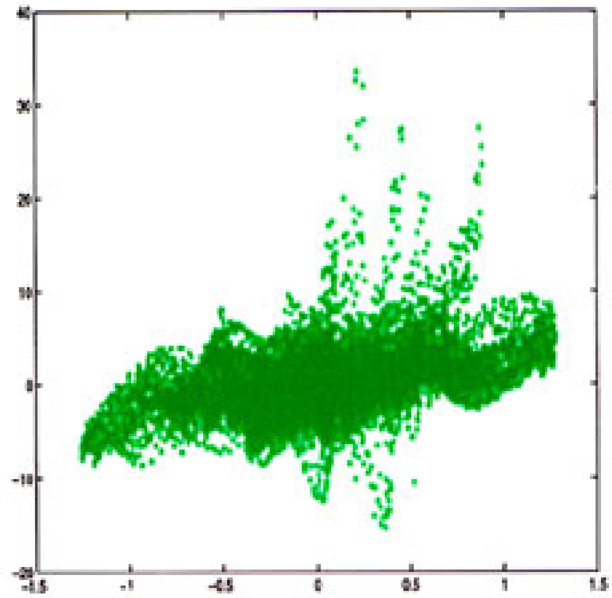
PDF of vorticity

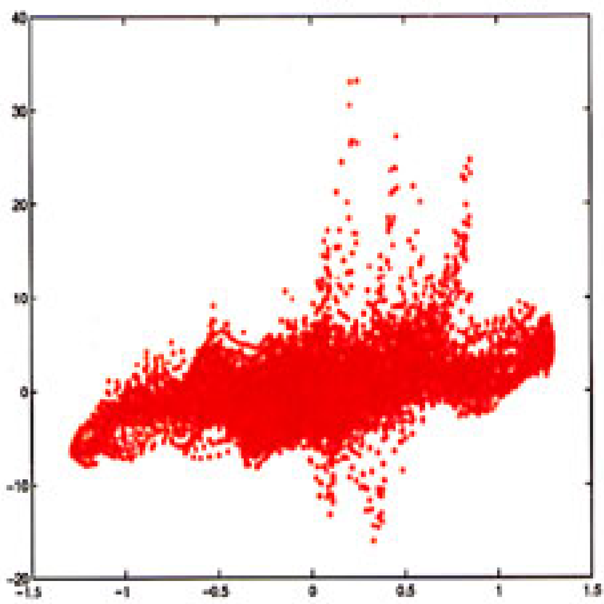
LABORATORY EXPERIMENT

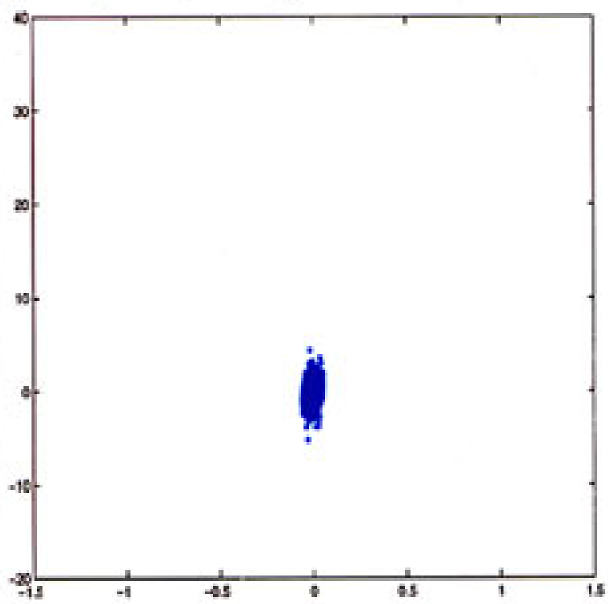
Coherent Function F

$$\omega = F(\psi)$$

$Ro = 0.5$ $\qquad$ $Re = 5 \cdot 10^4$



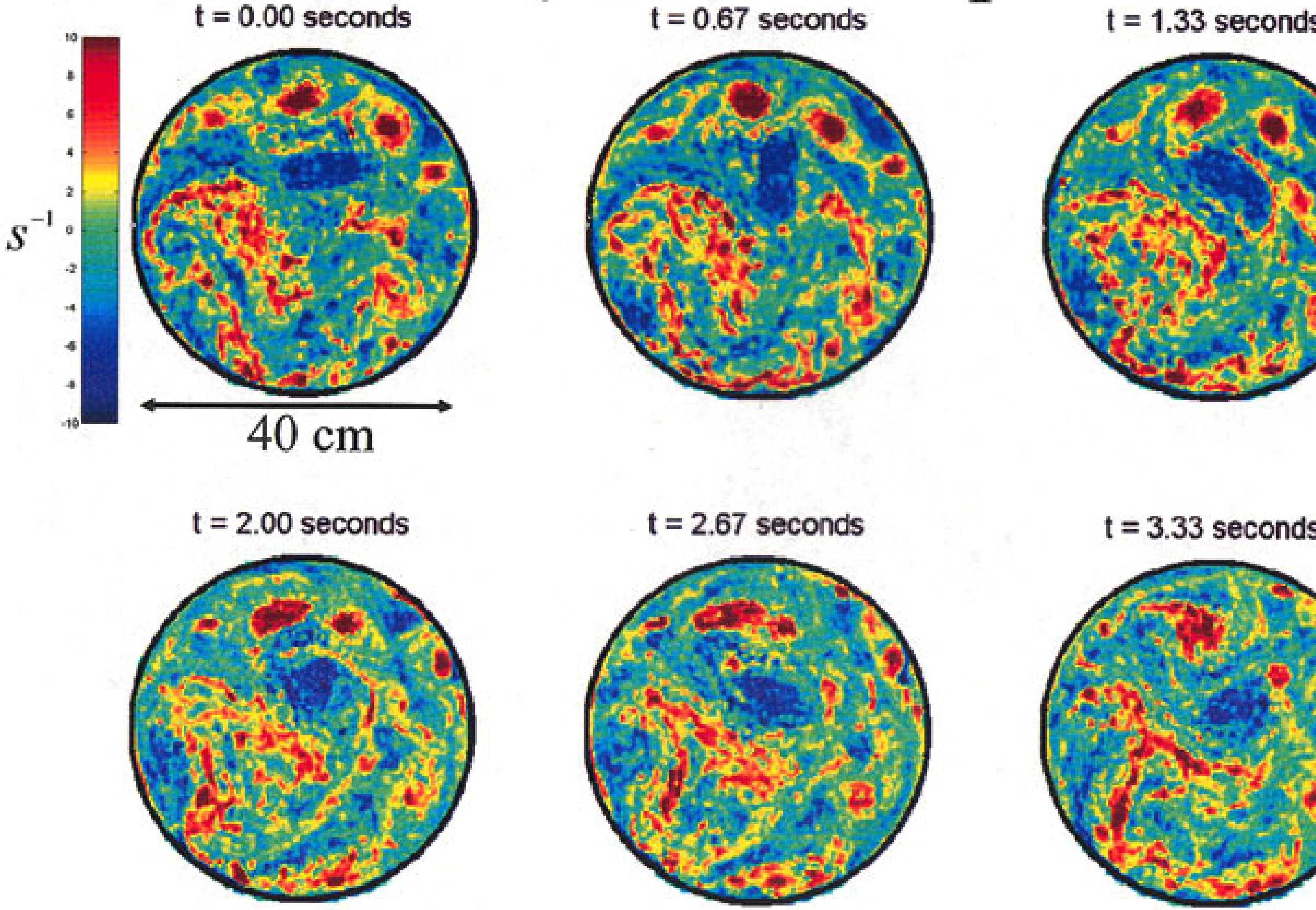




$\omega_c = F_c(\psi_c)$ $\qquad$ $\omega_I = F_I(\psi_I)$

Vorticity Field Snapshots

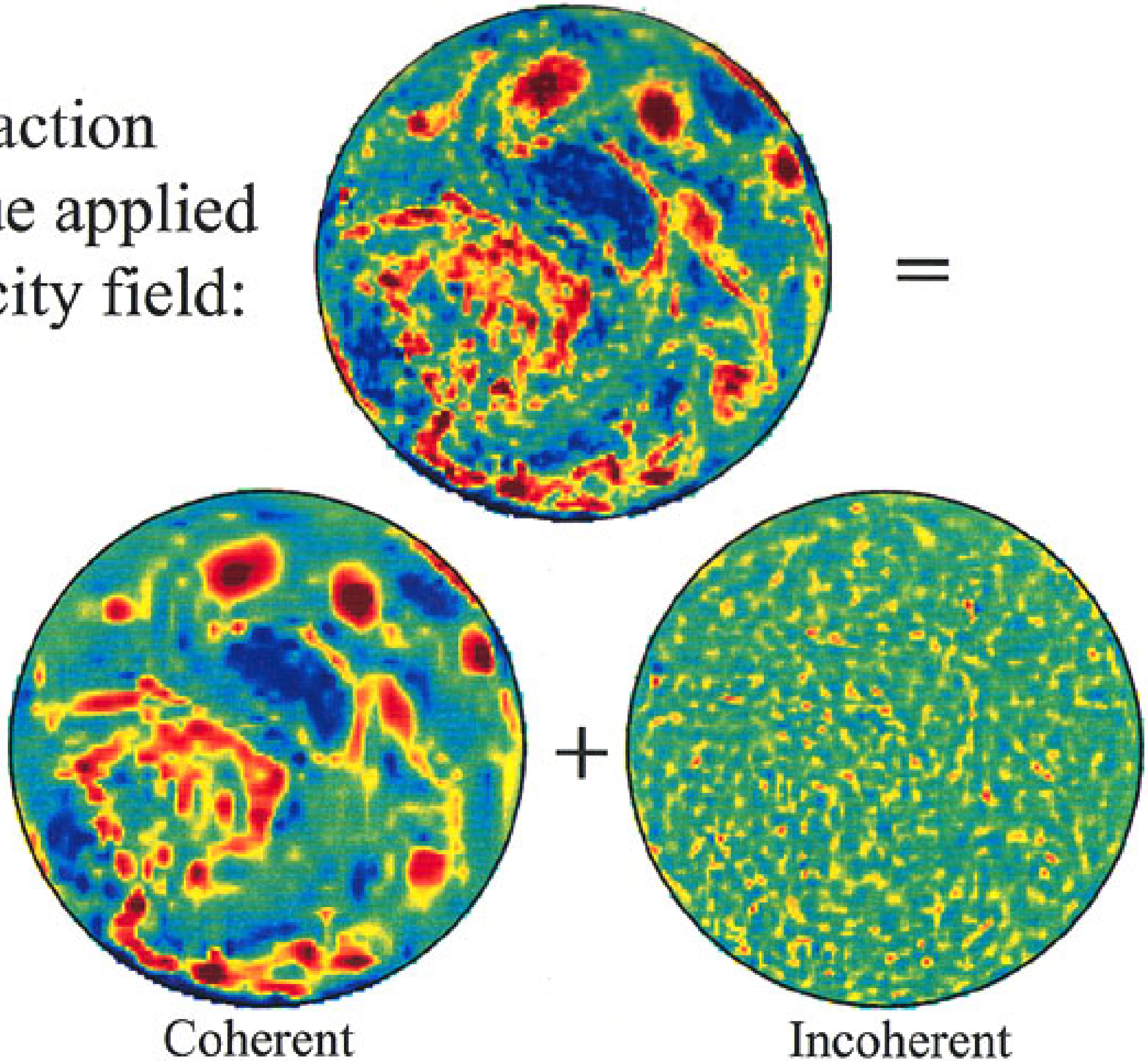
measured by PIV



LABORATORY EXPERIMENT IN ROTATING TANK

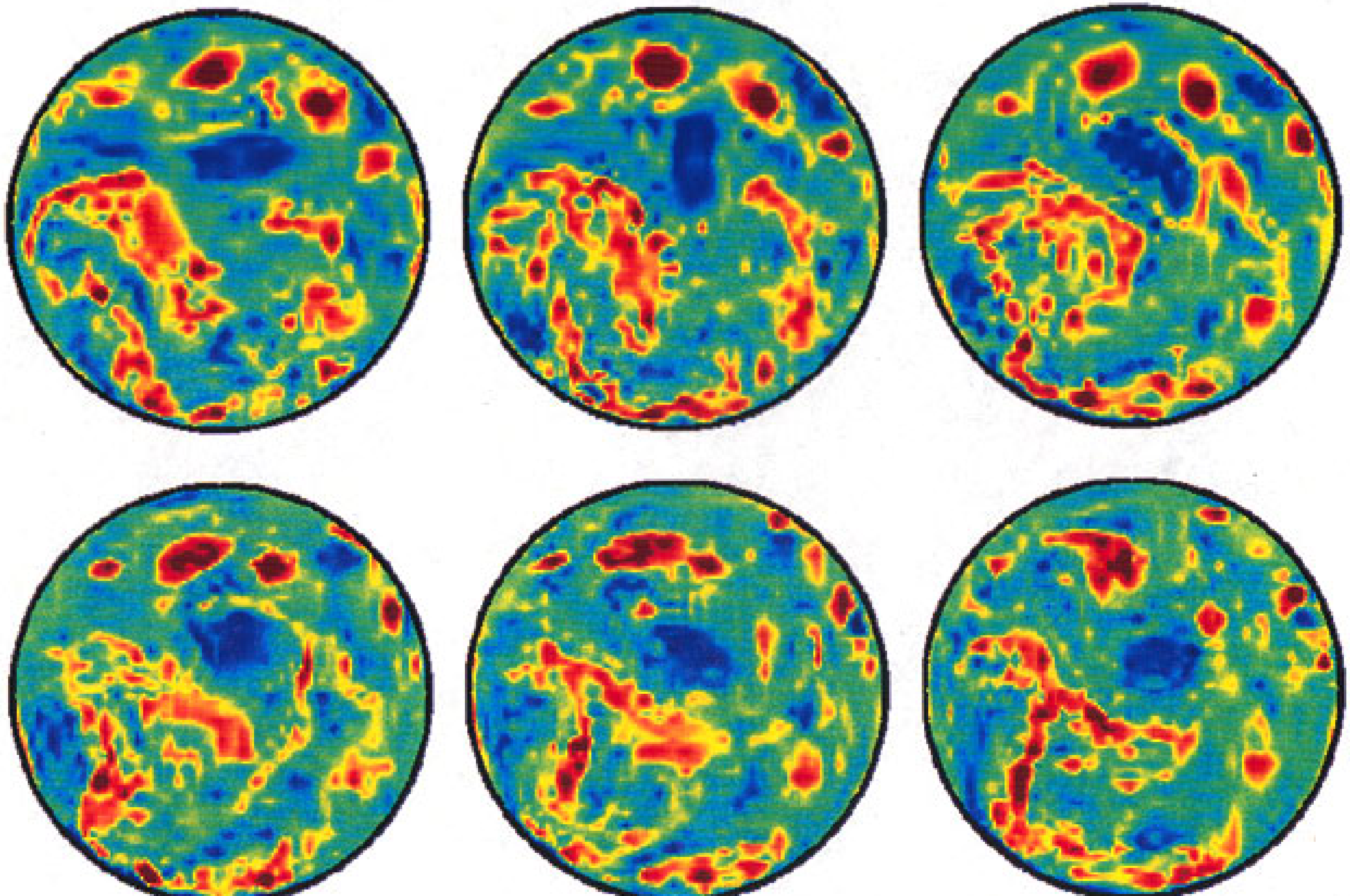
Jori Rupert-Felsot, Eran Sharon, Harry Swinney

Extraction technique applied to vorticity field:



Coherent Fields

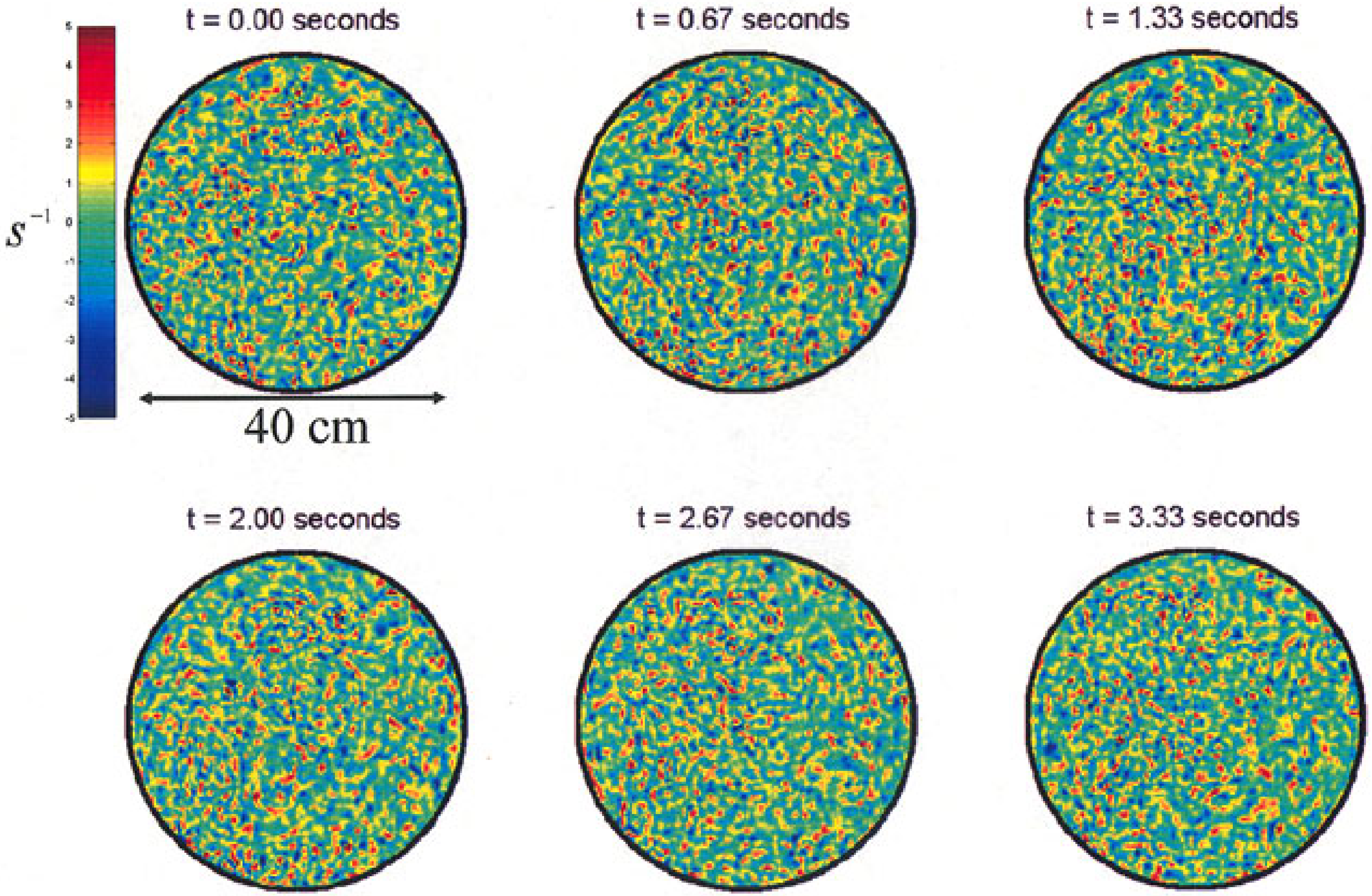
t = 0.00 seconds t = 0.67 seconds t = 1.33 seconds



t = 2.00 seconds t = 2.67 seconds t = 3.33 seconds

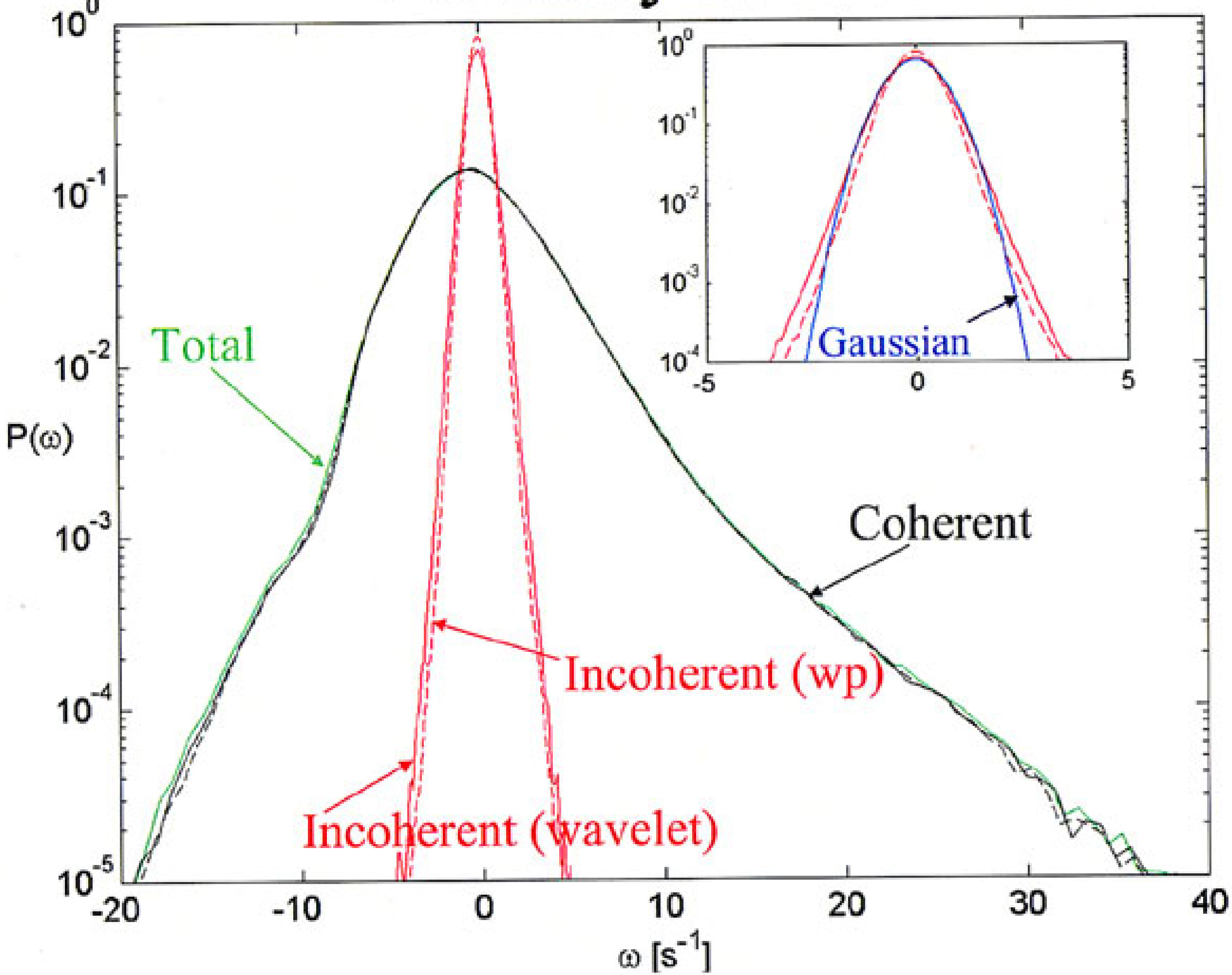
Coherent, long-range correlated, non Gaussian

Remainder Fields



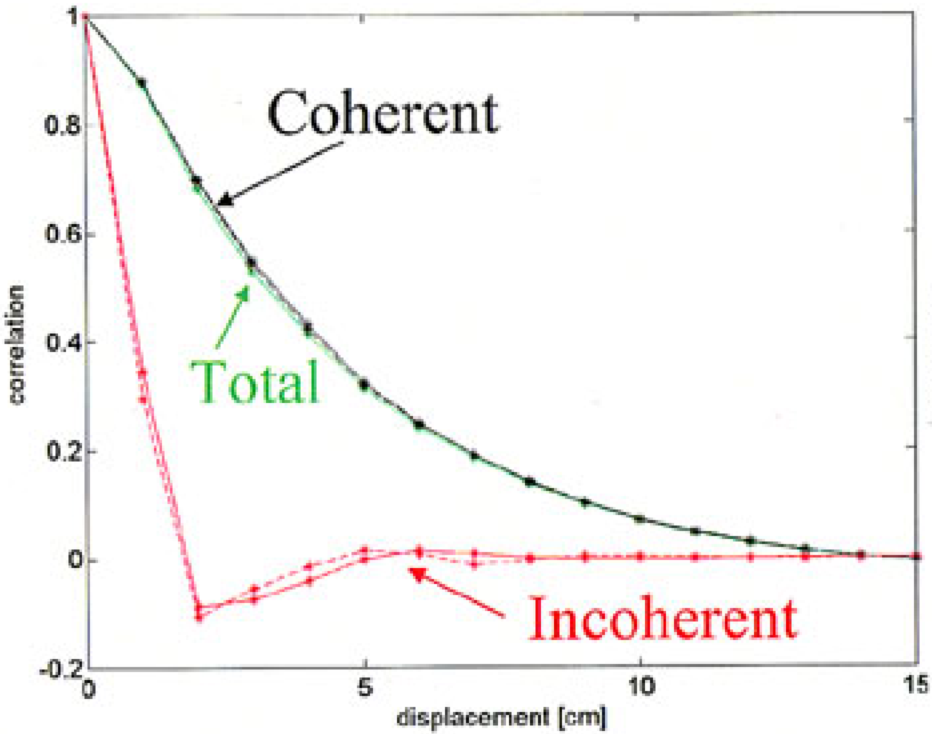
Incoherent, decorrelated, Gaussian

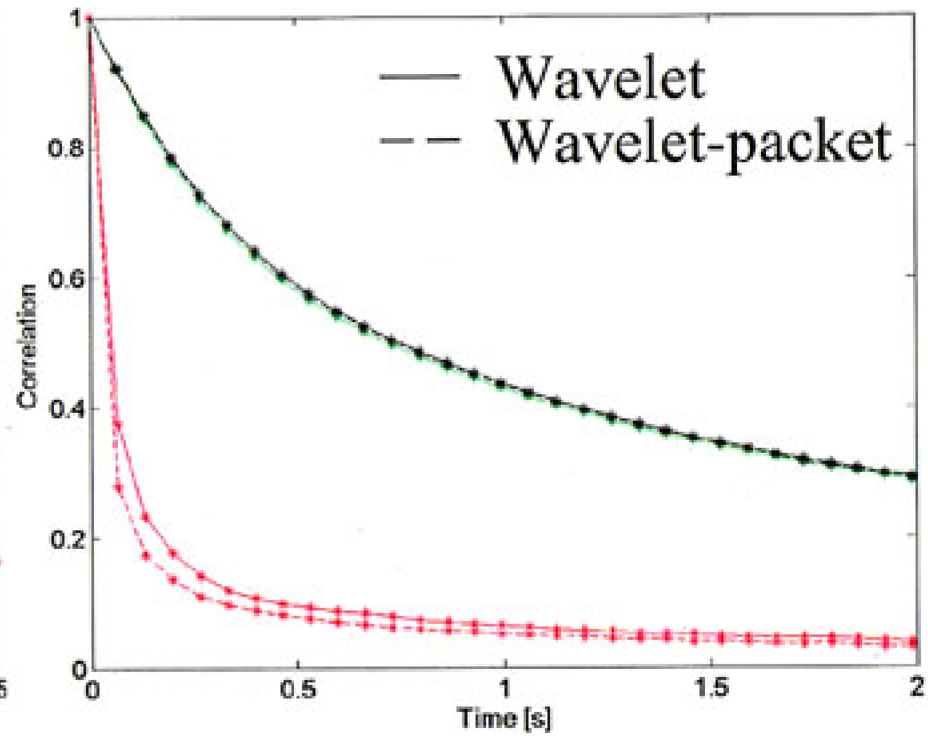
Vorticity PDFs



Temporal and spatial correlations

Both wavelet and wavelet-packet coherent component retain long time and spatial correlations

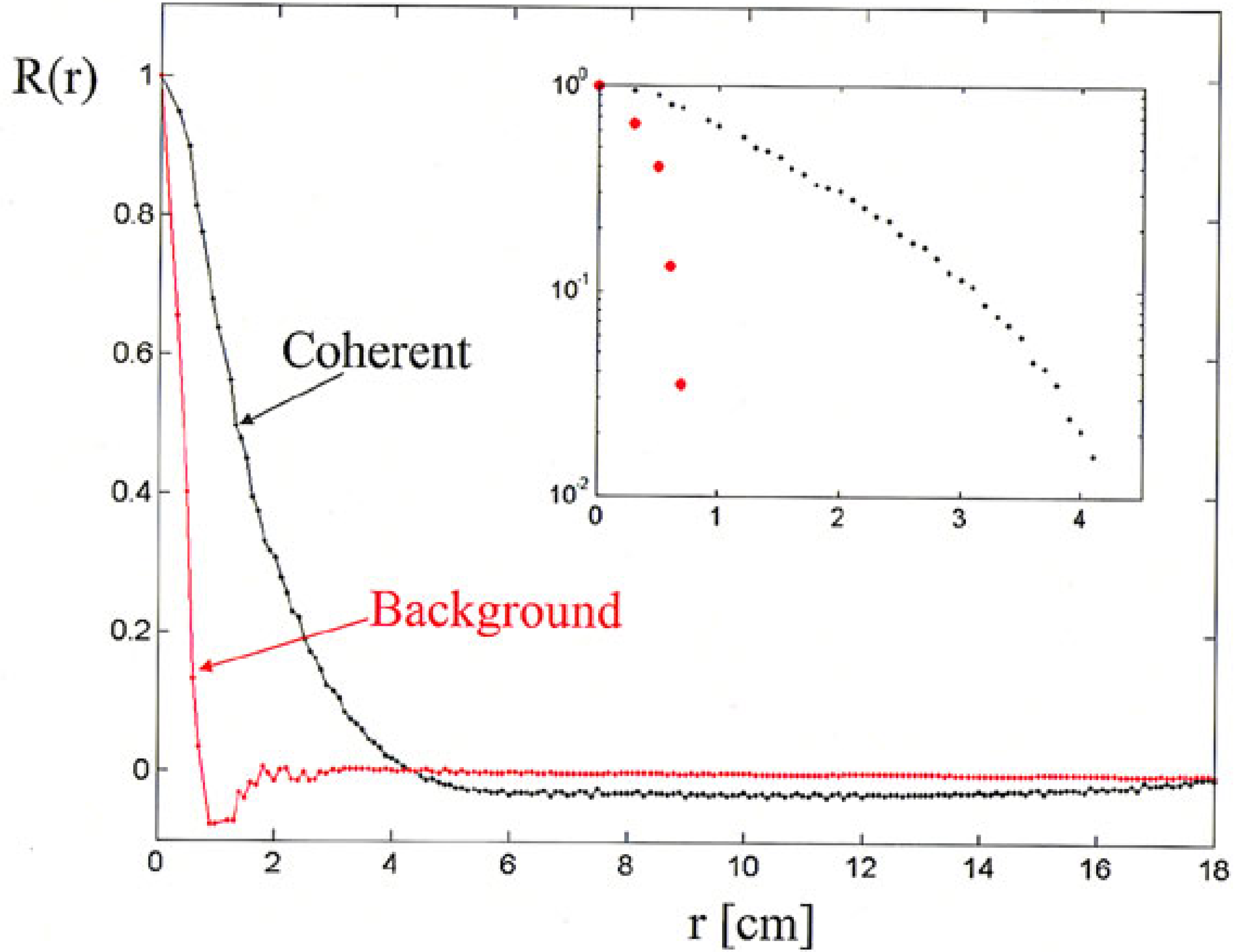




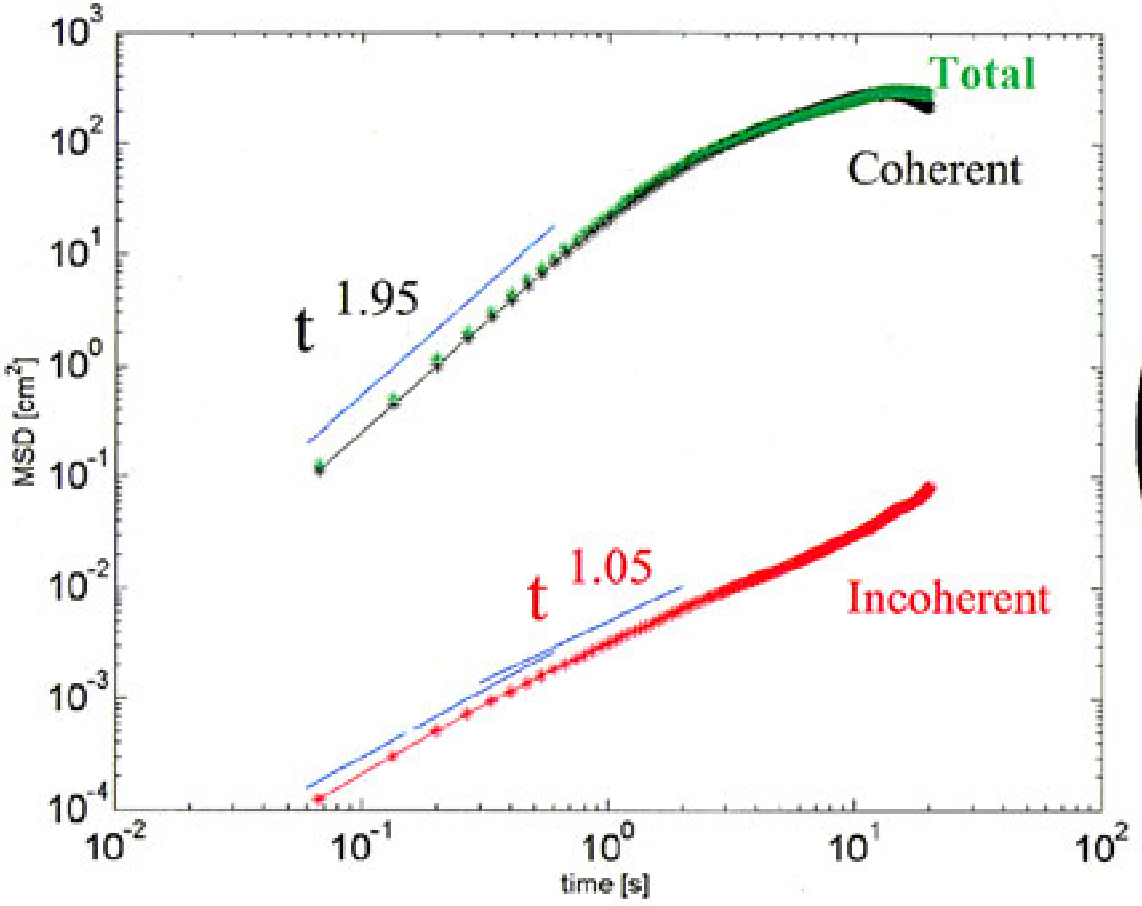
$$\frac{\langle \omega(|x|,t)\omega(|x|+r,t)\rangle}{\langle \omega^2\rangle}$$

$$\frac{\langle \omega(x,t)\omega(x,t+s)\rangle}{\langle \omega^2\rangle}$$

Spatial Correlation $\frac{\langle \omega(|x|,t)\omega(|x|+r,t)\rangle}{\langle \omega^2 \rangle}$



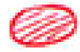
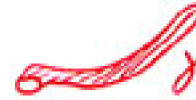
Tracer particle MSD



Coherent: flight-like dispersion

Incoherent: close to random walk

CHARACTERIZATION OF COHERENT VORTICES

2D :  ω_i 3D:  γ_i / $\tilde{\omega}_{>}$

- They correspond to elliptic regions which are stable (≤ 0 Lyapounov exponents)

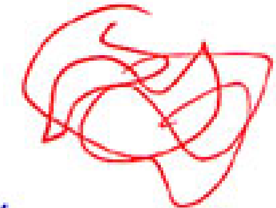
$\Rightarrow$ there is no mixing as long as they are isolated, but when they interact: filamentation & dissipation

- They are highly intermittent, i.e. their support is rare in space and time $\Rightarrow$ they disappear in low-order statistics (L^2 norm) for $k \to \infty$.

- They have non Gaussian PDFs $\Rightarrow$ they dominate the total PDFs.

- They can be described by a small number of degrees of freedom $\Rightarrow$ non equilibrium state which cannot be statistically modelled (the Central Limit Theorem cannot be used).

CHARACTERIZATION OF INCOHERENT BACKGROUND

2D:  3D:  / $\tilde{\omega}_<$

- It corresponds to hyperbolic regions which are unstable (>0 Lyapunov exponents) $\Rightarrow$ there is strong mixing

- It is non intermittent, i.e. its support is dense in space and time $\Rightarrow$ it dominates low-order statistics (L^2 norm) for $k \to \infty$

- It has narrower PDFs $\Rightarrow$ it disappears in the total PDFs.

- We need a large number of degrees of freedom of the basis we use $\Rightarrow$ they correspond to an equilibrium state which can be statistically modelled (the Central Limit Theorem is used to define averages)

EXTENSION TO 3D

Vorticity becomes a vector and there is now vortex stretching. But we still should rely on:

- Biot-Savart $\vec{v} = \nabla \times \nabla^{-2} \vec{\omega}$,
- Incompressibility $\nabla \cdot \vec{v} = 0$,
- Helmholtz' theorem

to design 3D numerical codes.

The basic elements should be vortex tubes Γ_i and the important dynamical quantity to compute should be circulation

$$\Gamma_i = \int_{\Omega_i} \vec{\omega} = \oint_{\partial\Omega_i} \vec{v}$$

Ω_i

Each vortex tube should be represented as a superposition of wavelets corresponding to its internal degrees of freedom.

3D TURBULENT FLOWS

CVS FILTERING

cf. { Phys. Rev. Lett., 87, 5, 30th July 2001
Phys. Fluids, 11, 8, 1999
Ann. Rev. Fluid Mech., 24, 1992 }

Nonlinear filtering of
the wavelet coefficients
of vorticity

Total Vorticity

$$\tilde{\omega}_i = \langle \omega_i | \psi \rangle \quad i=1,2,3$$

Coherent Vorticity
if $|\tilde{\omega}_C| > \varepsilon$

Incoherent Vorticity
if $|\tilde{\omega}_I| \leq \varepsilon$

with $\varepsilon = \left(\frac{4}{3} Z \log_e N\right)^{1/2}$
universal threshold

$$\vec{\omega}_C = |\psi\rangle\langle\tilde{\omega}_C|$$

$$\vec{\omega}_I = |\psi\rangle\langle\tilde{\omega}_I|$$

$$\vec{\omega} = \vec{\omega}_C + \vec{\omega}_I$$

Coherent Velocity

$$\vec{V}_C = \nabla \times (\nabla^{-2} \vec{\omega}_C)$$

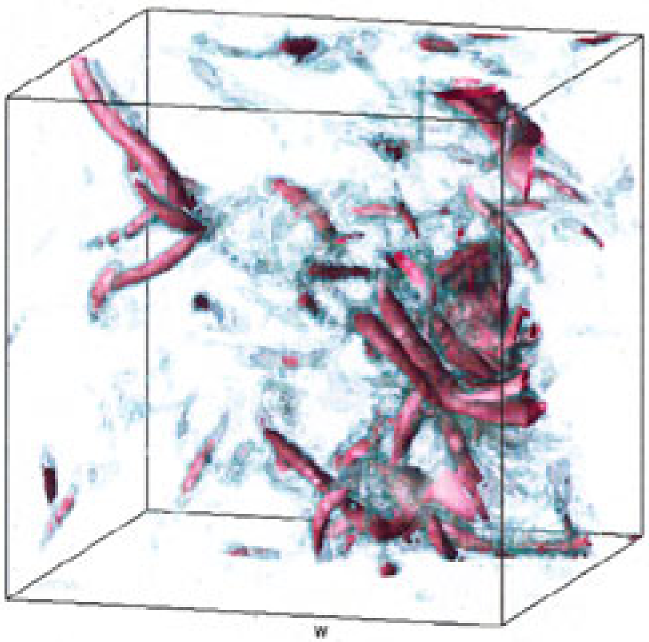
Incoherent Velocity

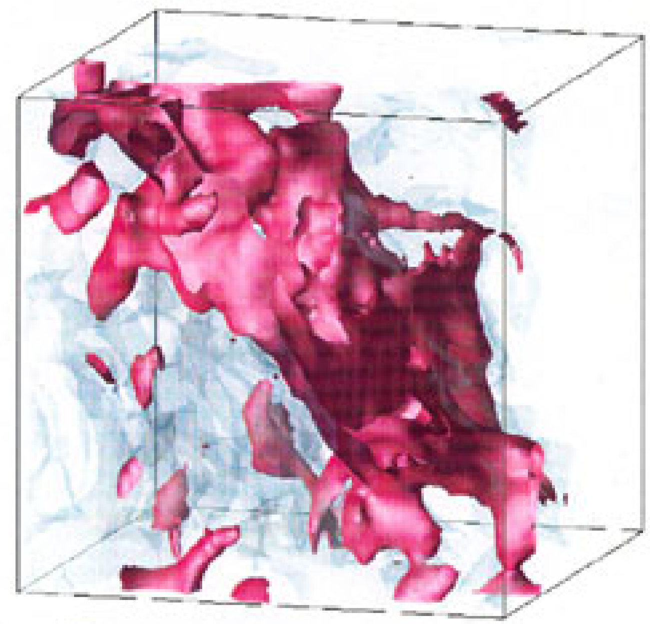
$$\vec{V}_I = \nabla \times (\nabla^{-2} \vec{\omega}_I)$$

$$\vec{V} = \vec{V}_C + \vec{V}_I$$

DNS

HOMOGENEOUS ISOTROPIC TURBULENT
FLOW AT $R_\lambda = 150$





computed by Vincent & Meneguzzi

Isosurfaces of vorticity
Resolution 256^3
$Re_\lambda = 150$

Isosurfaces of velocity

Resolution $N = 256^3$

Vincent & Meneguzzi, JFM 225, 1991

LES: Fourier filtering

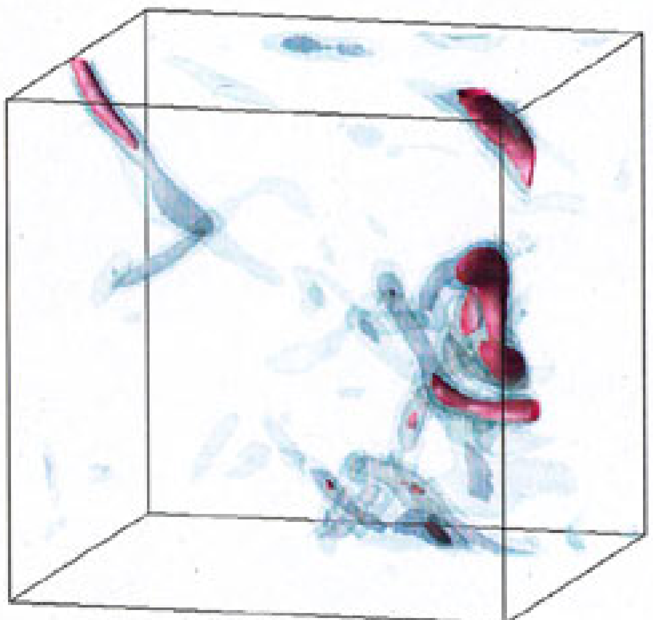
same as POD for homogeneous isotropic turbulent flow

large scale part

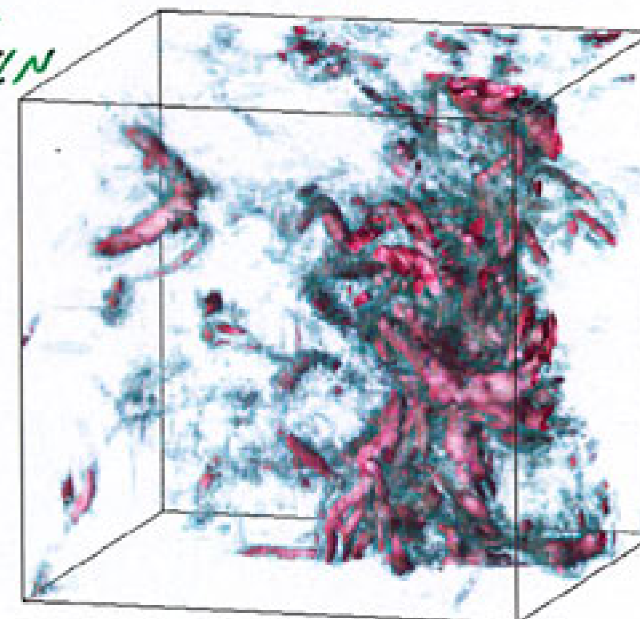
small scale part

$N = 256^3$

3% N   97% N



W (low pass filter)



W (high pass filter)

Isosurfaces of vorticity

3% # Coefficients ($|\vec{\omega}| = 3\sigma, 4\sigma, 5\sigma$, $\sigma = \sqrt{2Z}$)
64% Z
99.1% E

97% # Coefficients ($|\vec{\omega}| = 3/2\,\sigma$, 2σ, $5/2\,\sigma$)
36% Z
0.9% E

CVS: Wavelet filtering

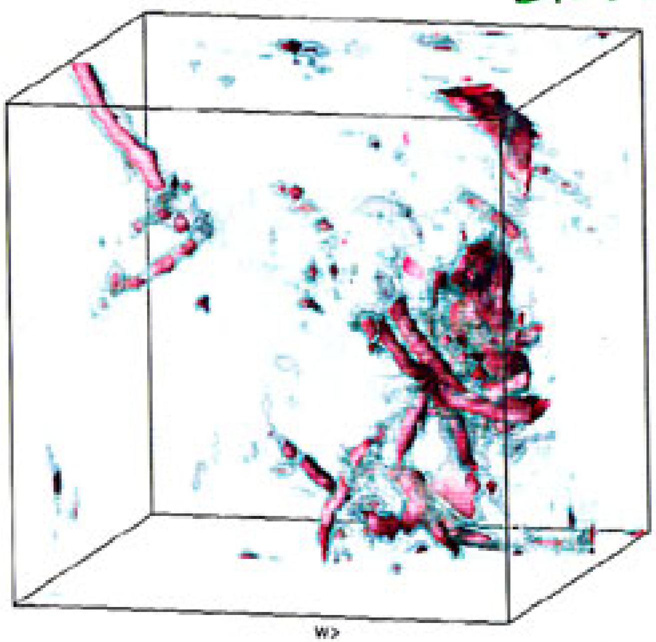
$N = 256^3$

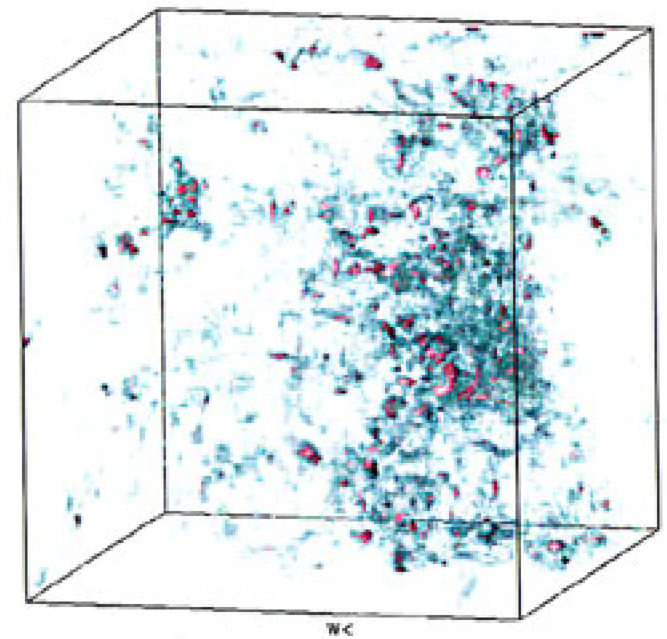
coherent part

$3\% \, N$

incoherent part

$97\% \, N$





3% # WLC
75% Z
98.9% E

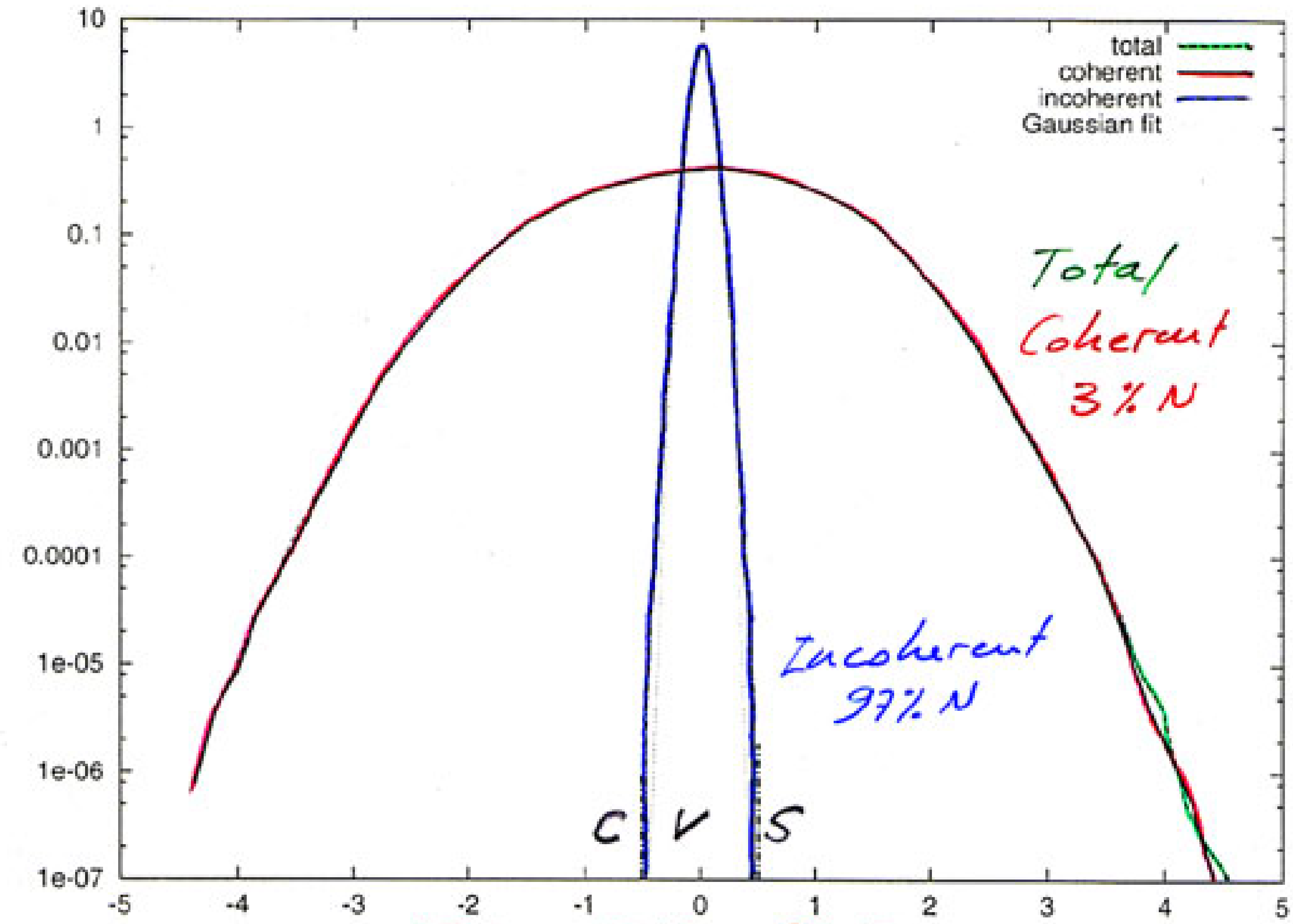
Isosurfaces of vorticity

$(3\sigma, 4\sigma, 5\sigma)$

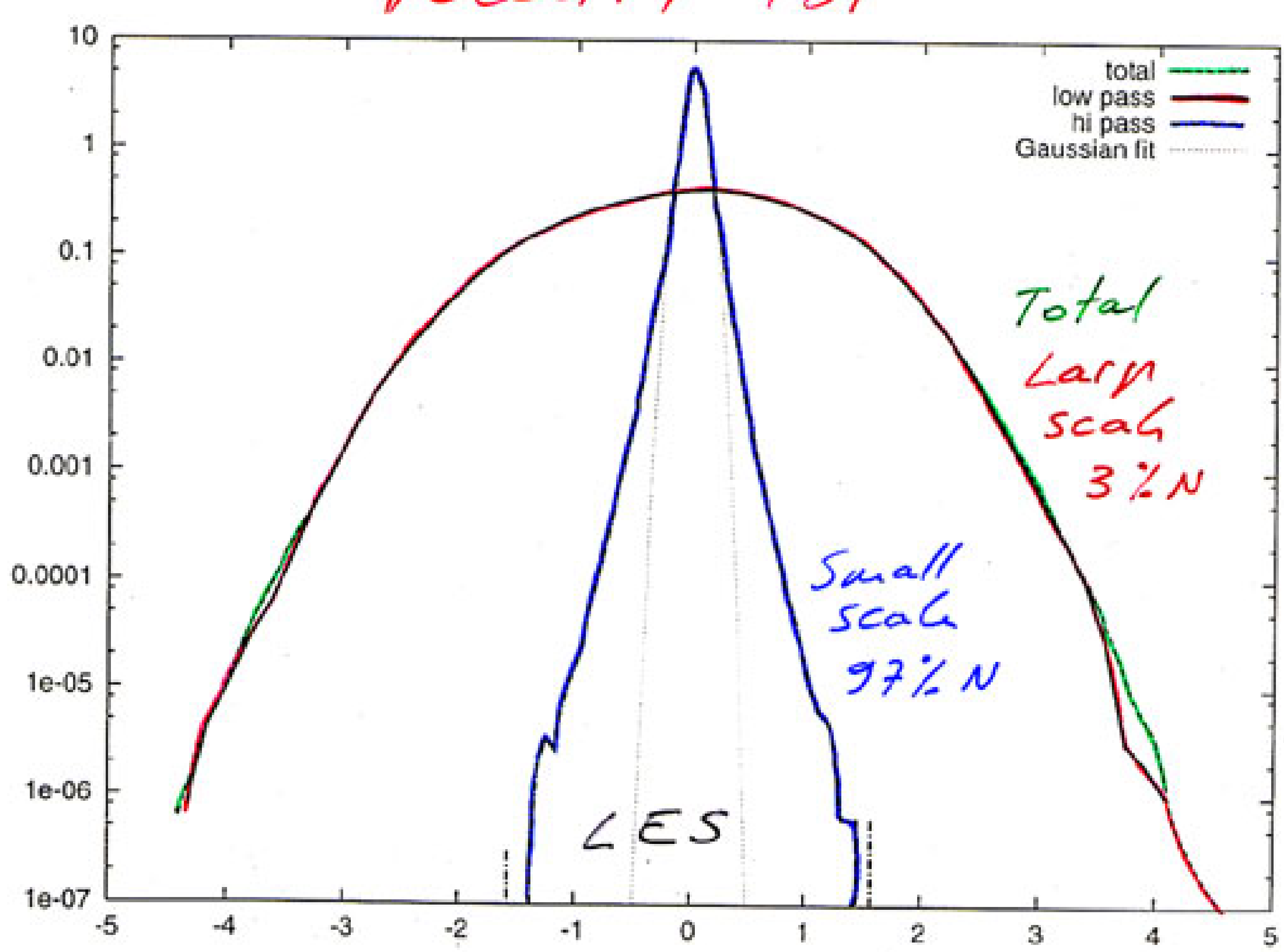
$\sigma = \sqrt{2Z}$

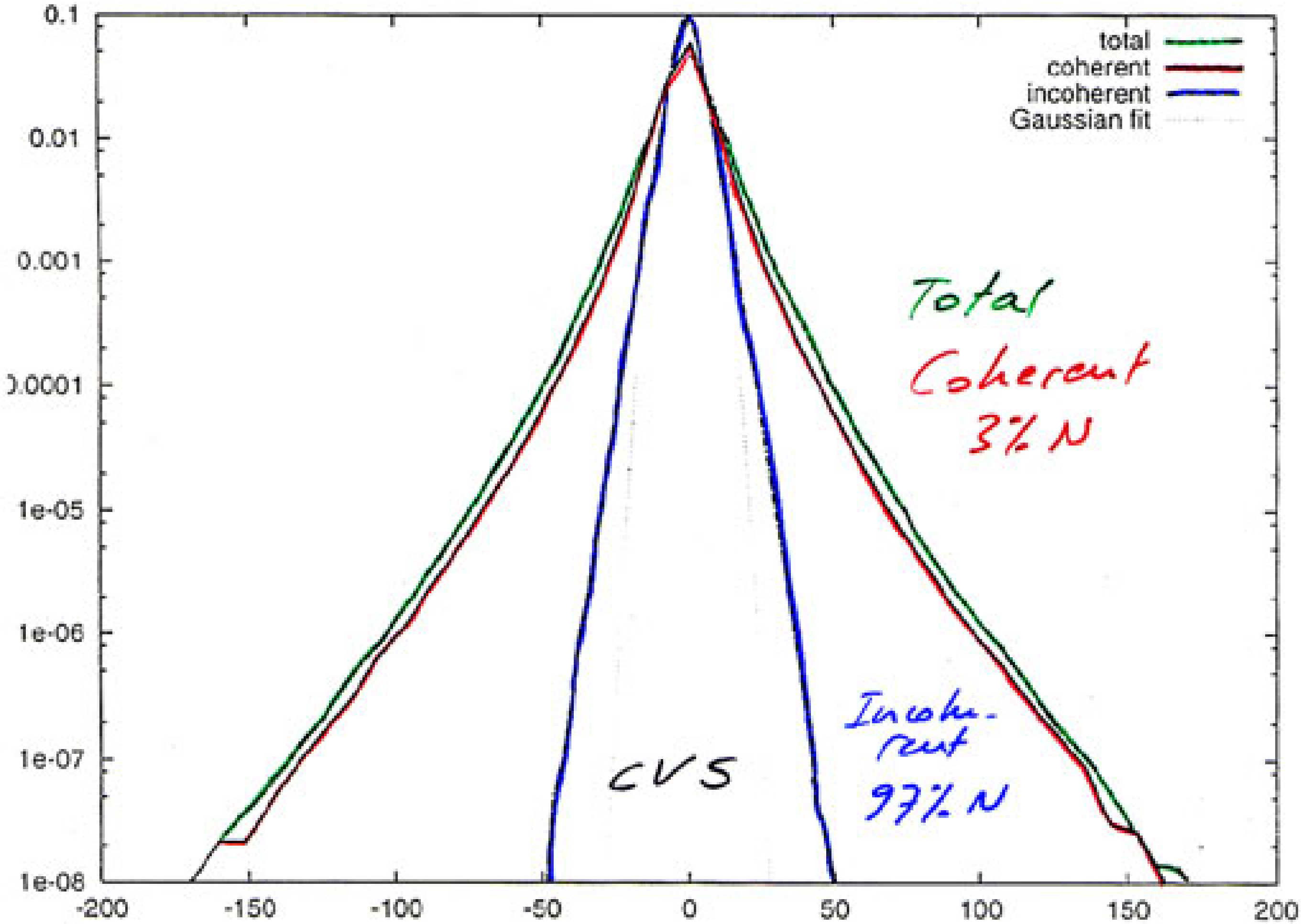
97% # WLC
25% Z
0.6% E

$(|\vec{\omega}| = 3/2\,\sigma, 2\sigma, 5/2\,\sigma)$

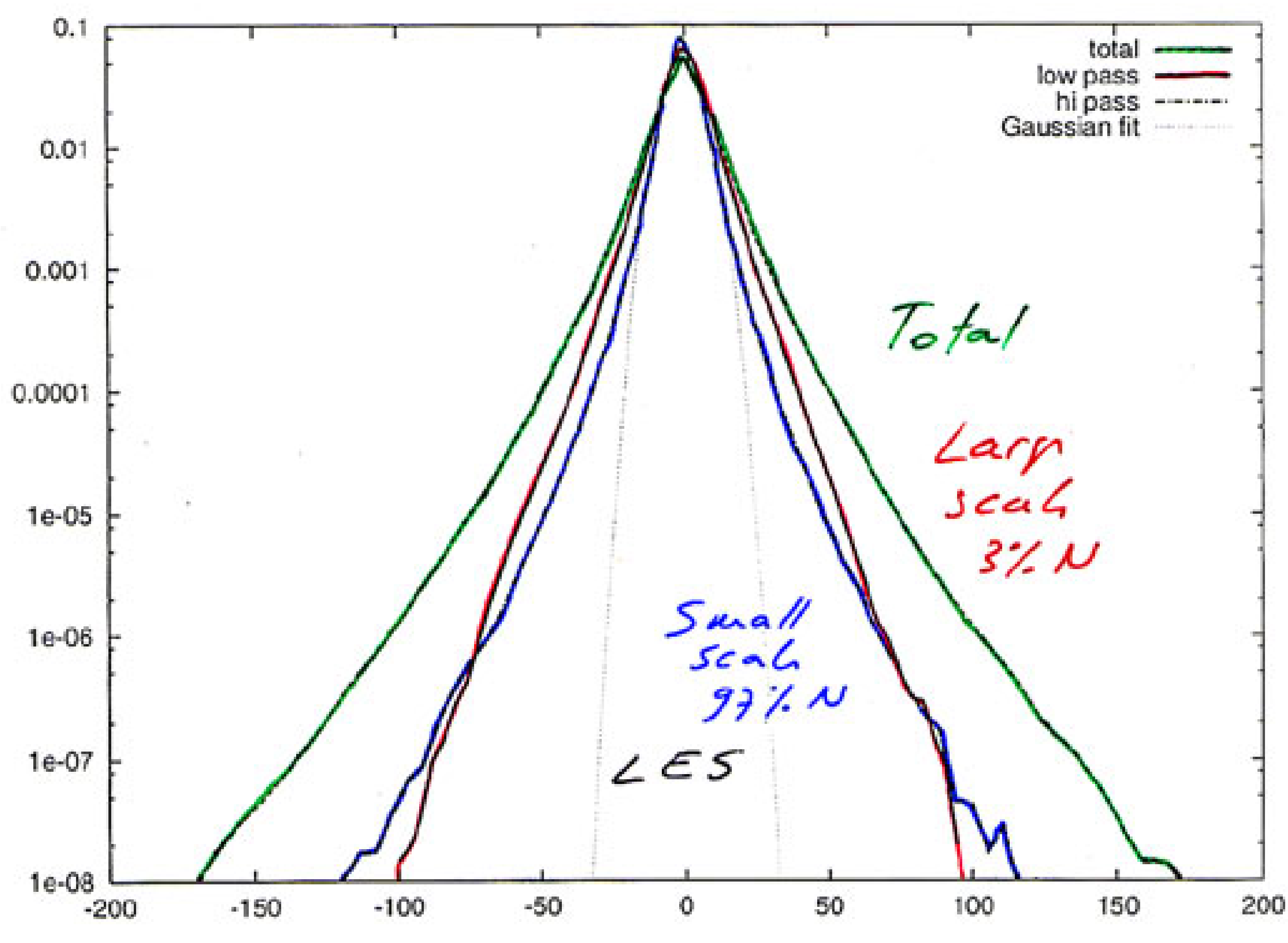


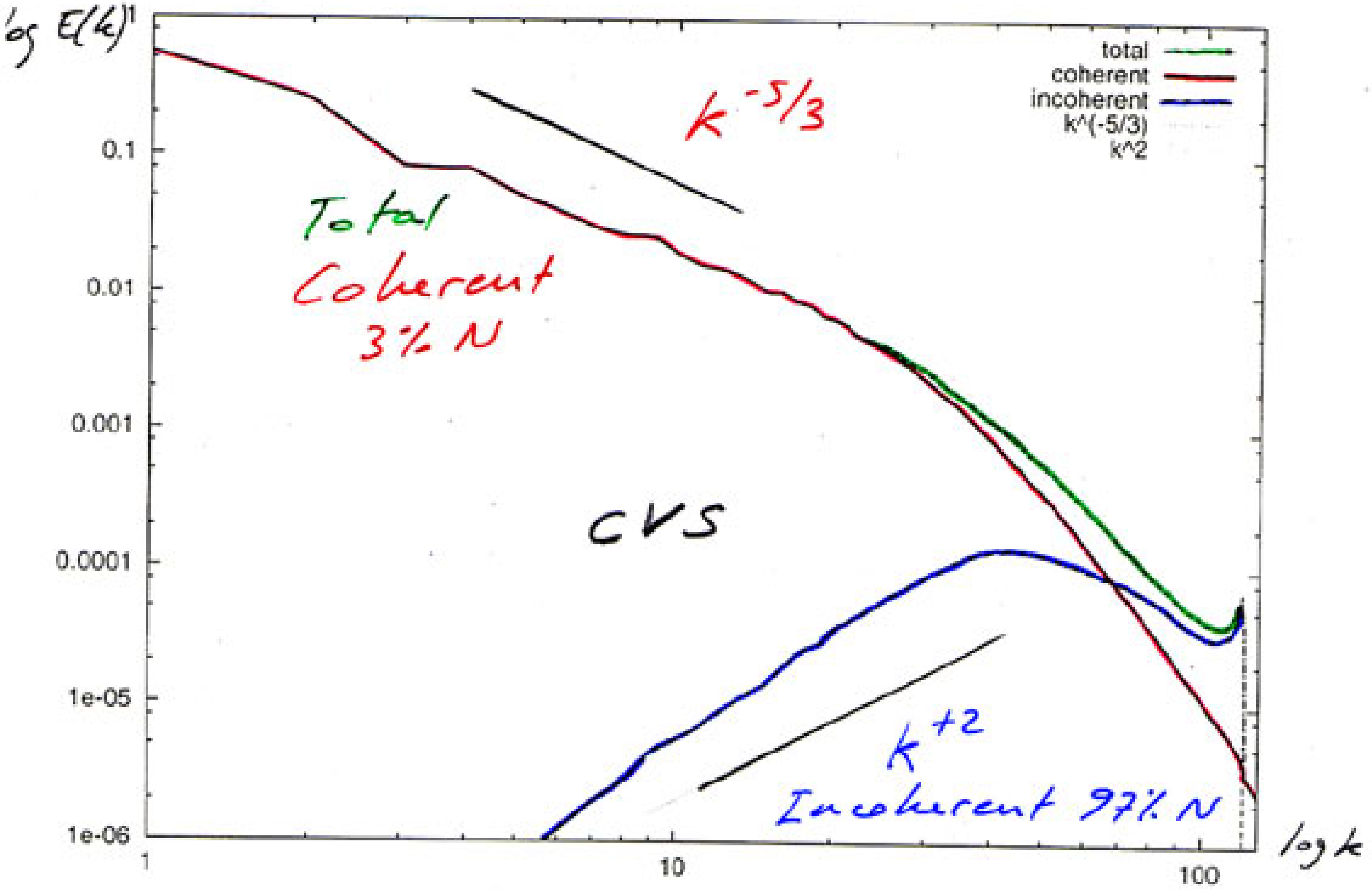
VELOCITY PDF



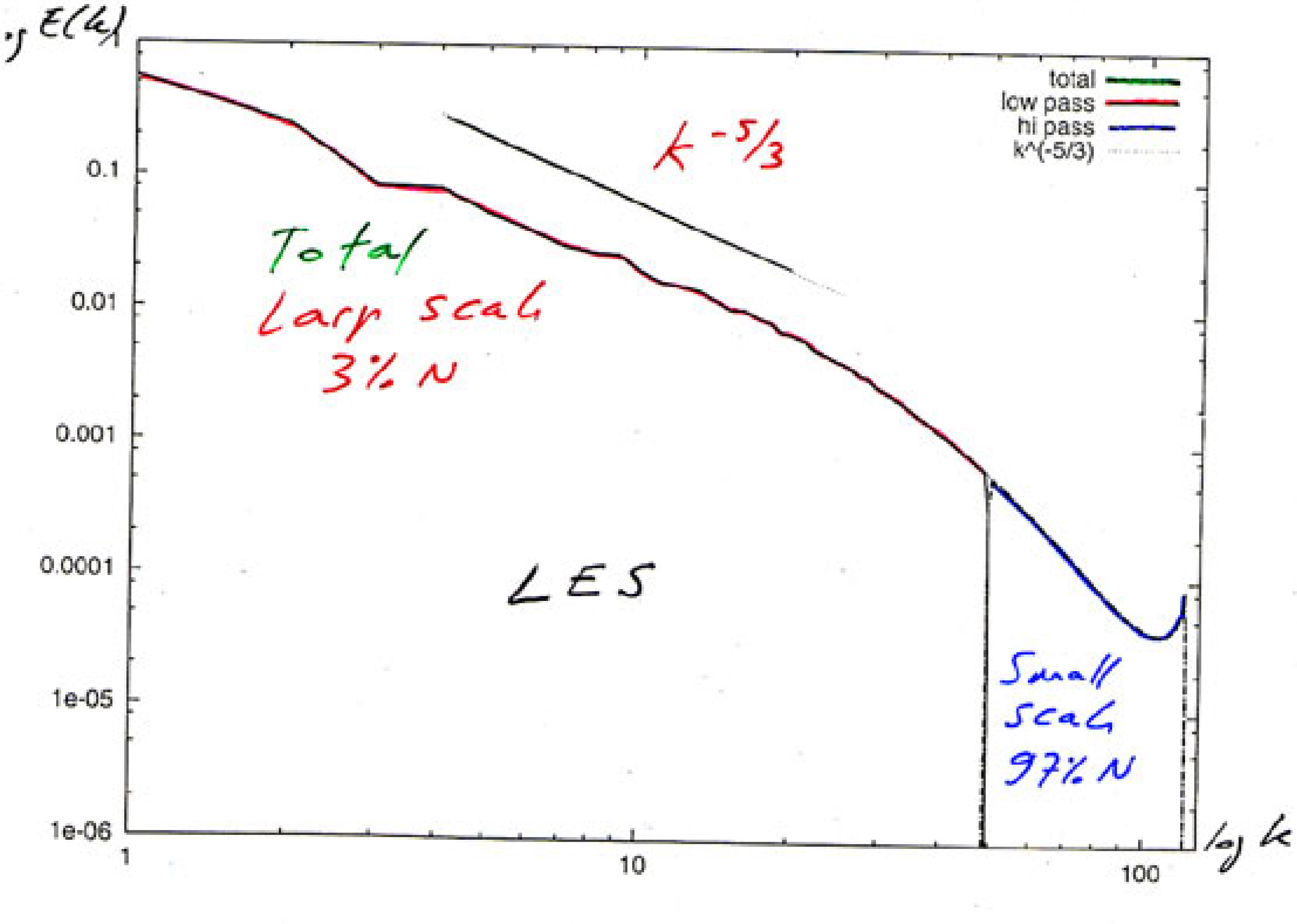


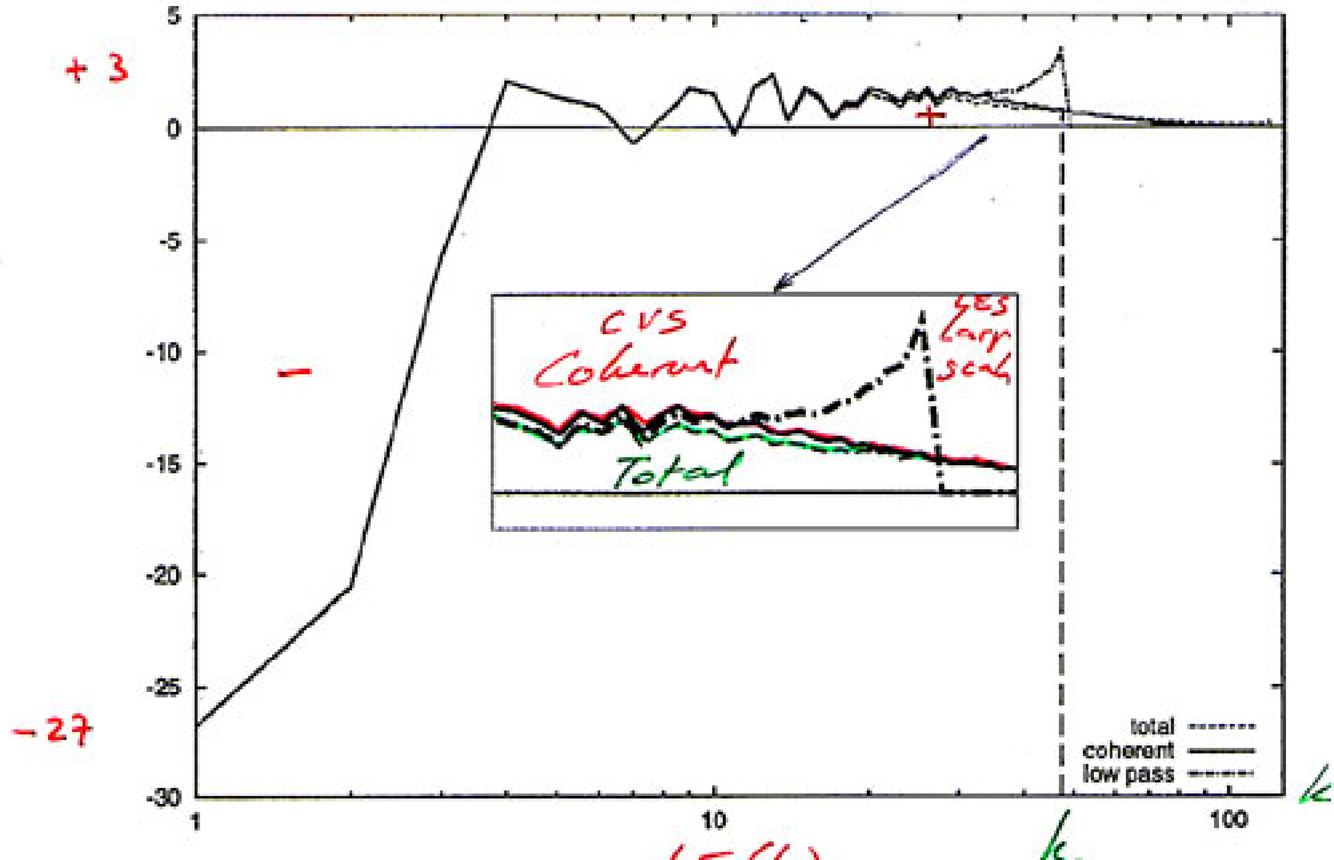
VORTICITY PDF



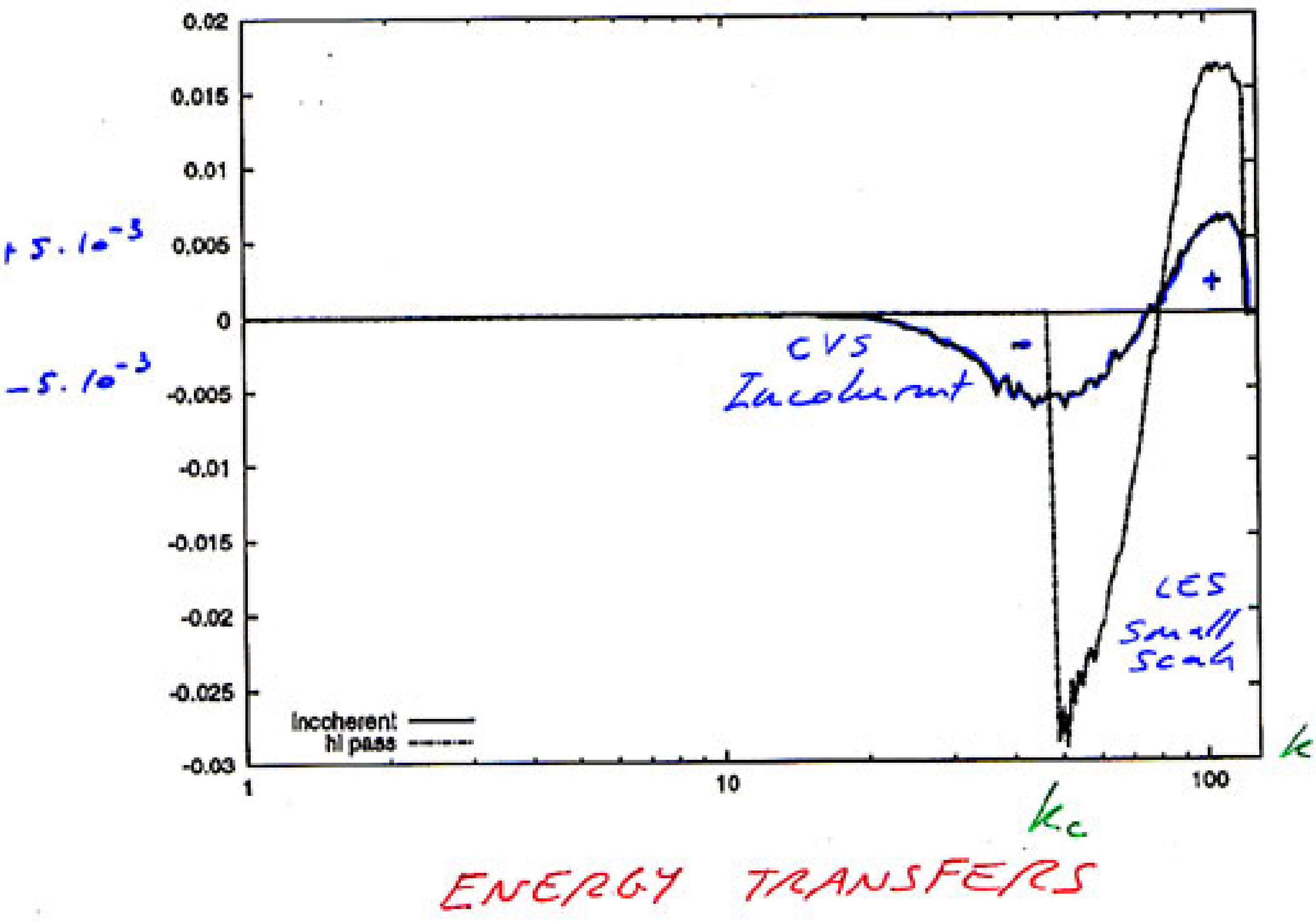


ENERGY SPECTRUM





$\frac{dE(k)}{dt}$

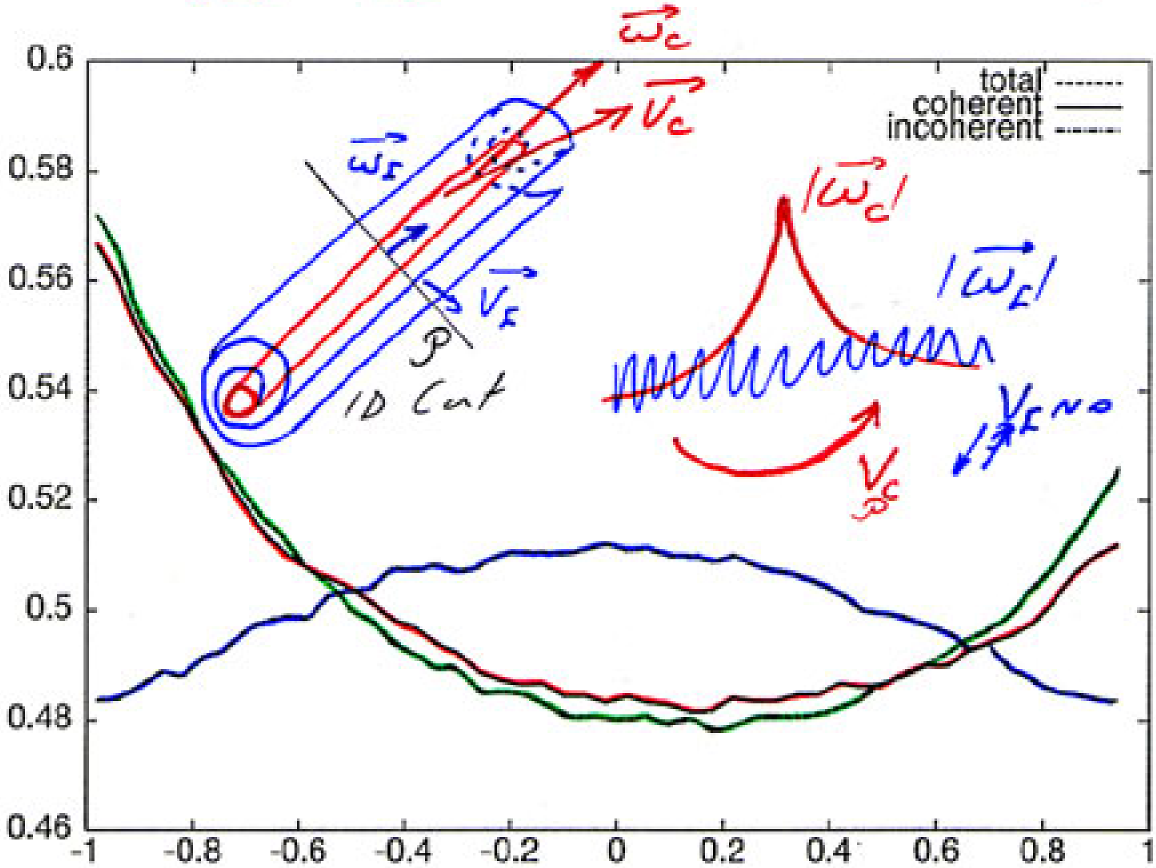


ENERGY TRANSFERS

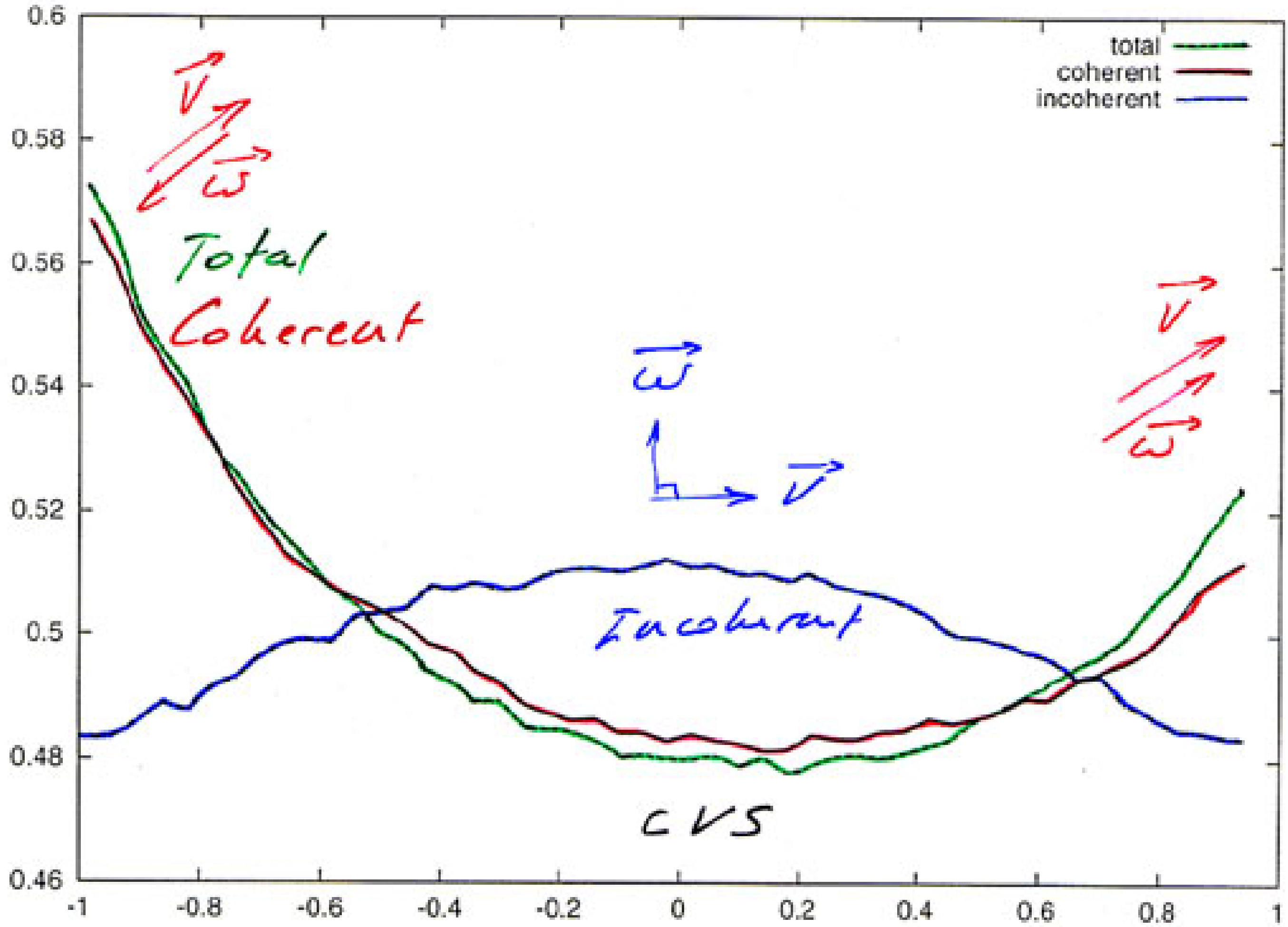
Keith MOFFATT

JFM, 150, 1985

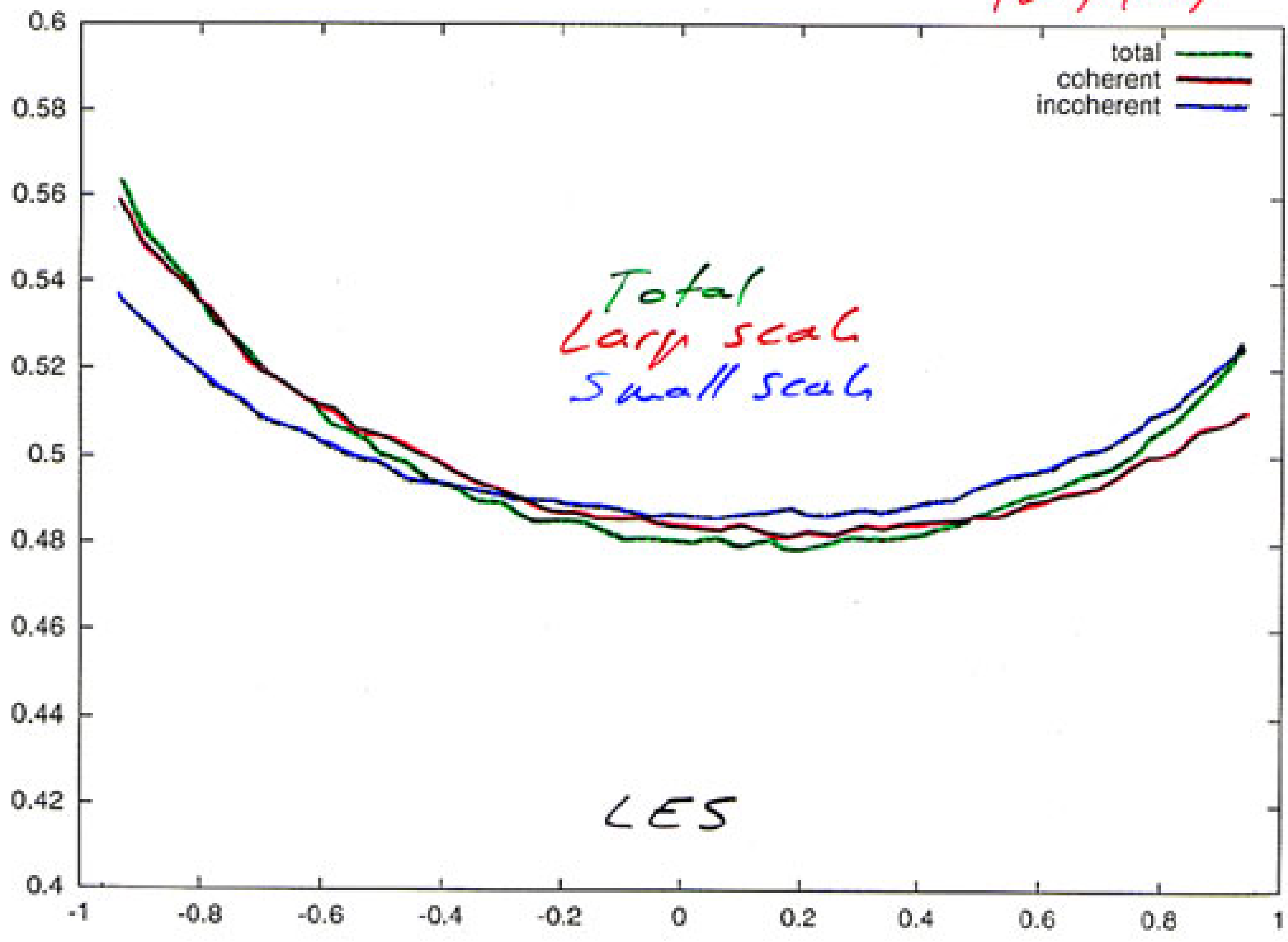
'Euler flows contain blobs of maximal helicity which may be interpreted as 'coherent structures,'

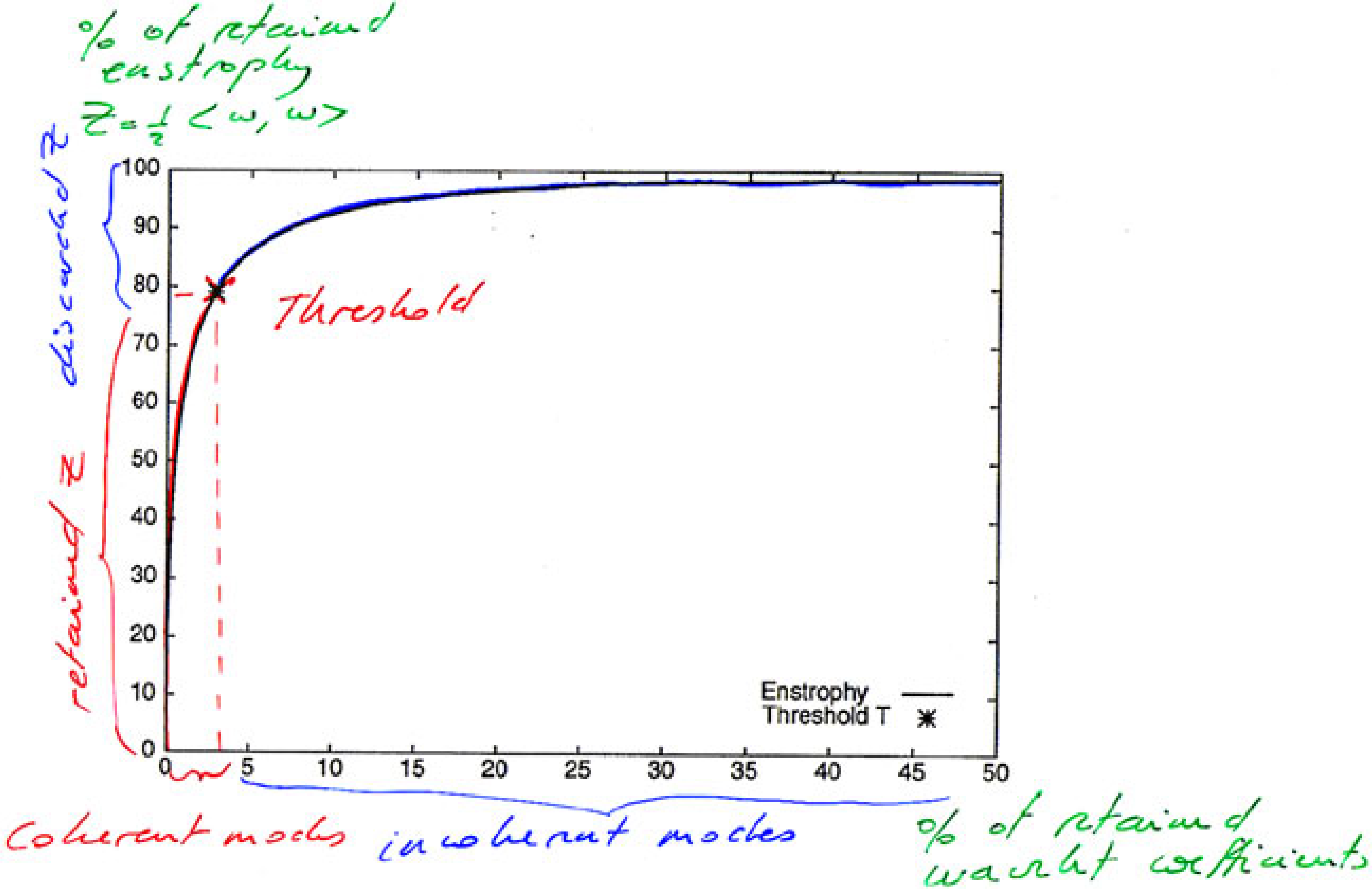


separated by regular surfaces on which vortex sheets, the site of strong dissipation, may be located.'



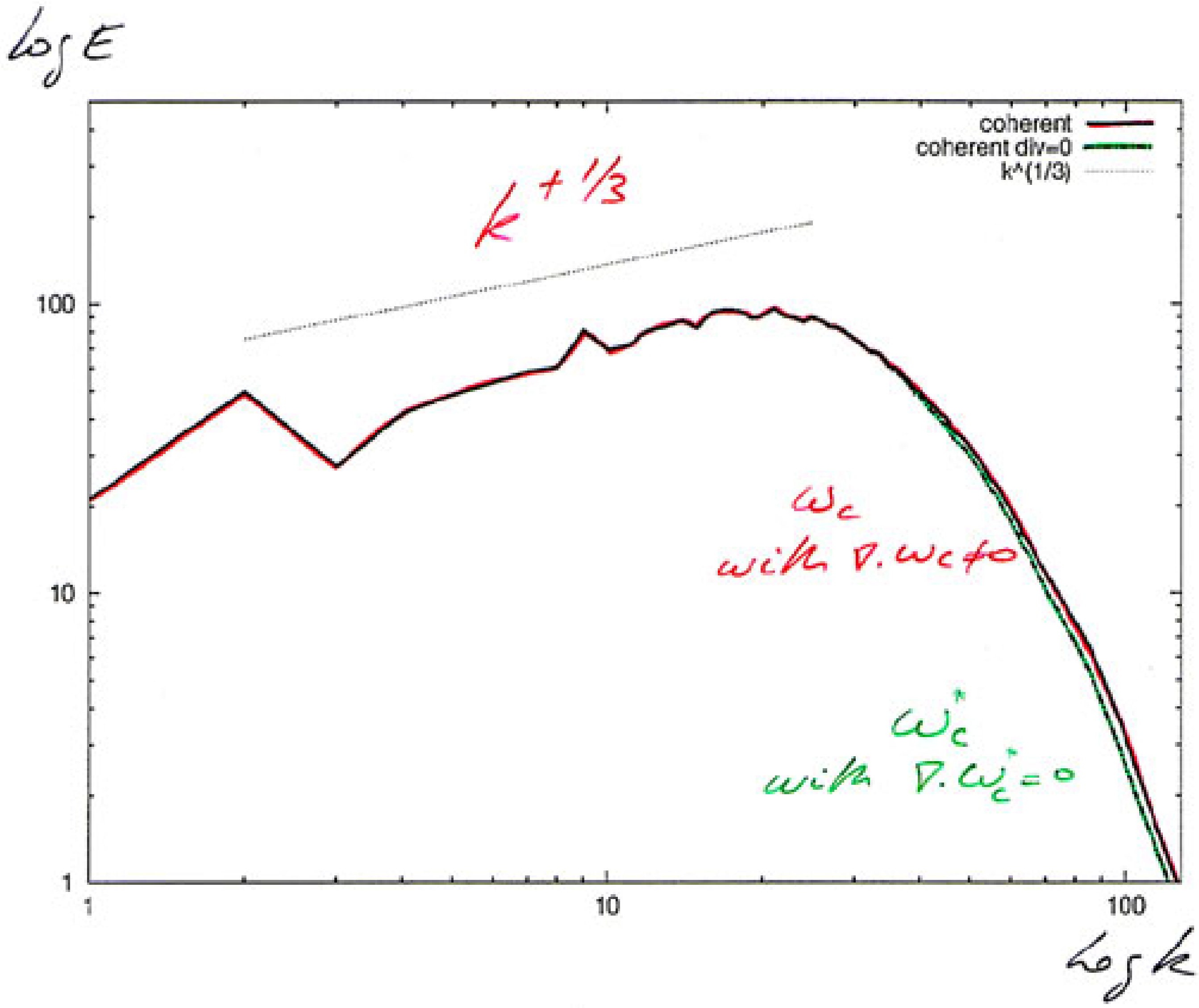
RELATIVE HELICITY $h = \dfrac{\vec{v} \cdot \vec{\omega}}{|\vec{v}| \cdot |\vec{\omega}|}$



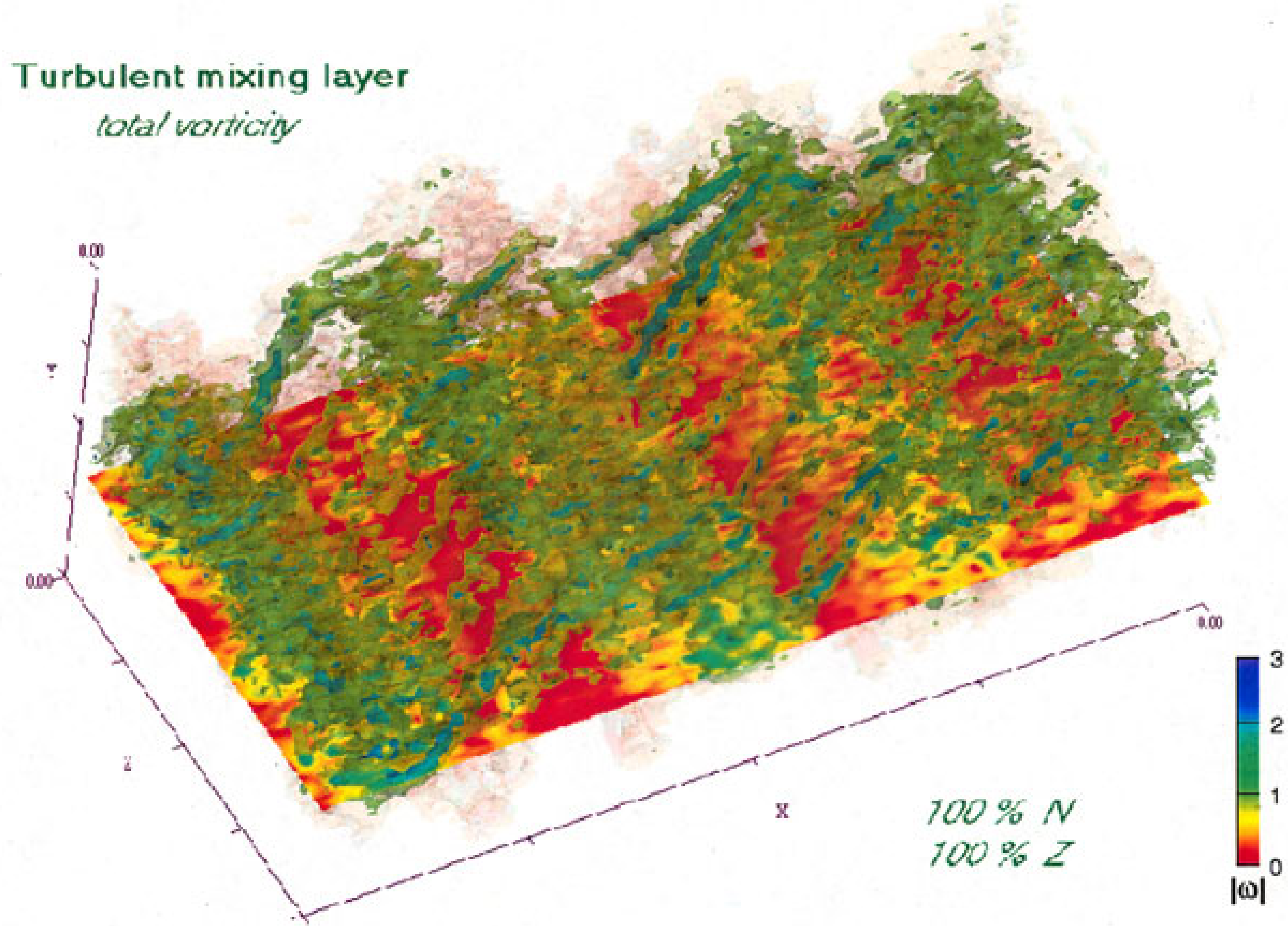


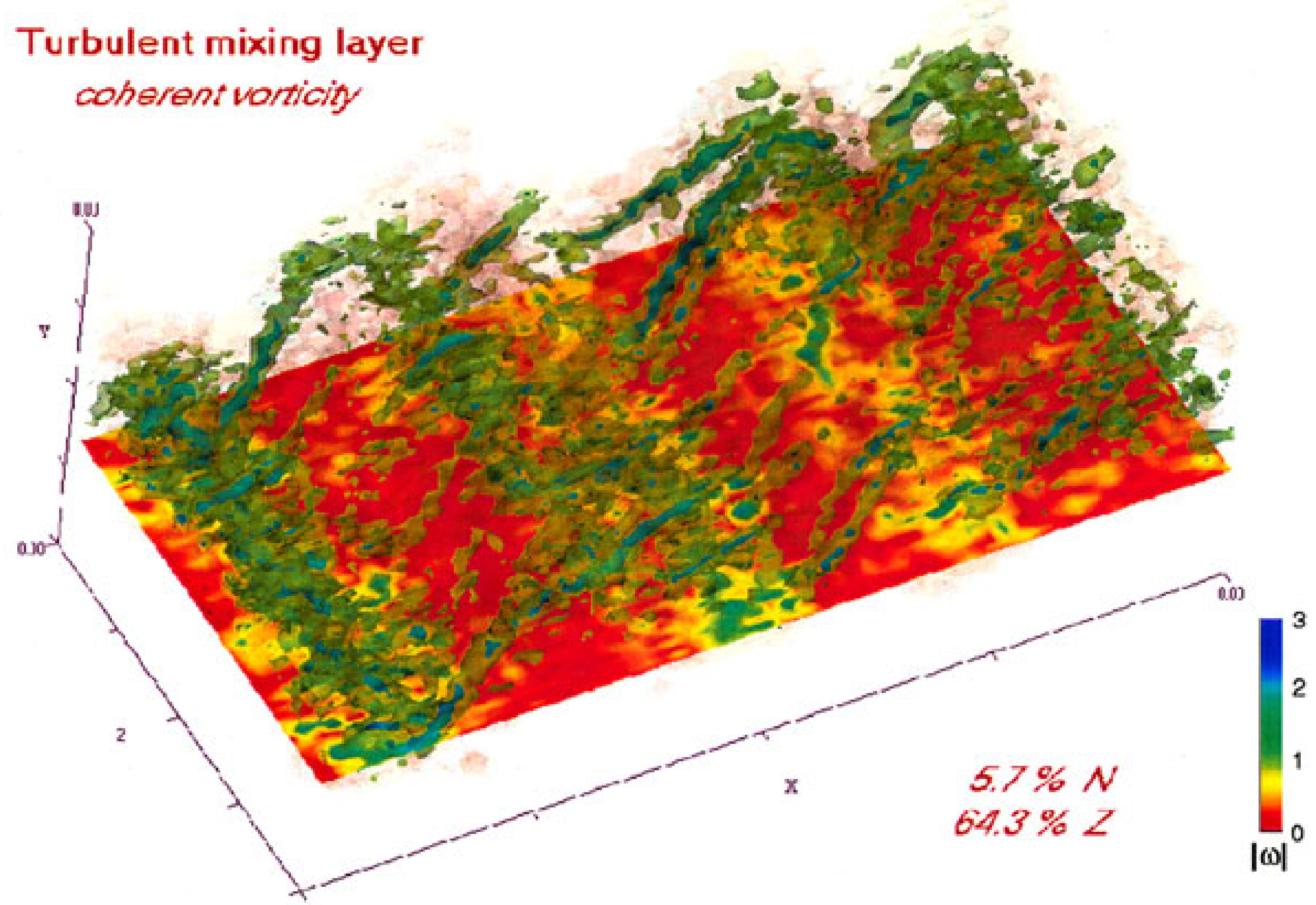
COMPRESSION

ENSTROPHY SPECTRUM

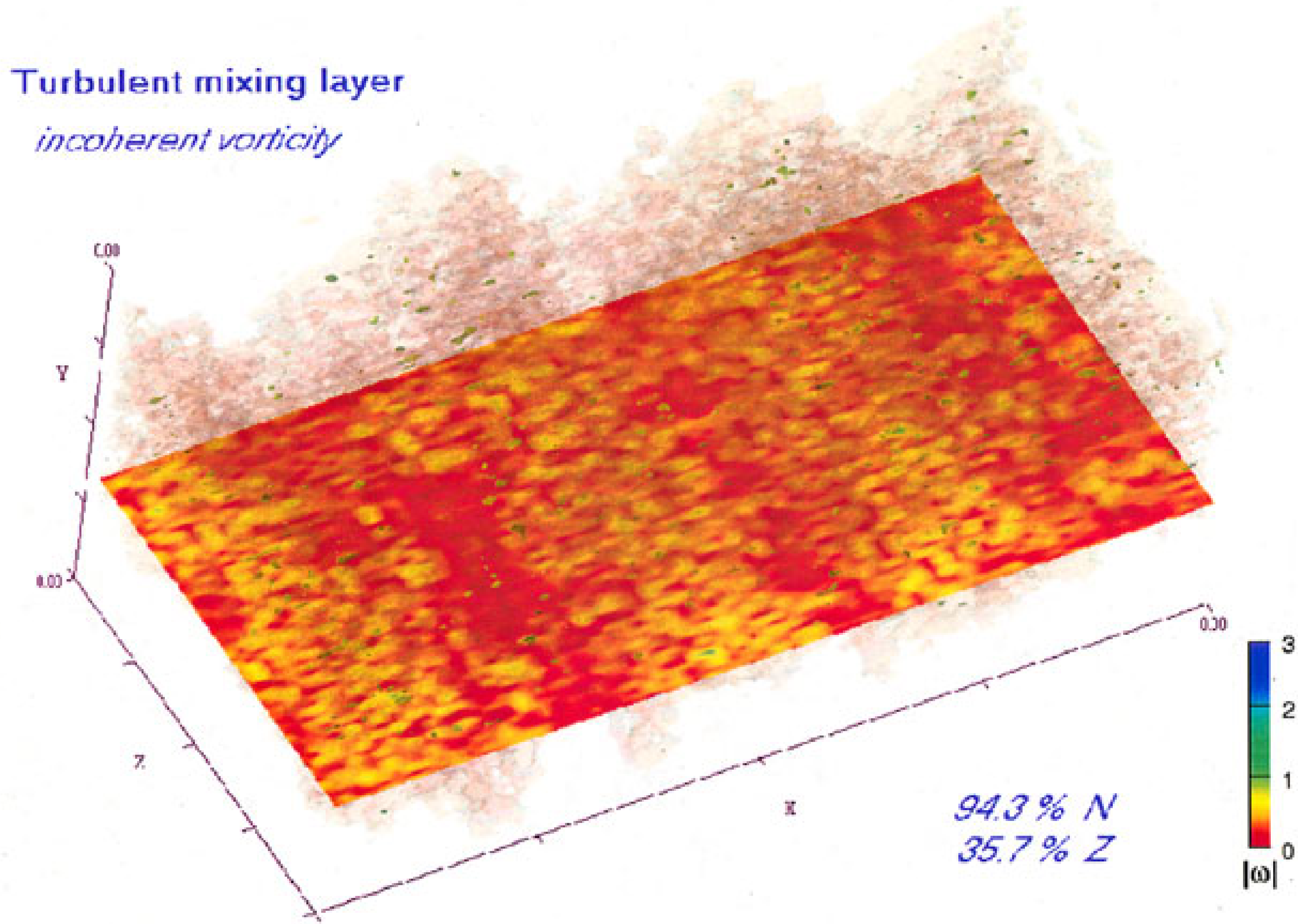


The contribution of the divergent part of the coherent vorticity field is negligible and affects only the dissipation scales


Turbulent mixing layer
total vorticity
0.00
0.00
0.00
y
z
x
100 % N
100 % Z
3
2
1
0
|ω|



Turbulent mixing layer
coherent vorticity
Y
2
X
5.7 % N
64.3 % Z
3
2
1
0
|ω|

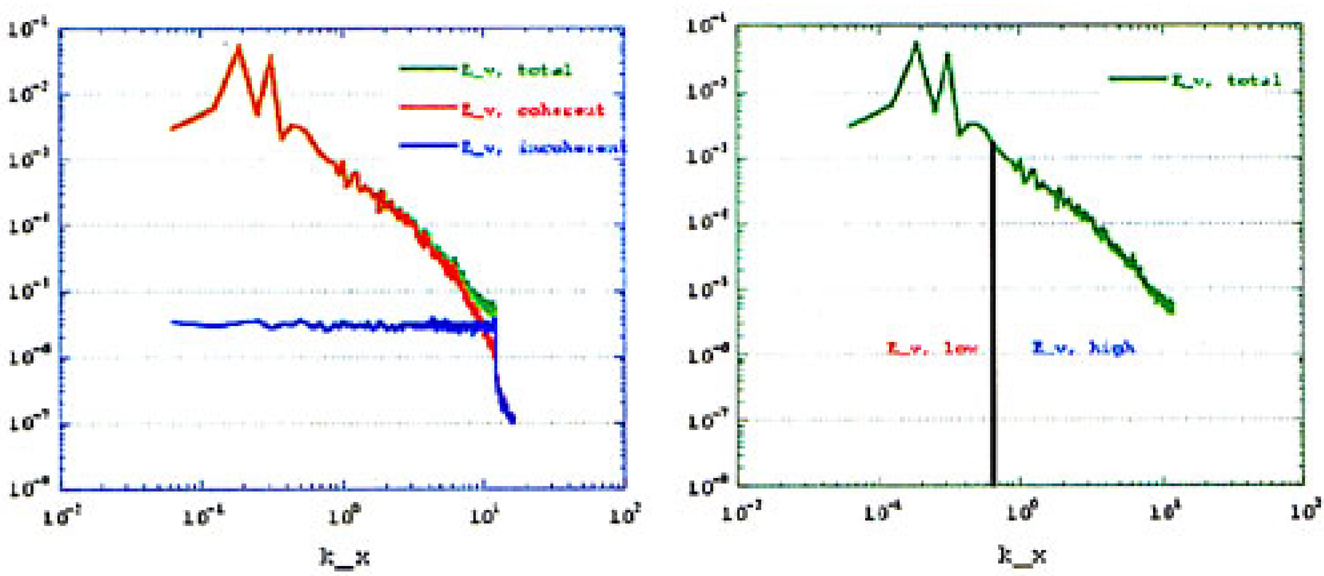

Turbulent mixing layer
incoherent vorticity
Y
Z
X
94.3 % N
35.7 % Z
3
2
1
0
|ω|

Mixing layer, forced

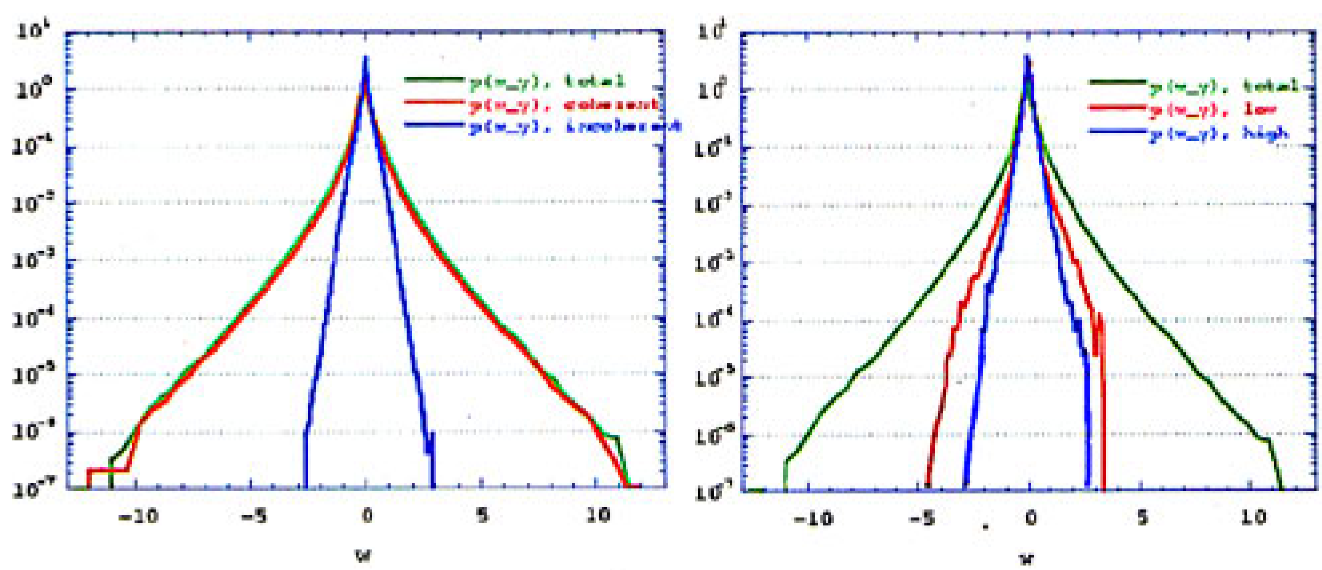
N = 512 x 256 x 128

CVS		LES
coherent	*3 % N*	*low wavenumber*
incoherent	*97 % N*	*high wavenumber*

1D energy spectra E(k_x)



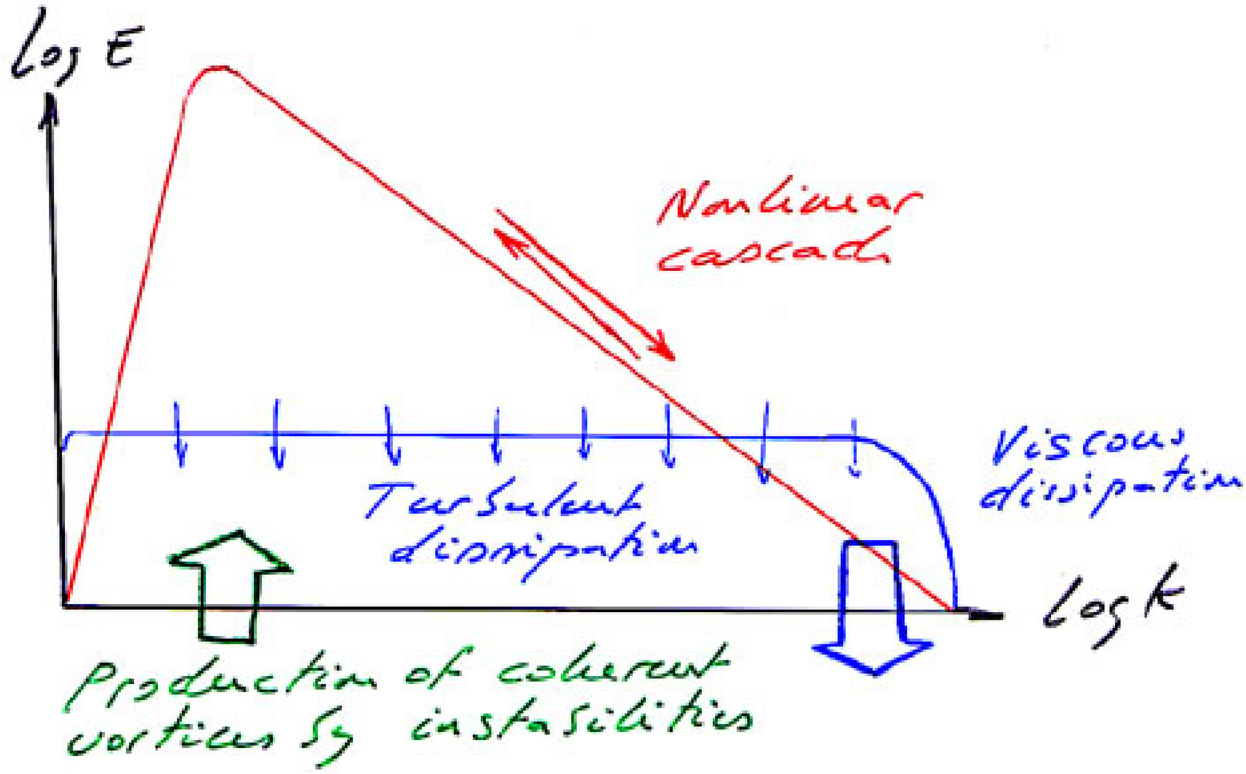
Vorticity PDF p(ω)

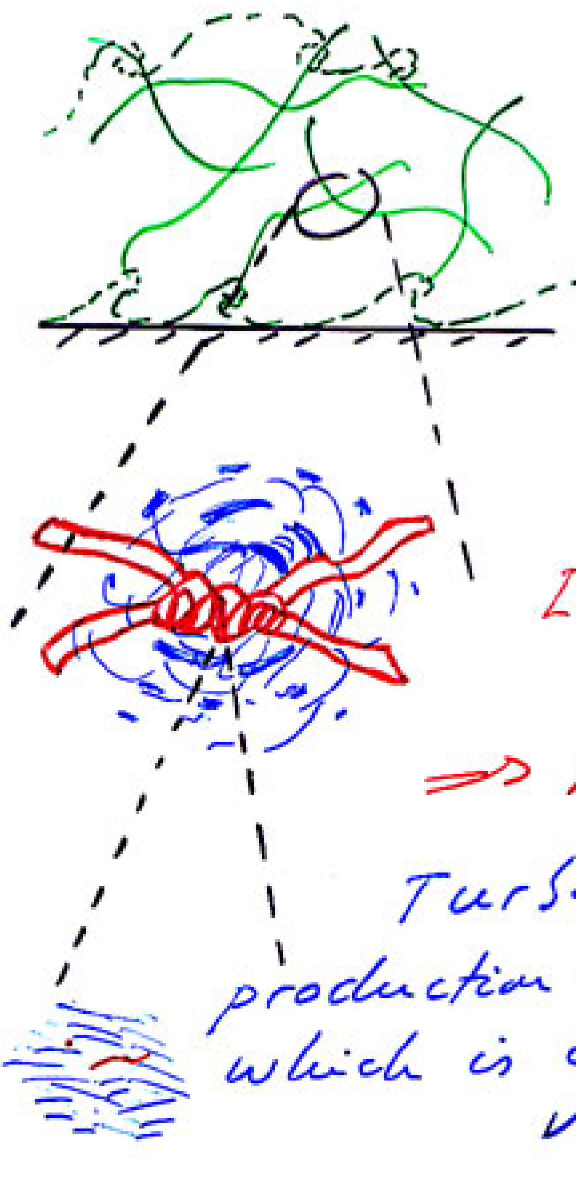


PROPOSED SCENARIO FOR THE TURBULENT CASCADE

The turbulent cascade comes from the nonlinear interactions between vortices out of equilibrium, which transfer coherent energy throughout the whole inertial range, and at the same time produce incoherent energy in quasi equilibrium which then dissipates.

INTERPRETATION





PRODUCTION

Webs of vortex tubes produced by instabilities in shear layers

INERTIAL RANGE

Intermittent vortex interactions

⇒ Nonlinear cascade

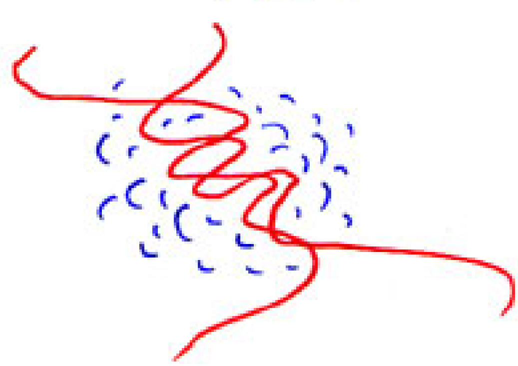
+

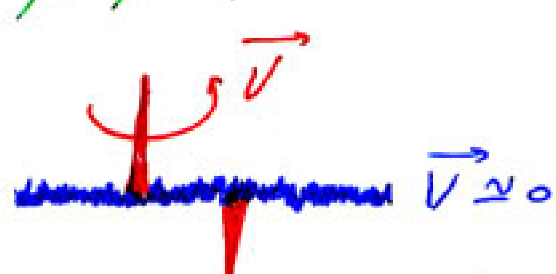
Turbulent dissipation

production of background noise which is damped in the viscous range

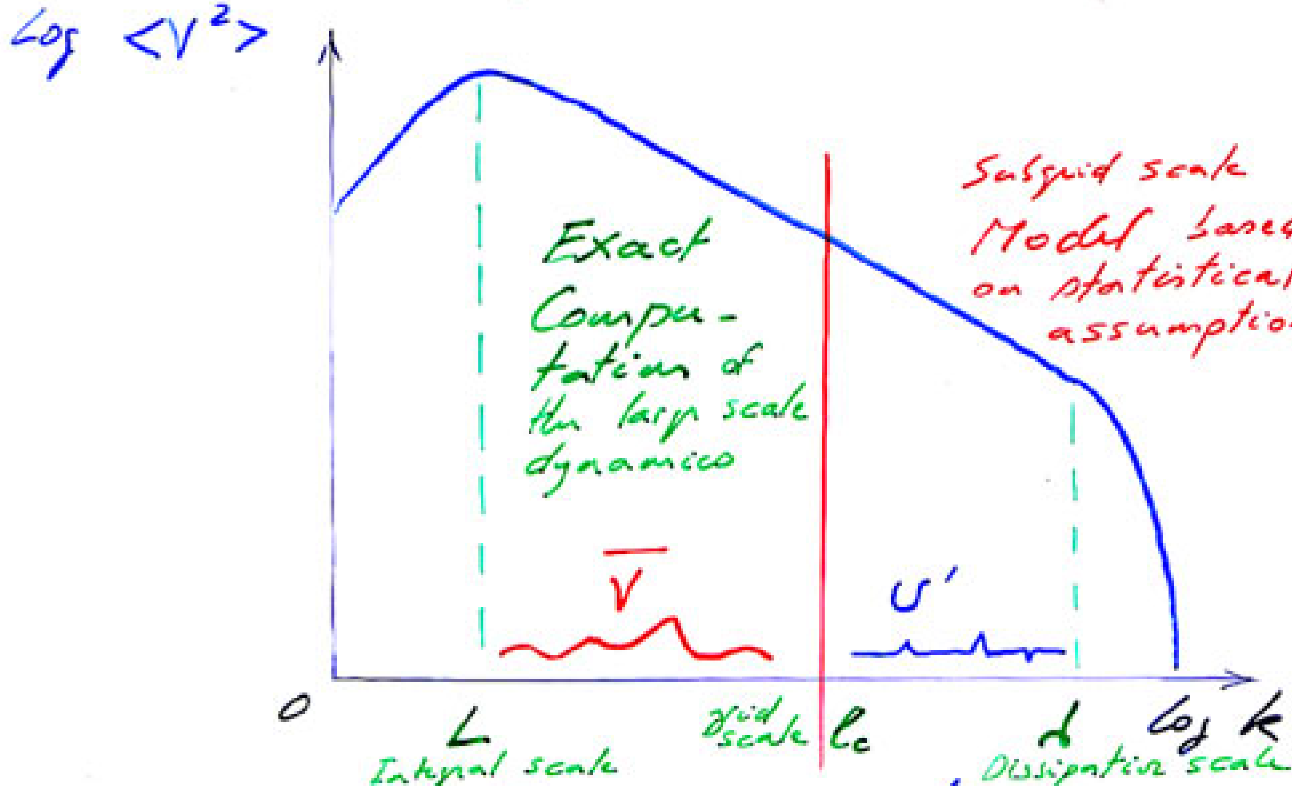
Hans LIEPMANN
Marseille,
1961

'These patterns, which show a nearly-random small-scale structure within an organized large-scale structure, result from nothing more than a relatively small number of line vortices attempting to rotate about each other. One gets here a fascinating glimpse into rapid randomization by vortex interactions and a demonstration of the statistical character of turbulence proper!'





TURBULENCE MODELLING WITH L.E.S.



Energy spectrum

$$V = \overline{V} + v', \quad P = \overline{P} + p'$$

$\Rightarrow$ equation for the Large Eddy motions:

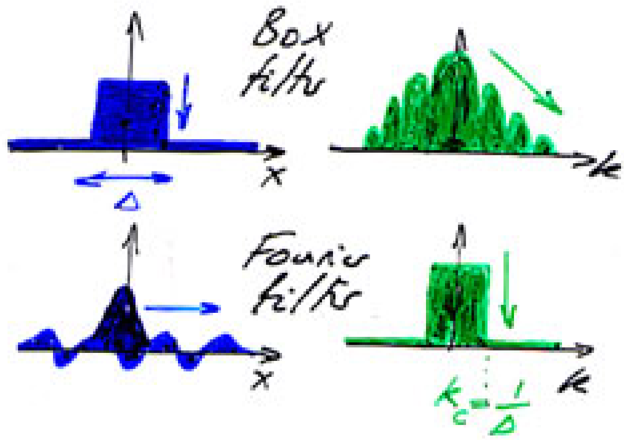
$$\frac{\partial \overline{V_i}}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{V_i V_j}\right) + \frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i} = \nu \frac{\partial}{\partial x_j}\left(\frac{\partial \overline{V_i}}{\partial x_j} + \frac{\partial \overline{V_j}}{\partial x_i}\right) - \frac{\partial \tau_{ij}}{\partial x_j}$$

Subgrid scale tensor τ_{ij}

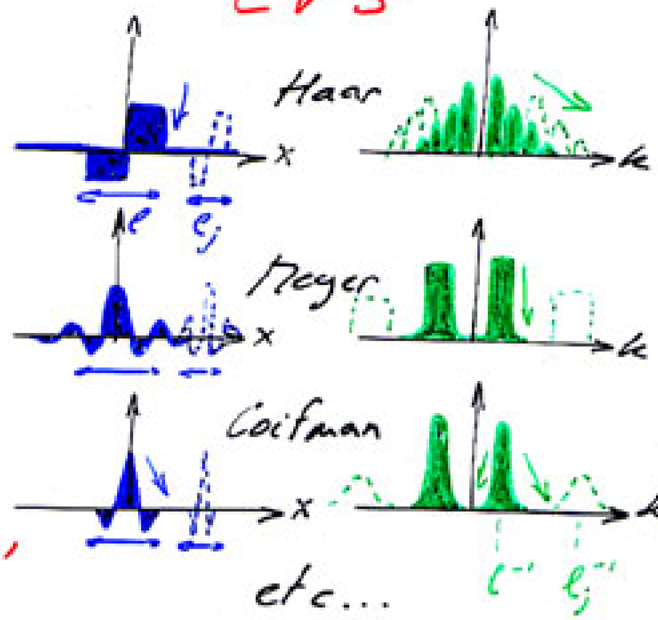
$$\tau_{ij} = \underbrace{\left(\overline{\overline{V_i}\,\overline{V_j}} - \overline{V_i}\,\overline{V_j}\right)}_{L_{ij}} + \underbrace{\left(\overline{\overline{V_i}\, v_j'} + \overline{\overline{V_j}\, v_i'}\right)}_{C_{ij}} + \underbrace{\overline{v_i' v_j'}}_{R_{ij}}$$

COMPARISON BETWEEN DIFFERENT FILTERS

LES



WAVELET CVS

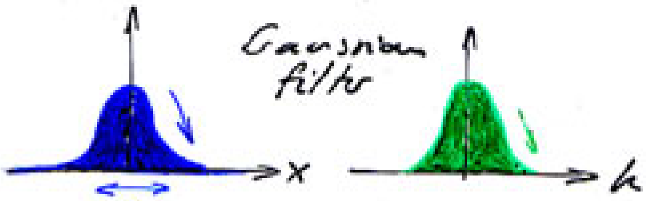


$$u = \bar{u} + u'$$

etc...

Linearity: $\begin{cases} \overline{u+v} = \bar{u} + \bar{v} \\ \overline{au} = a\bar{u} \\ \overline{Du} = D\bar{u} \end{cases}$

Orthogonality: $\begin{cases} \bar{\bar{u}} = \bar{u} \\ \bar{u}' = 0 \end{cases}$



It is not a projector

$\Rightarrow \begin{cases} \bar{\bar{u}} \neq \bar{u} \\ \bar{u}' \neq 0 \end{cases}$

Nonlinear term (product):

if Fourier filter

$$\overline{uv} \neq \bar{u} \cdot \bar{v}$$

if Wavelet filter

$$\overline{uv} \sim \bar{u} \cdot \bar{v}$$

RESOLUTION OF 2D NAVIER-STOKES IN WAVELETS

We want to solve:

$$\begin{cases} \dfrac{\partial \omega}{\partial t} + J(\psi, \omega) = \nu \nabla^2 \omega \\ \omega = \nabla^2 \psi \end{cases}$$

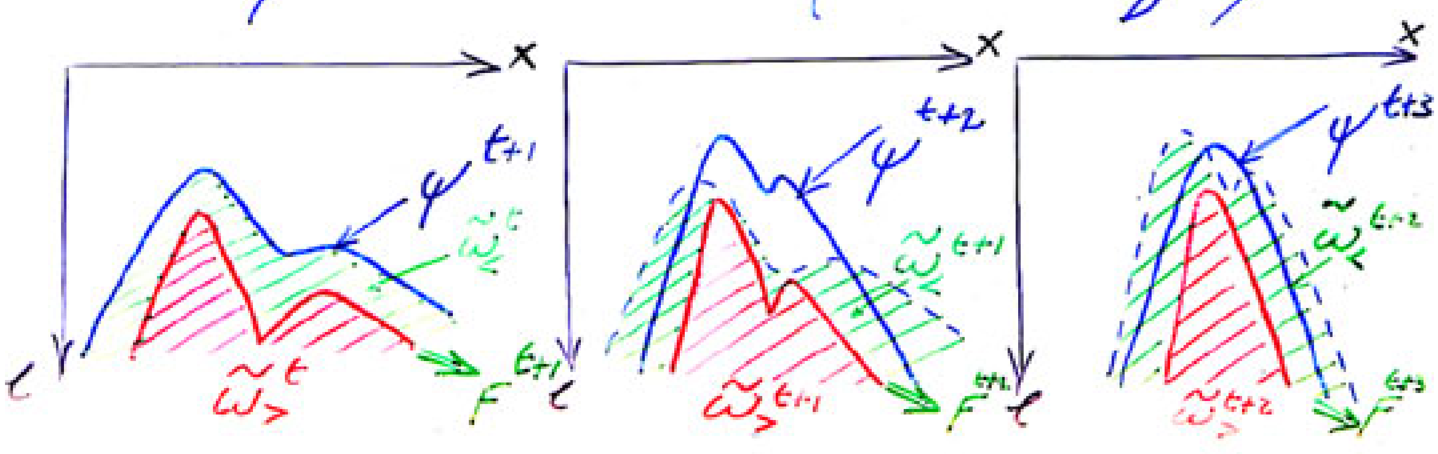
We choose an appropriate wavelet or wavelet packet basis $|\psi^t\rangle$ adapted for time t and we project onto it:

$$\left\langle \left[\frac{\partial}{\partial t} \tilde{\omega}_> + J(\nabla^{-2} \tilde{\omega}_>, \tilde{\omega}_>) \right] \middle| \psi^{t+1} \right\rangle = F^{t+1}$$

$\tilde{\omega}_>$ are the strongest wavelet coef.

F random phase forcing term such that $|F| = \langle \omega_<^2 \rangle (1-D)$ and $\hat{F} = |F| e^{i\theta}$, θ random

$0 \leq D \leq 1$ turbulent diffusion parameter $\left(R_e \propto \frac{1}{D} \right)$.



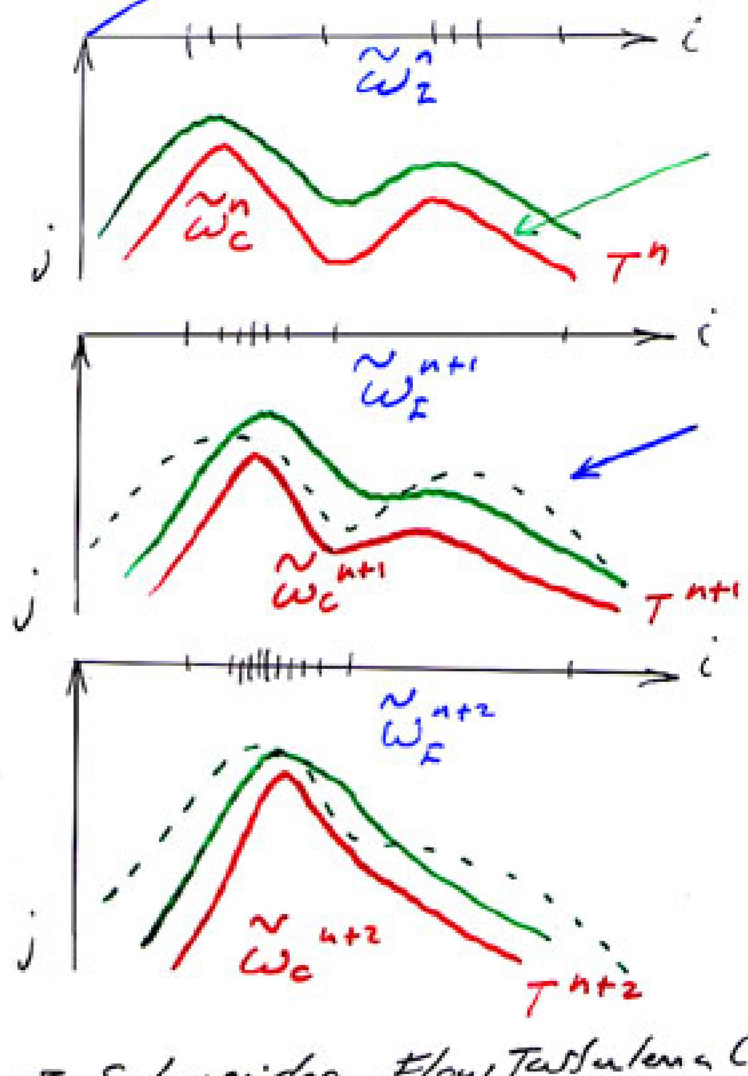
CVS FOR NAVIER-STOKES

$$\omega = \omega_c + \omega_I$$
$$\vec{V} = \vec{V_c} + \vec{V_I}$$

$$\|\omega_c\| \gg \|\omega_I\| \qquad \|\vec{V_c}\| \gg \|\vec{V_I}\|$$

$$\|\vec{\nabla}\omega_c\| < \|\vec{\nabla}\omega_I\|$$

$$\frac{\partial \omega_c}{\partial t} + \cancel{\frac{\partial \omega_I}{\partial t}} + \vec{V_c}\cdot\vec{\nabla}\omega_c + \vec{V_c}\cdot\vec{\nabla}\omega_I + \vec{V_I}\cdot\vec{\nabla}\omega_c + \cancel{\vec{V_I}\cdot\vec{\nabla}\omega_I} = \nu\nabla^2\omega_c + \cancel{\nu\nabla^2\omega_I}$$



Security zone

Incoherent Modes below security zone are discarded

$\Longrightarrow$ model turbulent dissipation

cf. Farge & Schneider, Flow Turbulence Combustion, 66, 2001

ADVANTAGES OF WAVELETS TO SOLVE PDEs

- Good compression of functions with inhomogeneous regularity,

- Many differential and integral operators applied to wavelets are almost again wavelets ⇒ vaguelettes diagonalize the stiffness matrix and thus avoid solving the linear system for each time step,

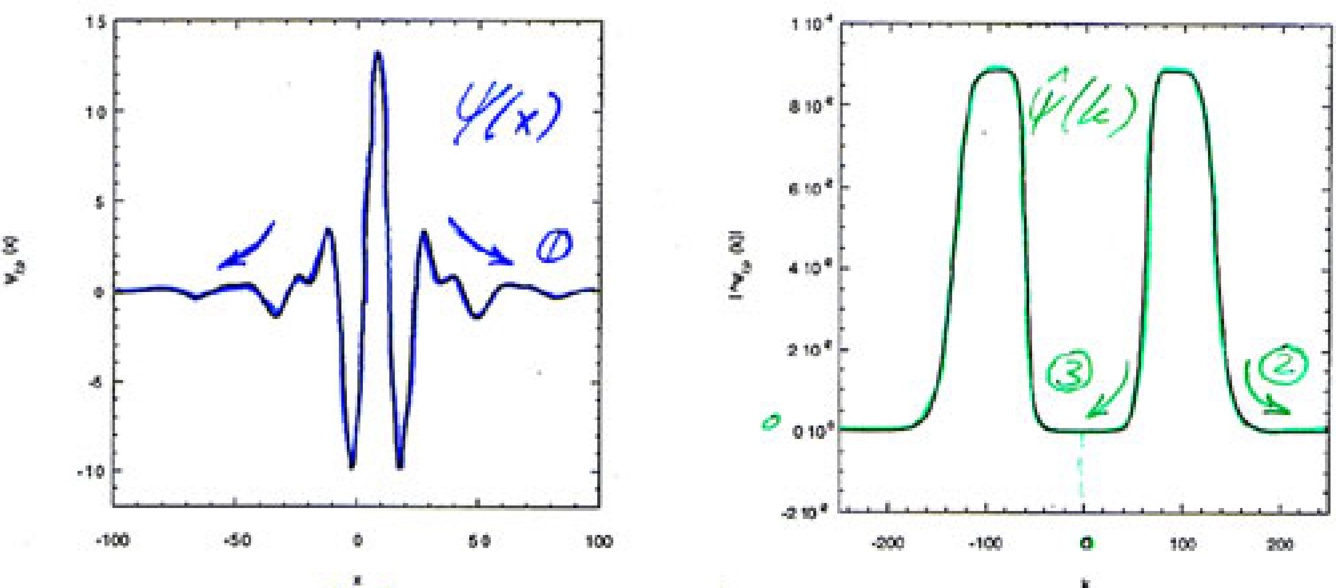
- Good localization of wavelets in both space and scale ⇒ simple adaption strategy for evolution problems by switching on neighbour wavelets,

- Hierarchical organization of wavelet bases ⇒ fast pyramidal algorithm in $O(N)$ operations.

ORTHOGONAL WAVELETS

QUINTIC SPLINE WAVELET at scale $\ell = 2^{-7}$

ASSOCIATED FILTER at scale $\ell = 2^{-7}$



Δx

Δk

$$\Delta X(\ell) \cdot \Delta k(\ell) = C$$

$$\frac{\Delta k}{k} = cst$$

Figure 1: Quintic spline wavelet $\psi_{j,i}(x)$ for scale $j = 7$ and position $i = 0$ in physical space (top) and in Fourier space (bottom).

Wavelet properties:

① - Space localization $\Longrightarrow$ spatial adaptivity,

② - Smoothness, i.e. spectral localization for $k \to \infty$ $\Longrightarrow$ differentiability,

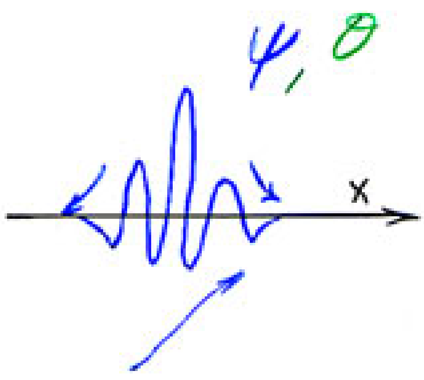
③ - Vanishing moments, i.e. spectral localization for $k \to 0$ $\Longrightarrow$ integrability.

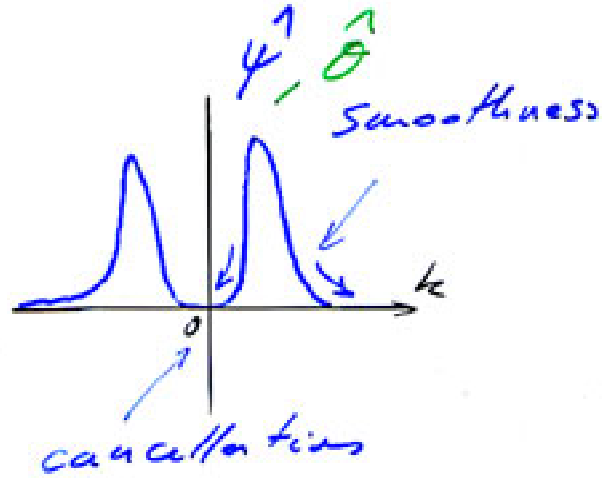
CHOICE OF THE BASIS FUNCTIONS

The test functions θ_{ji} are chosen to be the eigenmodes of the Green's function of heat kernel:

$$\left(\mathbb{1} - \Delta t\, \nu\, \nabla^2\right)^{-1}$$

with periodic b.c.



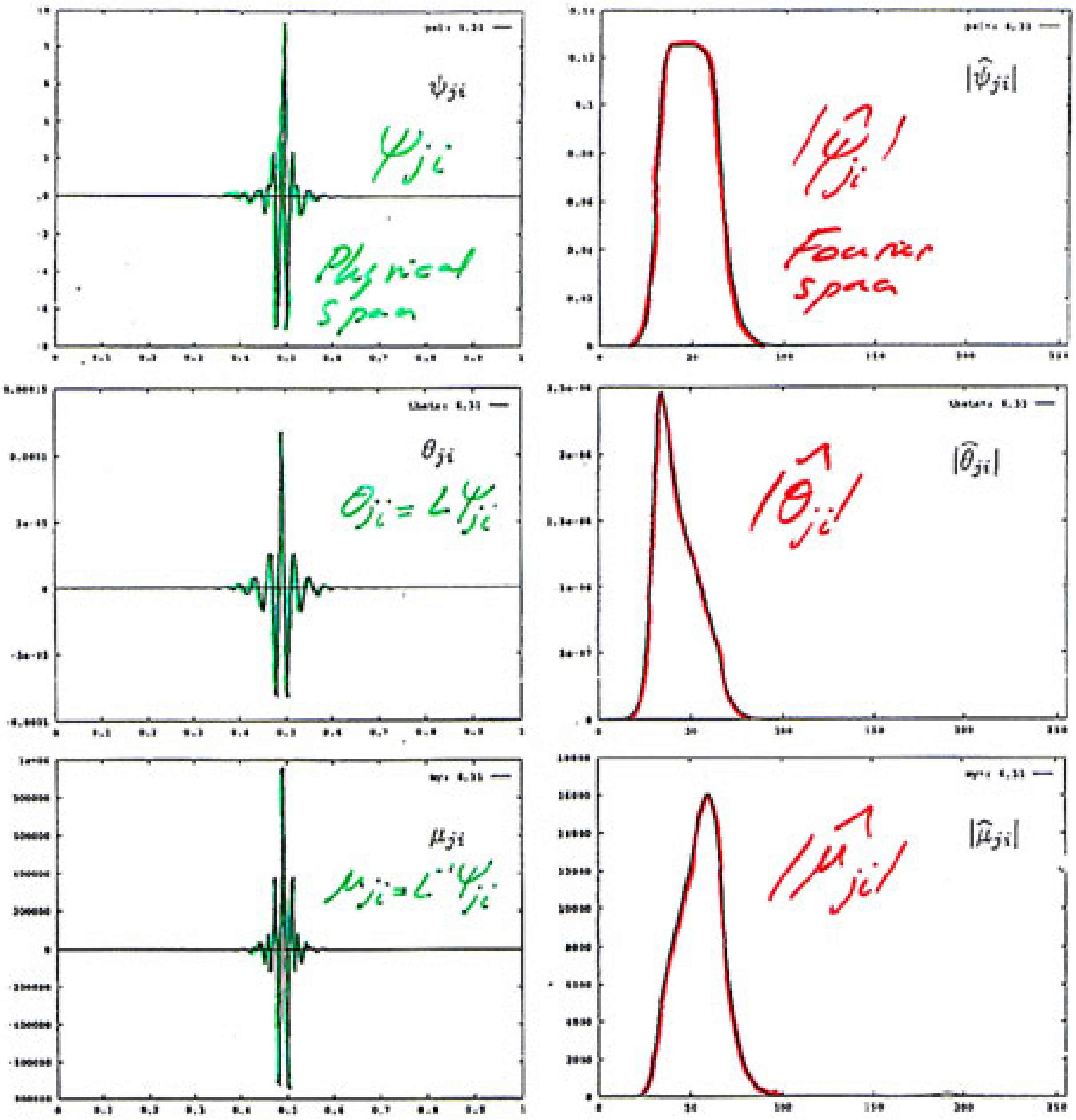


- enough space locality to keep boundary effects local,

- enough smoothness to derive $(\nabla \omega)$ while avoiding UV divergence,

- enough cancellations $\int x^n \psi(x)\, dx = 0$ to integrate $(\nabla^{-2} \omega)$ while avoiding IR divergence.

We choose basis functions which diagonalize the linear operator



ADAPTED BASIS FUNCTIONS

eigenmodes of the Green's function of Stokes operator

INTEGRATION OF THE LINEAR OPERATOR

Method of weighted residuals, Petrov-Galerkin scheme with

wavelets ψ as trial functions:

$$\omega^n(x,y) = \sum_\lambda \tilde{\omega}_\lambda \, \psi_\lambda(x,y)$$

with $\lambda = j, i_x, i_y, \mu$
$J = \log_2 N$

$$\begin{cases} j \in [0, J-1] \\ i_x, i_y \in [0, 2^j - 1] \\ \mu \in [1, 3] \end{cases}$$

Vaguelettes θ as test functions:

$$\theta_\lambda = L^{-1} \psi_\lambda$$

We operate a change of basis:

$$\tilde{\omega}_\lambda^{n+1} = \langle (\omega^n - V^n \cdot \nabla \omega^n), \theta_\lambda \rangle$$

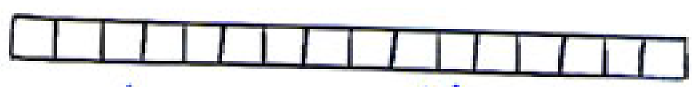
and we obtain the solution:

$$\omega^{n+1} = \sum_\lambda \tilde{\omega}_\lambda^{n+1} \psi_\lambda$$

PRINCIPLE OF THE ADAPTIVE PSEUDO-WAVELET METHOD

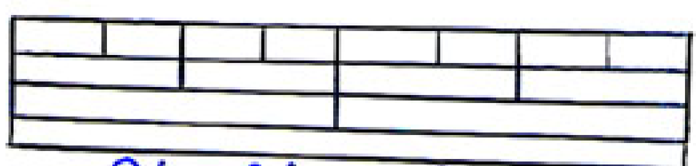
$$L \omega = N$$

Discretization
onto N grid points:



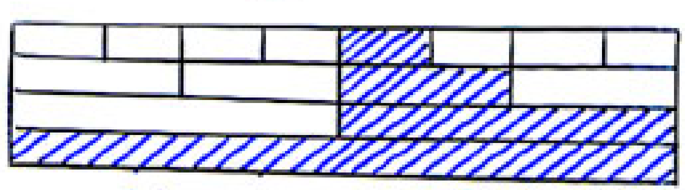
$$L_N \omega_N = N_N$$

Projection
onto N wavelets:



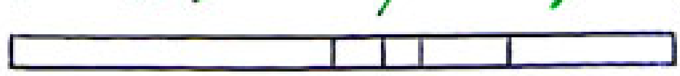
$$\tilde{L}_N \tilde{\omega}_N = \tilde{N}_N$$

Compression
onto $N_> \ll N$ wavelets:



$$\tilde{L}_{N_>} \tilde{\omega}_{N_>} = \tilde{N}_{N_>}$$

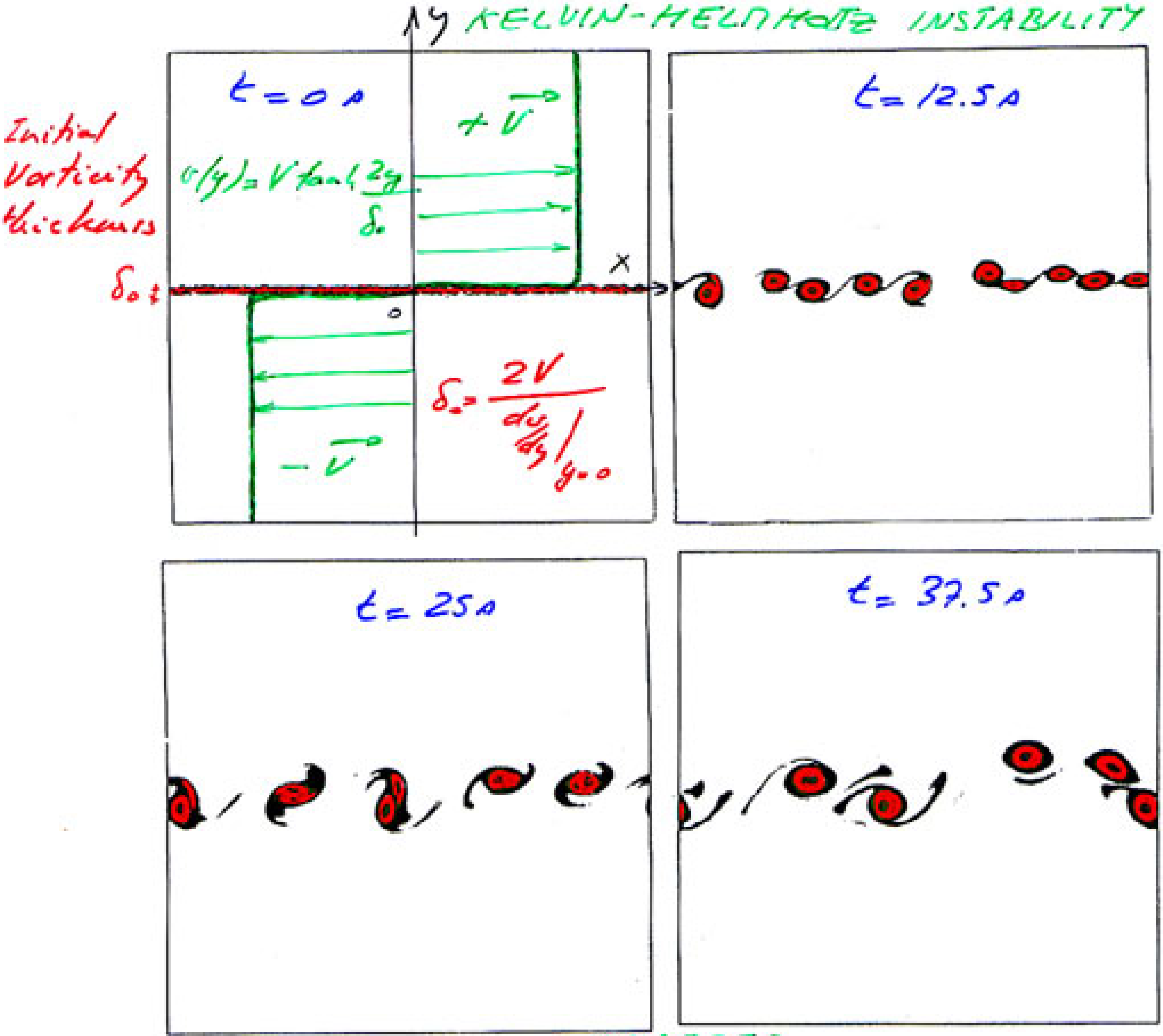
Reconstruction
onto $N_>$ adapted grid points:



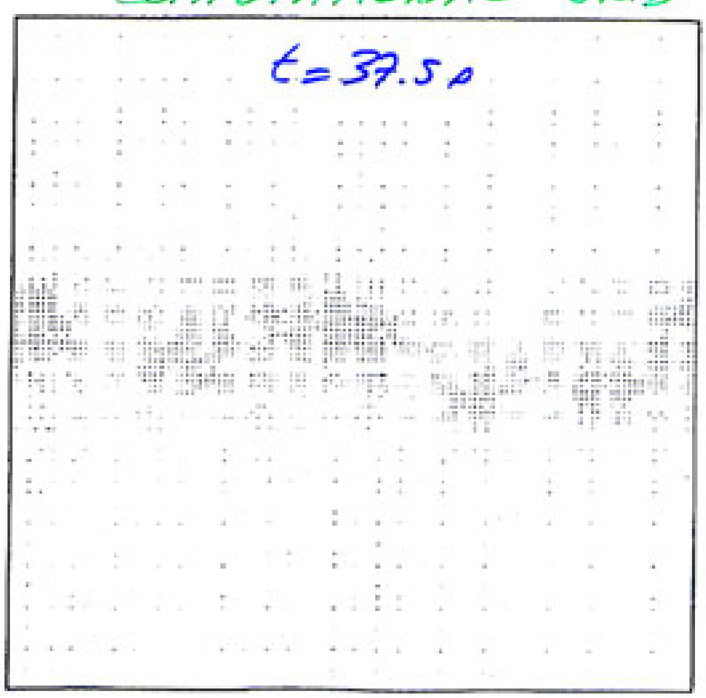
$L_{N_>} \omega_{N_>} = N_{N_>}$ # operations $O(N_>)$

TEMPORALLY DEVELOPING MIXING LAYER COMPUTED WITH CVS

KELVIN-HELMHOLTZ INSTABILITY



ADAPTED COMPUTATIONAL GRID



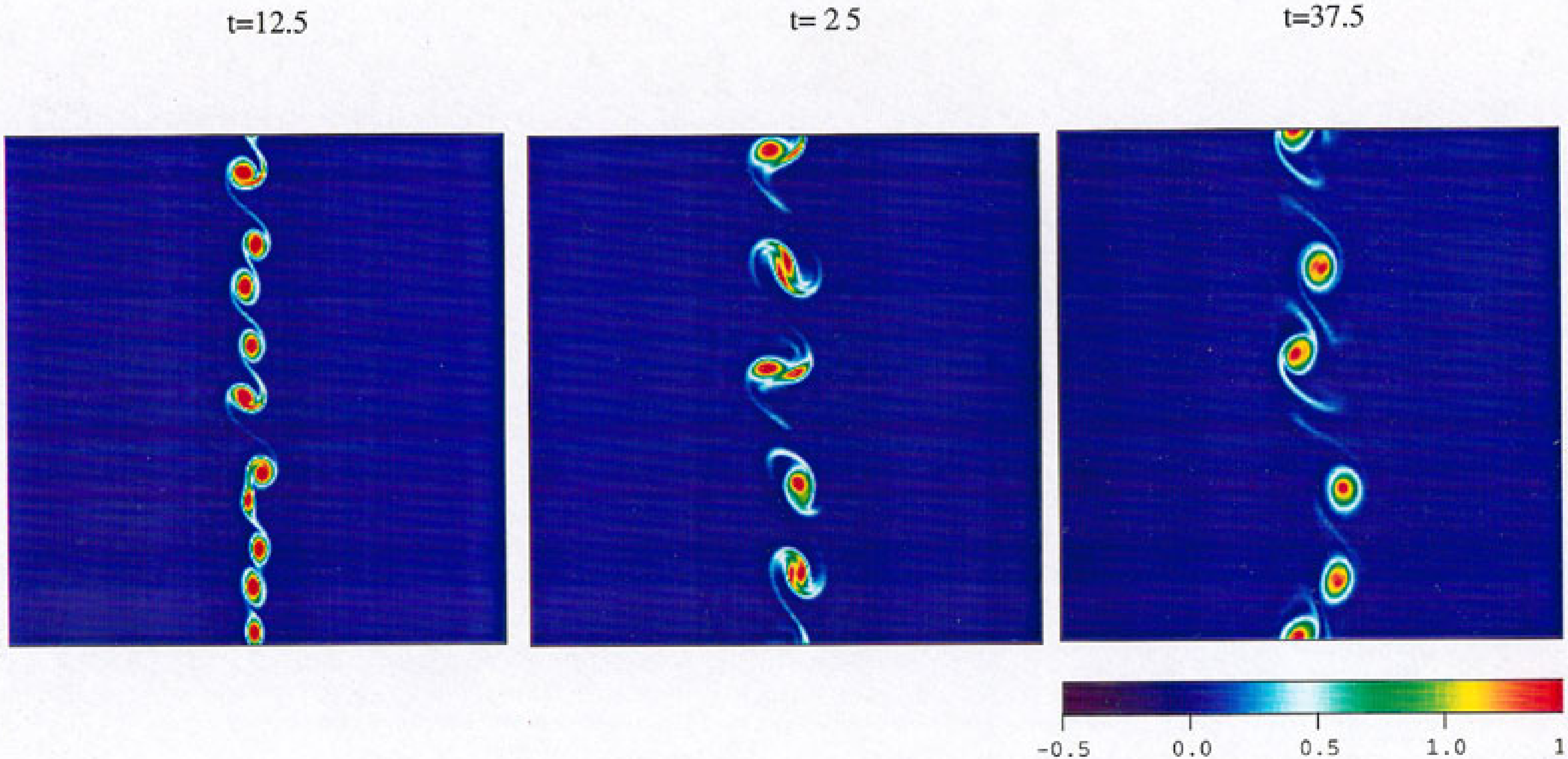
cf. Numerical simulation of a mixing layer in an adaptive wavelet basis

Schneider & Farge

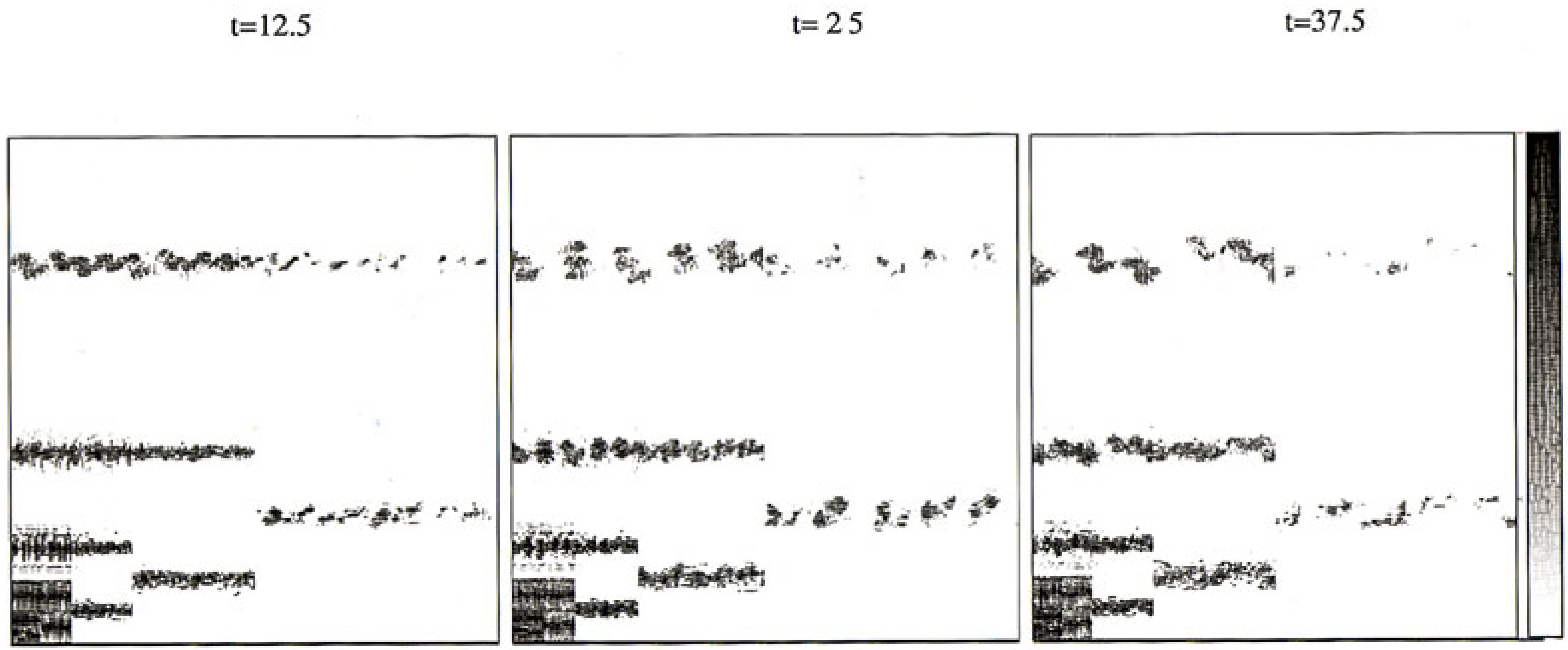
C.R.A.S.

Submitted, 1999

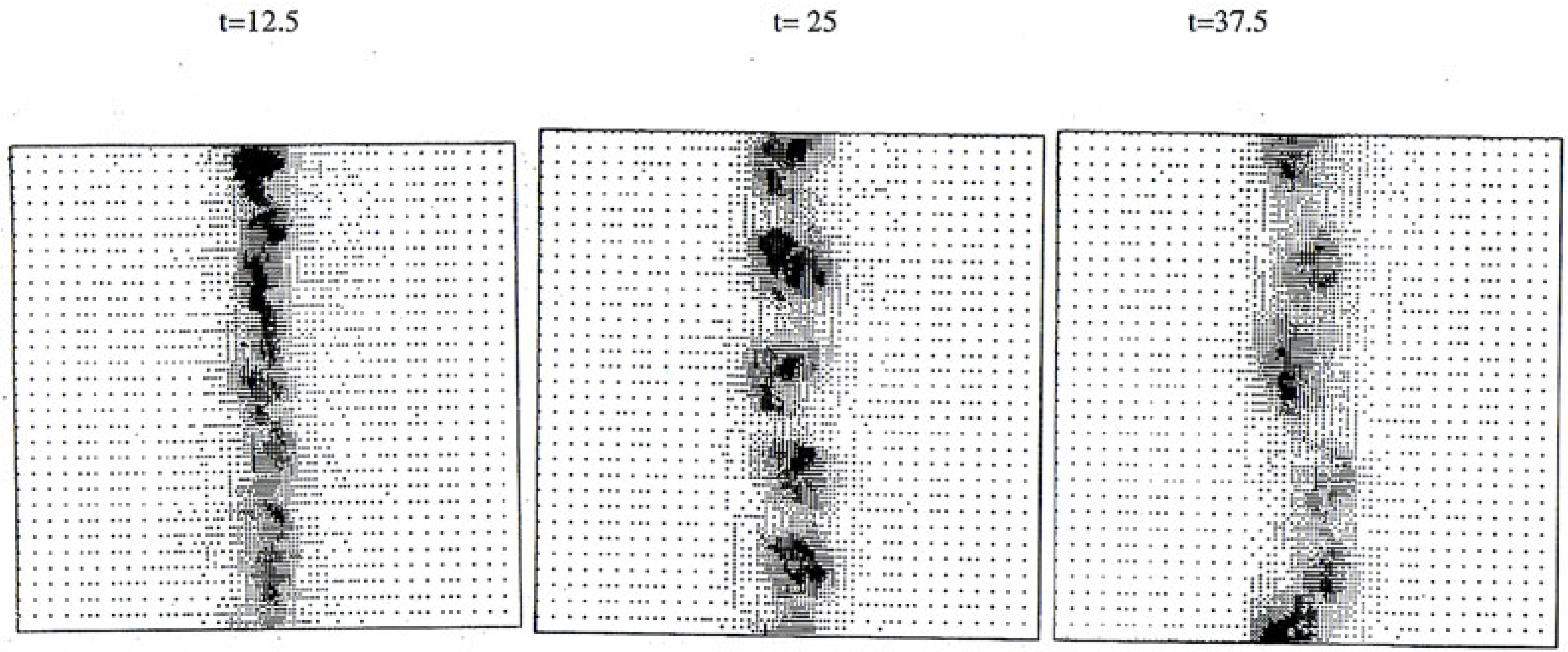
Mixing Layer: adaptive wavelet calculation (eps=10^{-6})



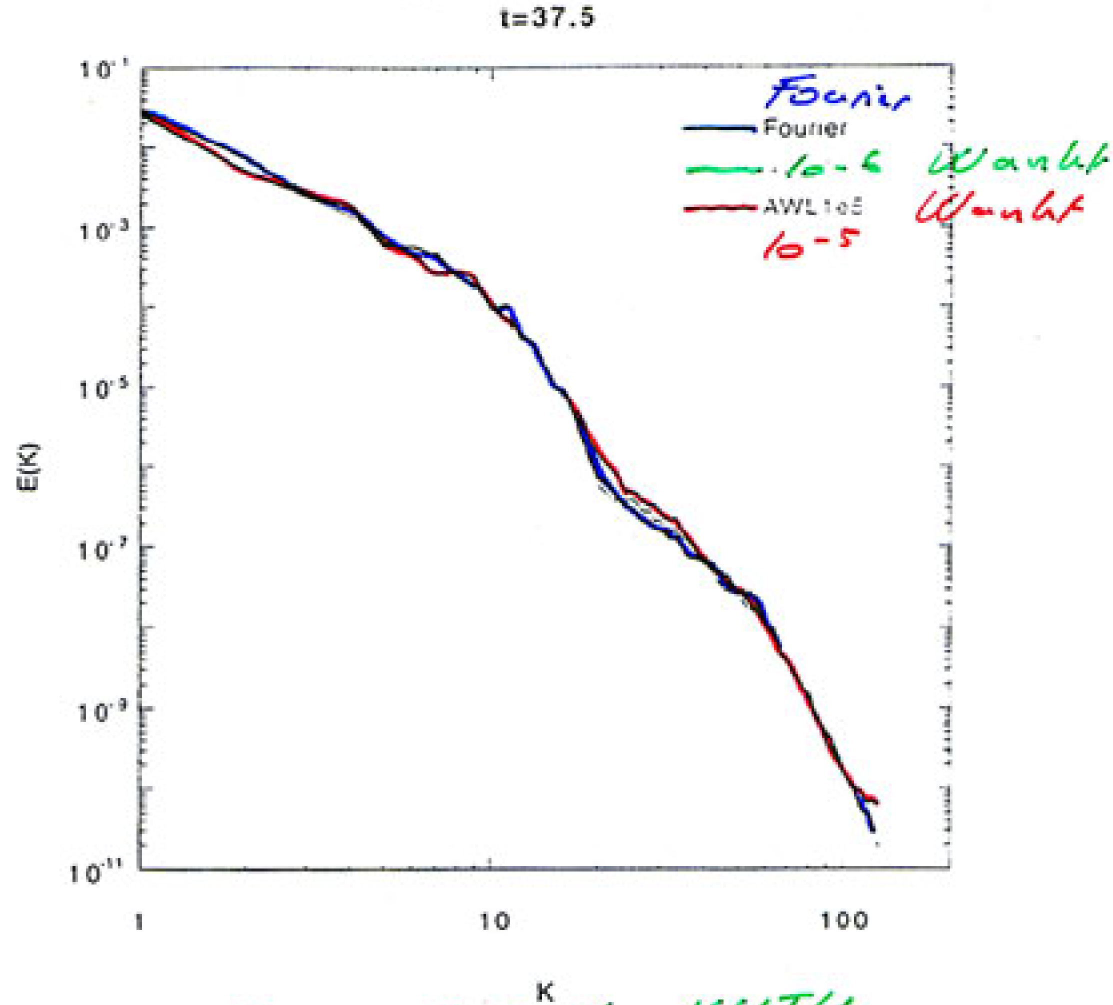
Mixing Layer: adaptive wavelet calculation (eps=10^{-6})



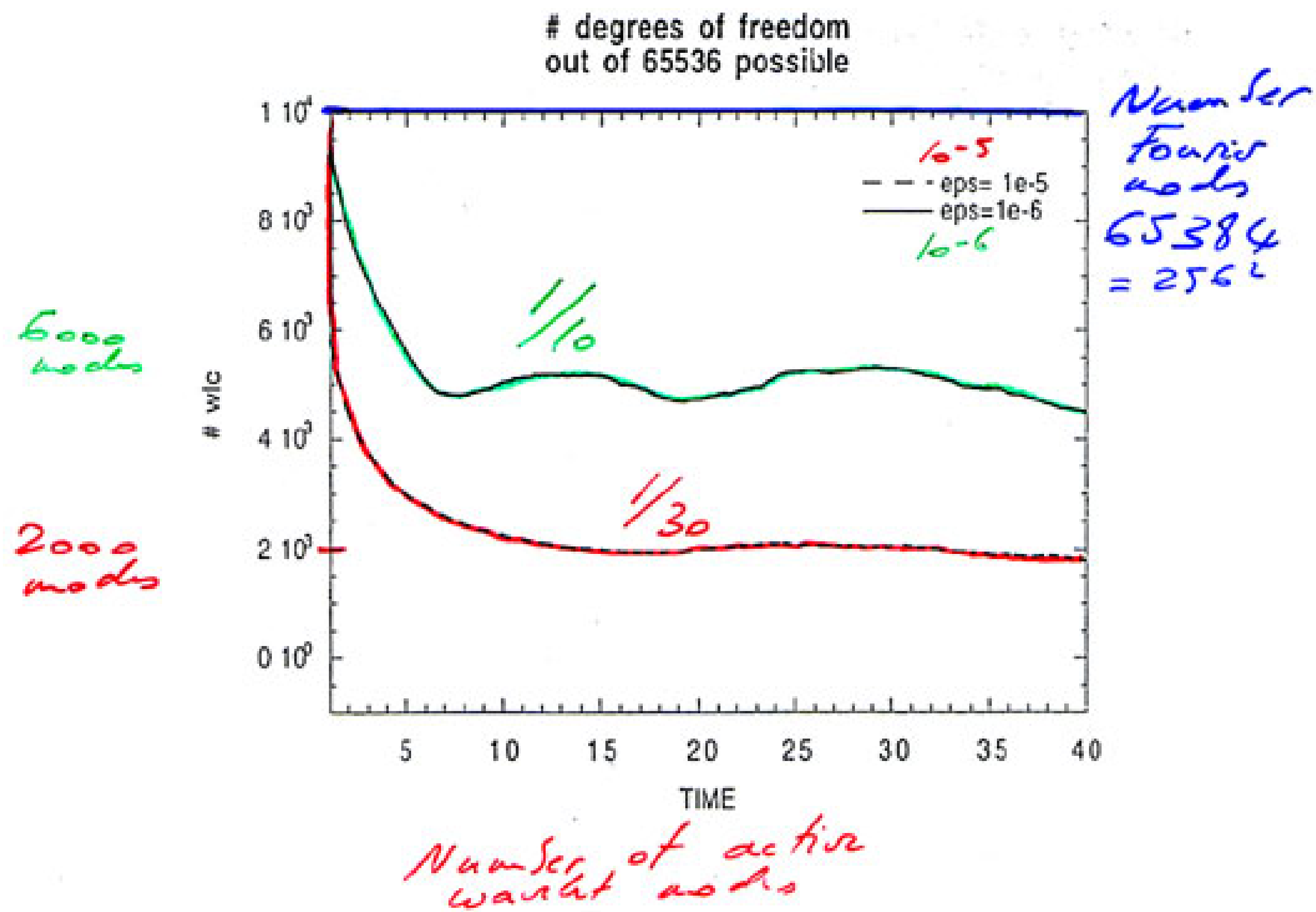
Mixing Layer: adaptive wavelet calculation (eps=10^{-6})



adaptive grid



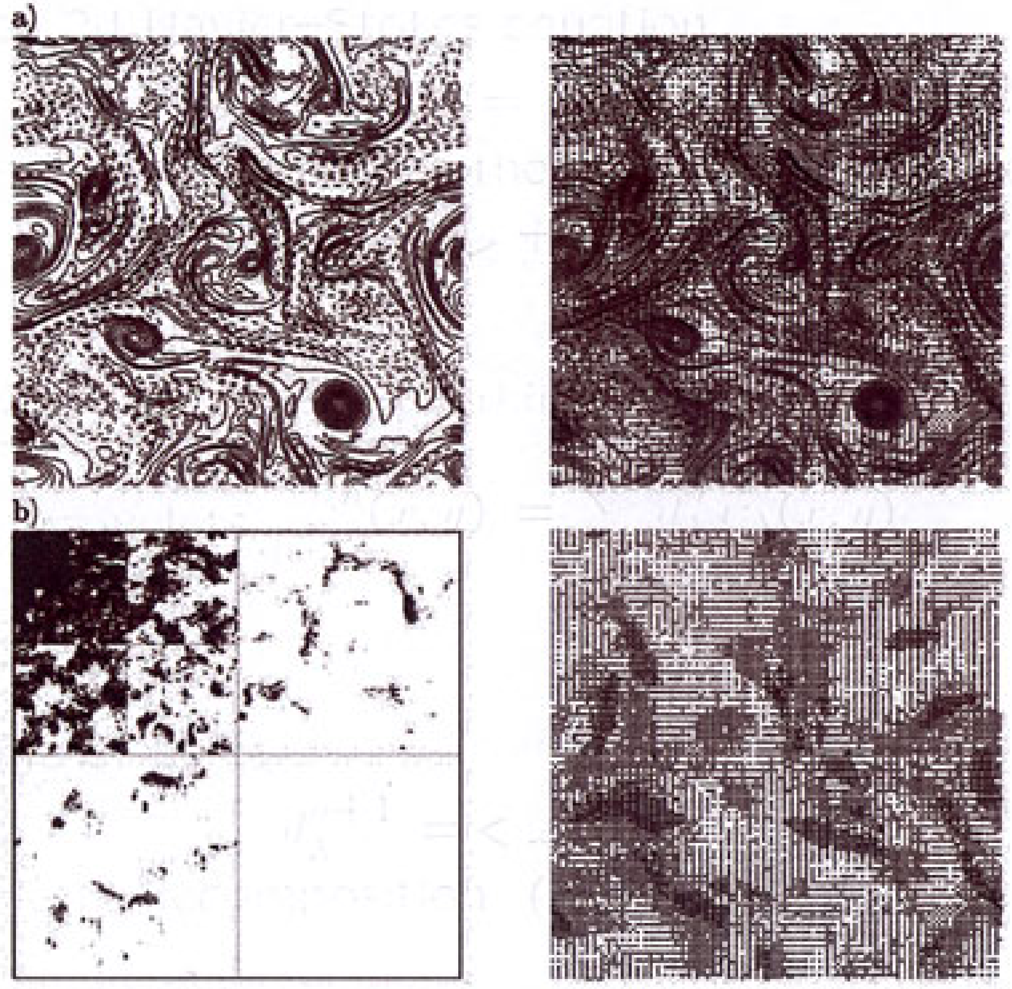
COMPARISON WITH
A FOURIER INTEGRATION



Decaying 2d homogeneous and isotropic turbulence

Vorticity

Vorticity
+ grid



Wavelet
coefficients

Adaptive
grid

SHORT-TERM PERSPECTIVES

Application of
CVS filtering to
3D rotating stratified
mixing layer

cf. Tubes, sheets and
Singularities in Fluid
Dynamics, ed. K. Bajer
and K.H. Moffatt, Kluwer, 211-216

Checking for
diffusive / non diffusive
dynamics in 3D flows
using passive scalars
and particles

Analysis of the nonlinear
term using CVS filtering
$\vec{V} \cdot \nabla \vec{\omega} - \vec{\omega} \cdot \nabla \vec{V}$

LONG-TERM PERSPECTIVES

Space-time coherent structures extraction using 4D wavelets:

- Checking that there is no scale separation in space and time,
- Checking for long-time correlation of the coherent contribution and short-time correlation of the random contribution

$\Rightarrow$ adaptive computation in space and time.

Combine CVS and penalization for 3D turbulent flows.

// wavelets.ens.fr $\rightarrow$ publications

HANS LIEPMANN'S THOUGHTS

Santa Barbara, 12th February 97

"As long as we will not be asked to predict the drag on a sphere or the pressure drop in a pipe from continuous, incompressible and Newtonian assumptions without any other complications (first principle), we will not have made it!"

$\Longrightarrow$ study the production of vorticity at the boundaries and in shear layers

production of vortices

interaction of vortices

Well-mixed region