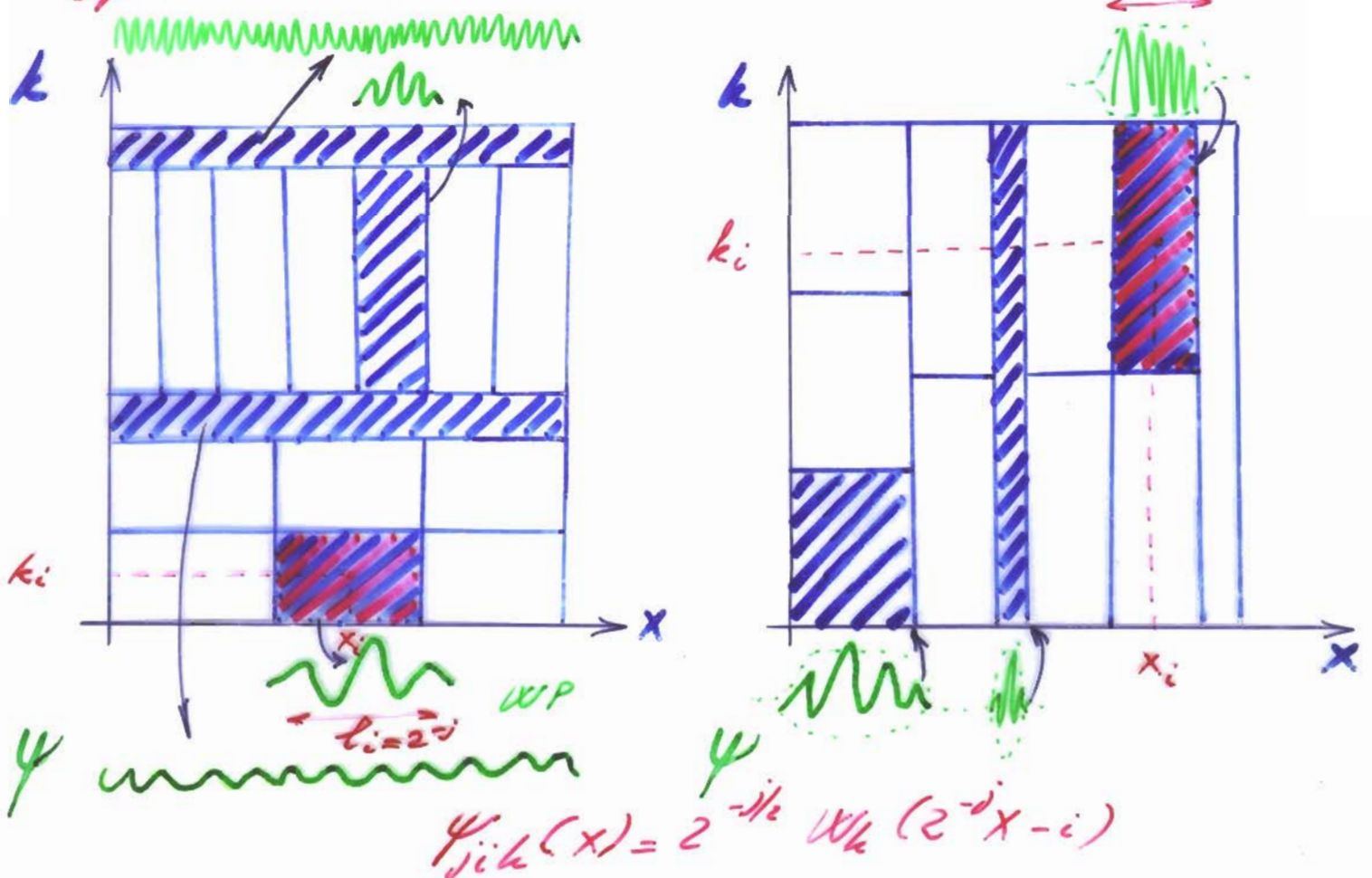


WAVELET PACKETS and LOCAL COSINES

Coifman, Meyer & Wickerhauser (1991)

Family of orthogonal bases
which unifies Dirac, Fourier,
Windowed Fourier (Malvar)
and wavelets

→
large flexibility to tile phase space
with phase-space 'atoms' and obtain
space-wavenumber-scale decomposition $(l_i = 2^{-i})$



ORTHOGONAL BASES OF PHASE-SPACE 'ATOMS'

The Balian-Low obstruction theorem
(C.R.A.S., 292, 1991)
prevents the wave-packets to form
an orthogonal basis



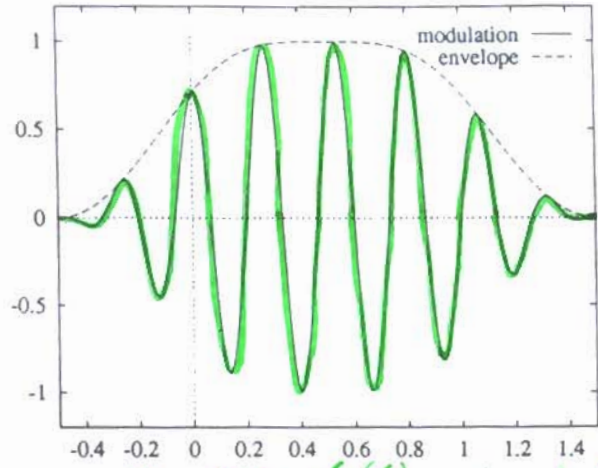
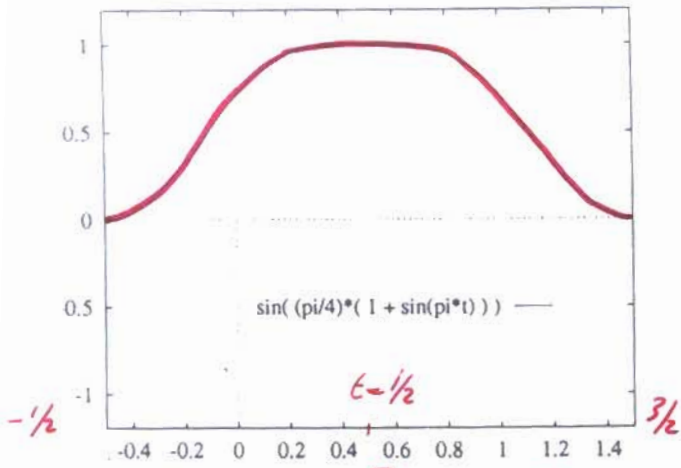
(C.R.A.S., 312, 1991)
Coifman, Meyer and Wickerhauser
circumvent this problem by:

- replacing exponentials e^{ikx}
by cosines $\cos \pi(k + \frac{1}{2})x$ and an
appropriate window shape

→ adaptive local
cosine transform

or - using quadrature mirror
filters → wavelet packet
transform.

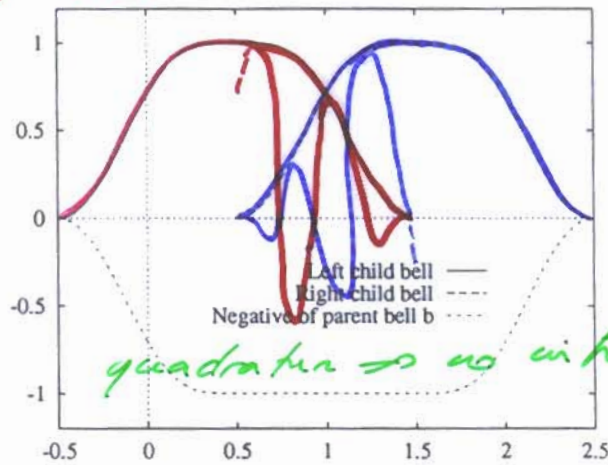
We have a library of
modulated waveforms out of
which various orthogonal bases
can be extracted.



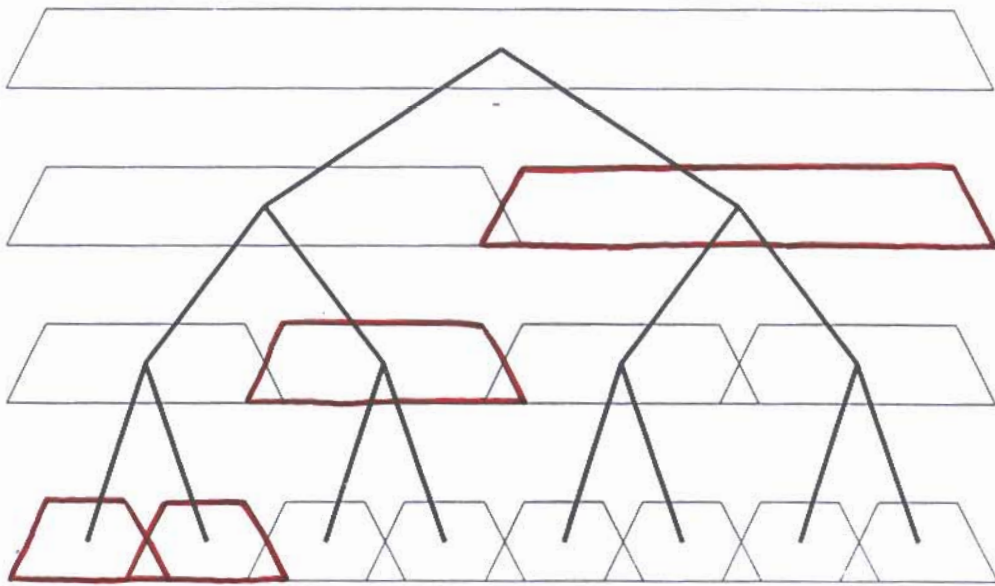
$$\beta(x) = \begin{cases} \sin \frac{\pi}{4} (1 + \sin \frac{\pi}{2} t) & \text{if } -1/2 \leq t \leq 3/2 \\ 0 & \text{if } t < -1 \\ 1 & \text{if } t > 1 \end{cases}$$

$$\psi_{nh}(t) = S_h(t) m_h(t + \frac{1}{2}, t)$$

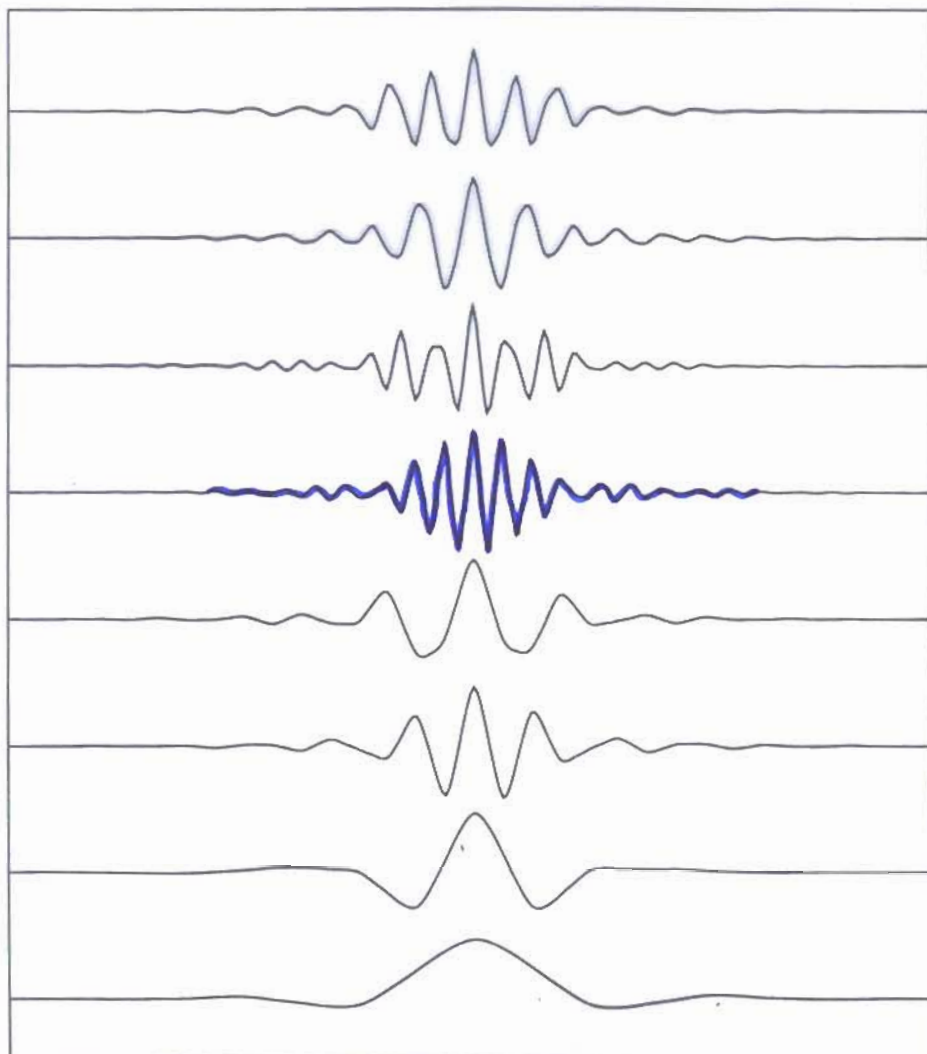
with $m(t) = \sqrt{2} \cos \pi t$



Adjacent orthogonal windows



Our possible basis made of natural wavelets (local adapted combs)



*Orthogonal Wavelet
Packets built with Coifman 18:*

*Each WP is orthogonal
to all of its translated, modulated
and dyadic rescaled versions.*

LIBRARY OF PHASE-SPACE 'ATOMS'

We define 3 operators

$$\begin{array}{ll} \text{translation} & T_x \quad \psi(x') = \psi(x'-x) \\ \text{modulation} & M_k \quad \psi(x') = e^{ikx} \psi(x) \\ \text{dilatation} & D_\ell \quad \psi(x') = \ell^{-1/2} \psi(\ell x) \end{array}$$

These transformations conserve energy so the wave-packets can be normalized to be unit vectors in L^2 .

Analysis

$$\tilde{f}_{xke} = \langle \psi_{xke} | f \rangle$$

Synthesis

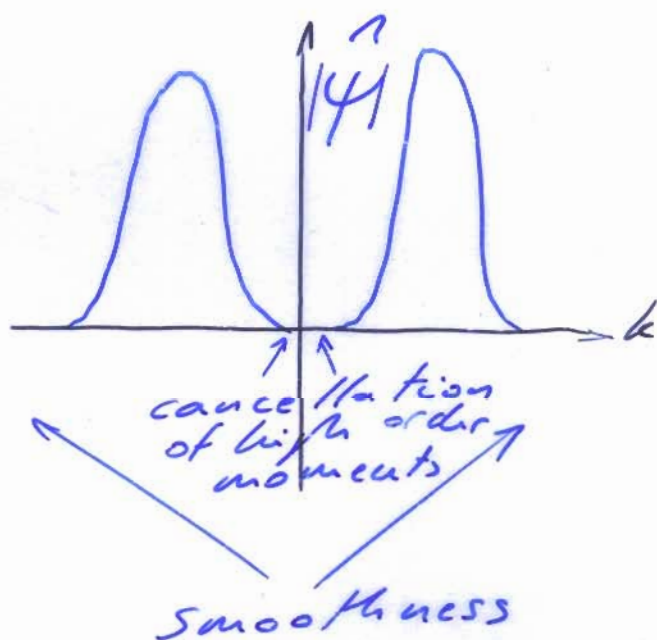
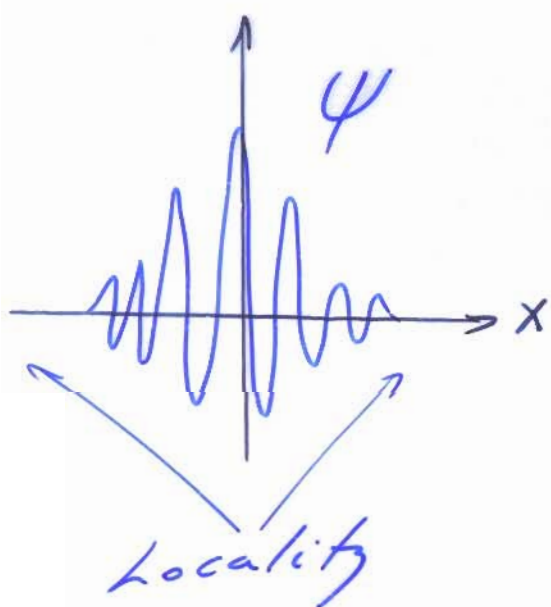
$$f(x) = \sum \tilde{f}_{xke} \psi_{xke}$$

Energy Conservation

$$\int |f(x)|^2 dx = \sum |\tilde{f}_{xke}|^2$$

LIBRARY OF ORTHOGONAL BASES

We build a library \mathcal{B} of many orthogonal bases B_n made of elementary functions ψ which have the spatial (compactness) and spectral (smoothness) locality properties we wish:



$\mathcal{B} \left\{ B_n: \{ \psi_{ni} \} \right\}$ with

$$\psi_{\ell} = \psi_{j,i,k} = 2^{-j/2} W_k (2^{-j}x - i)$$

i position, j scale, k wavenumber
 ℓ index of the wavepacket

WAVELET PACKET BASIS

A wavelet packet basis of $L^2(\mathbb{R})$ is any orthogonal basis selected from among the functions $\psi_{jik}(x) = 2^{-j/2} \psi_k(2^{-j}x - i)$.

The choice of the QMF ψ_k influences the properties of the wavelet packets in terms of smoothness, vanishing moments.

The computational complexity grows linearly with the filter's length.

We want to select the basis ψ which allows the most efficient representation of the signal f .

We obtain a new class of orthogonal bases of $L^2(\mathbb{R}^n)$ by constructing a library of modulated waveforms of prescribed smoothness, out of which various bases may be extracted, in particular the Walsh basis, the Windowed Walsh basis, the Wavelet basis and various Wavepacket bases.

QMF
PHASE-SPACE
'ATOMS'

Set of orthogonal functions:

$$\psi_{jik}(x) = 2^{-j/k} W_k(2^{-j}x - i)$$

characterized by 3 parameters:

$$\begin{cases} j & \text{scale} \\ i & \text{position} \\ k & \text{wavenumber} \end{cases}$$

W_k are not constructed explicitly but are computed from two conjugate quadratic filters applied to the N sampled values of the field w .

$\psi_{jik}(x)$ are square-integrable functions with prescribed smoothness depending on W .

Projection of w :

$$\tilde{w}_{jik} = \langle \psi_{jik} | w \rangle$$

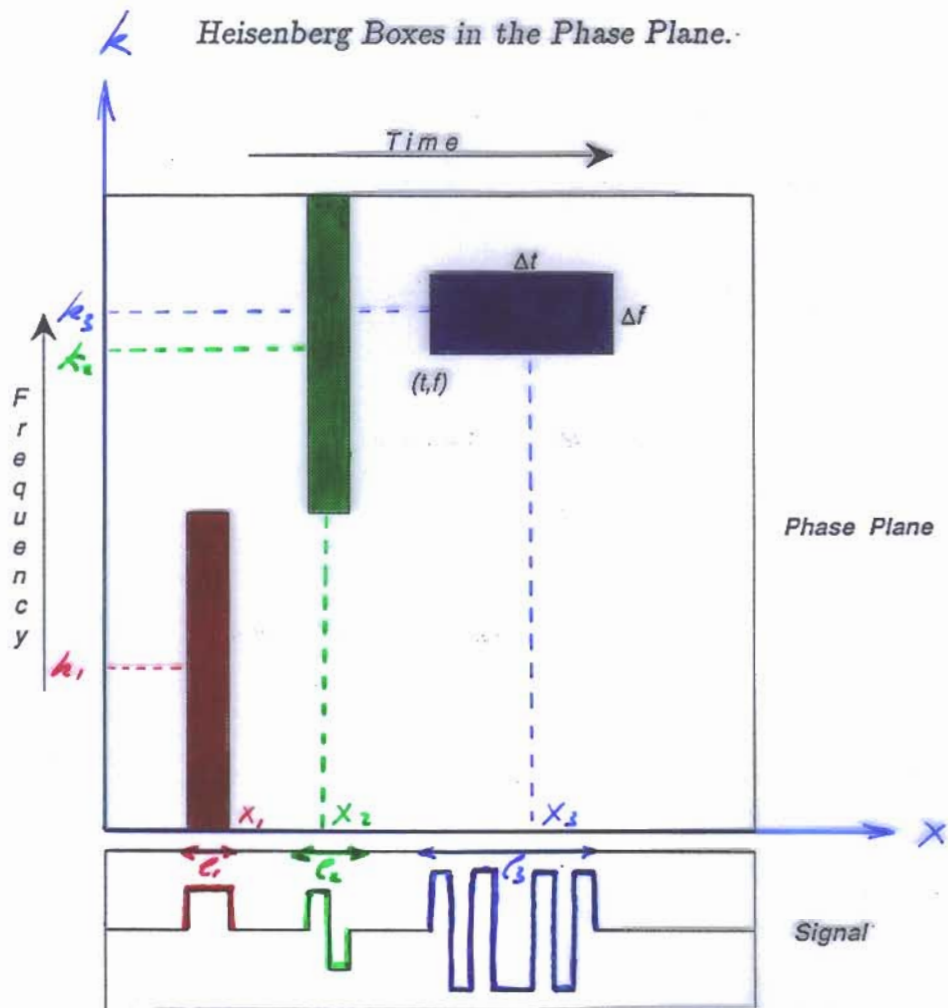
Reconstruction of w :

$$w(x) = \sum \tilde{w}_{jik} \psi_{jik}$$

Both require N operations.

In 2D:

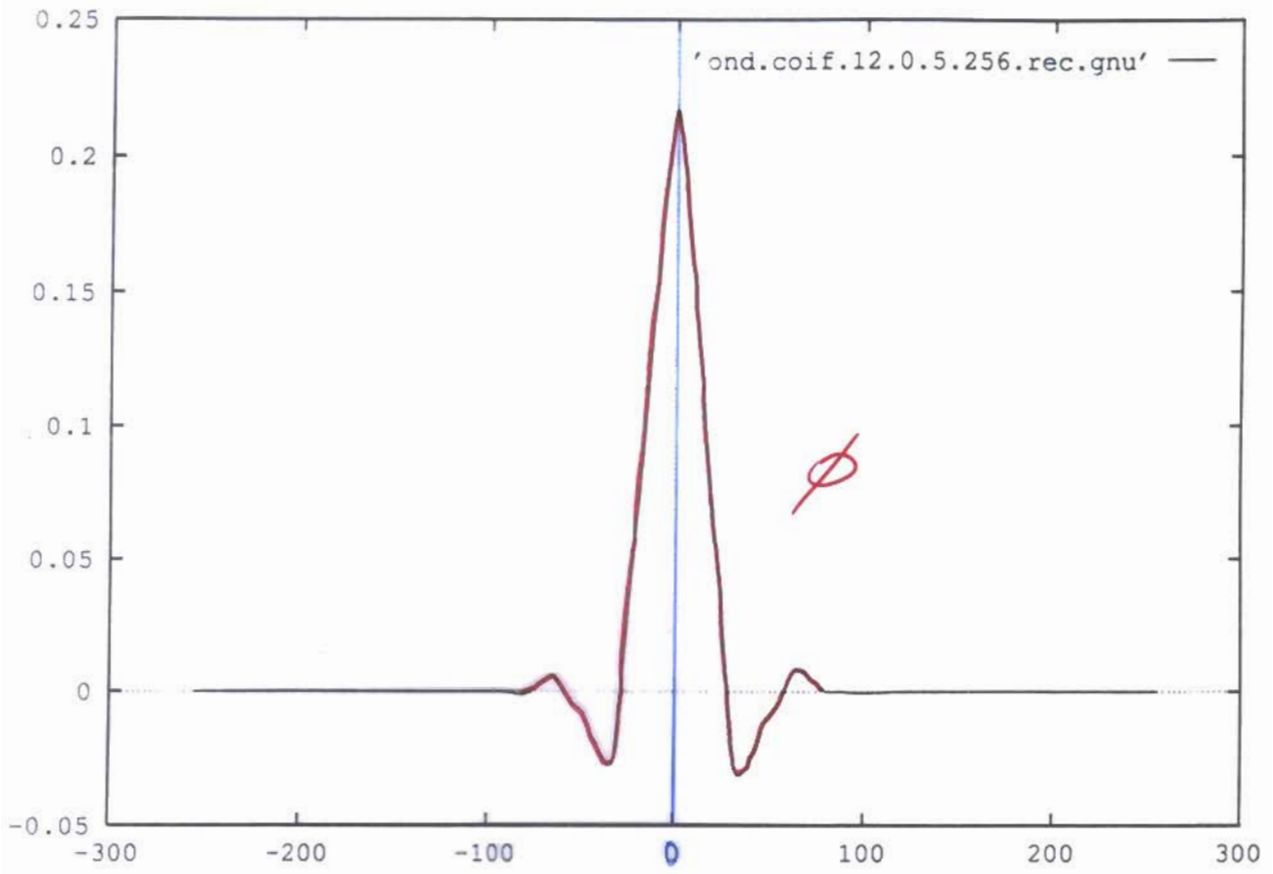
$$\psi_{j_1 k_1 j_2 k_2}(\vec{x}) = 2^{-j_1/k_1} W_{k_1}(2^{-j_1}x - i_1) W_{k_2}(2^{-j_2}y - i_2)$$



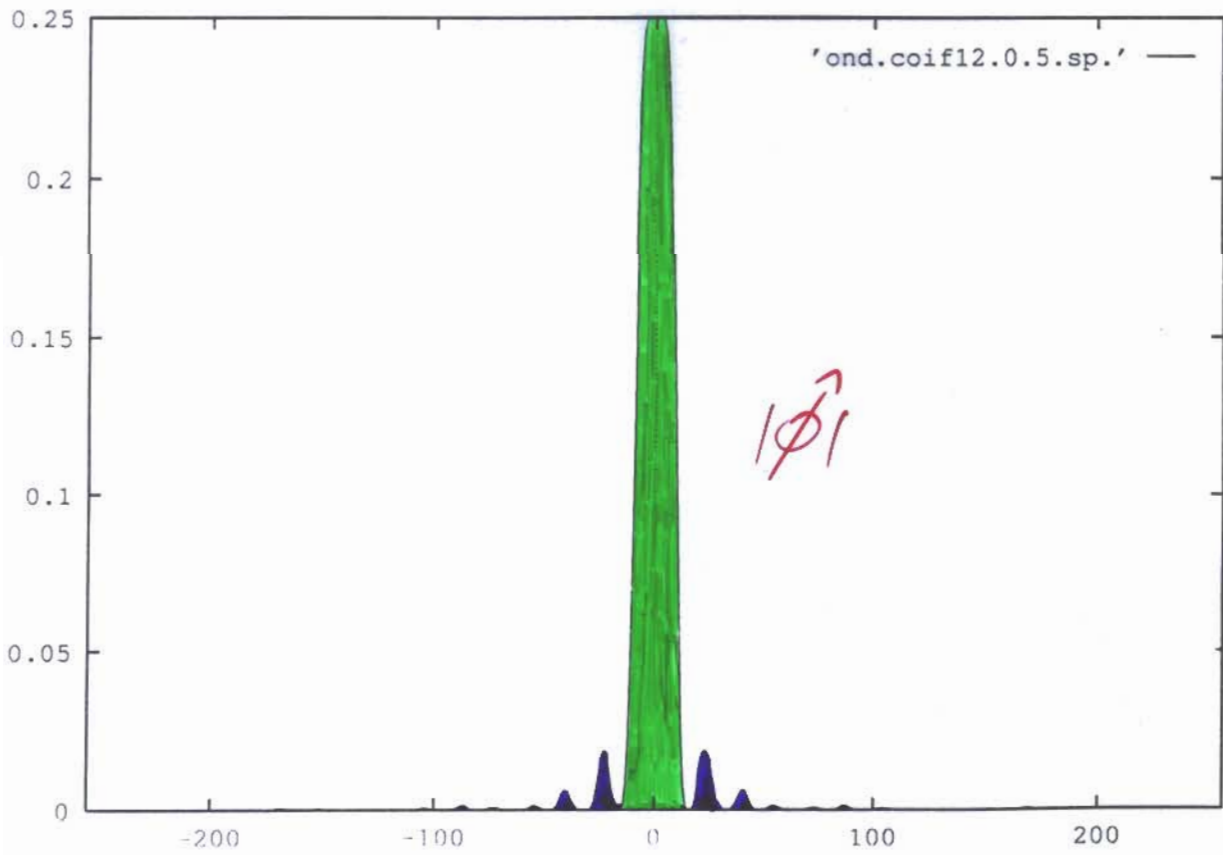
Examples of
wavelet packets
built with
the Haar filter W .

$$\psi_{j,k}(x) = 2^{-j/2} W_k(2^{-j}x - i)$$

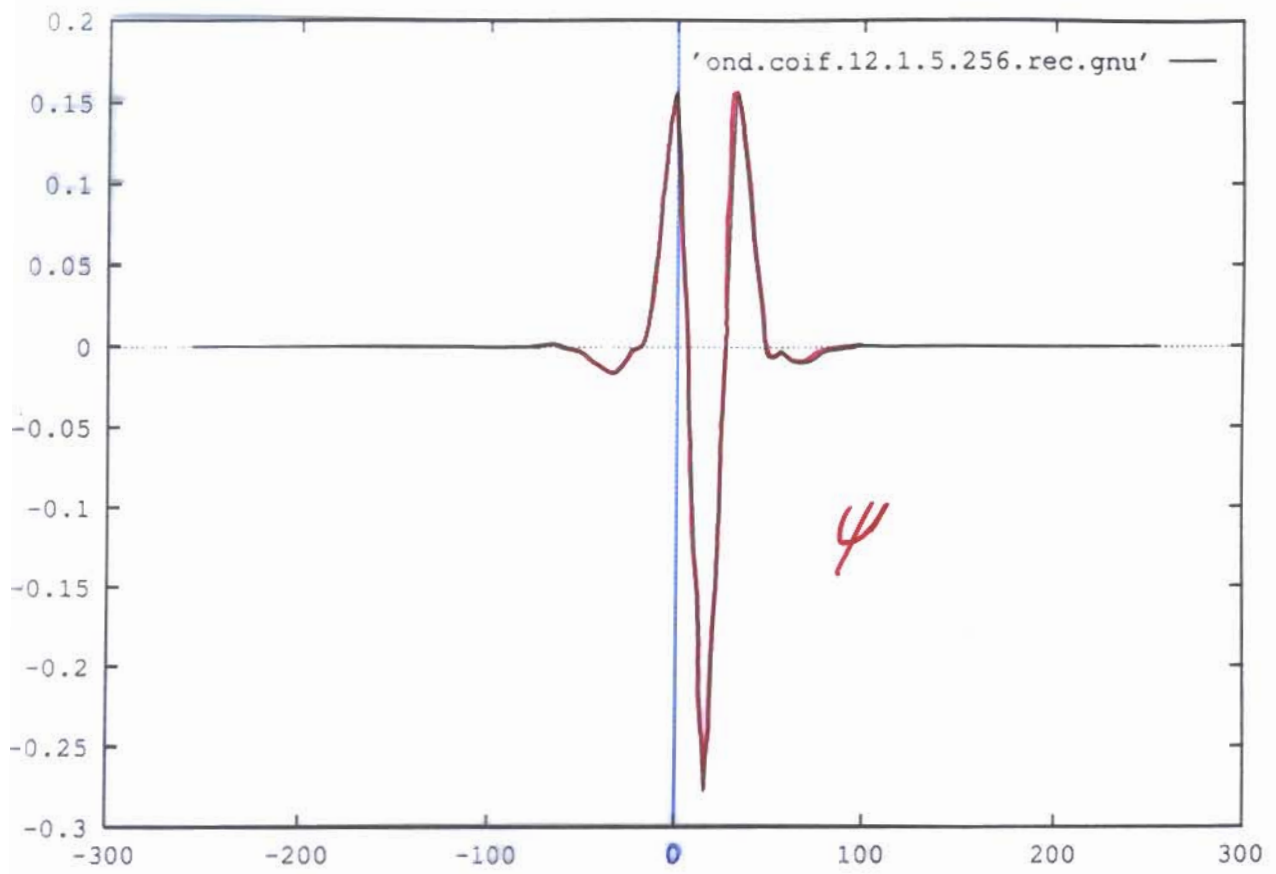
Coifman 12



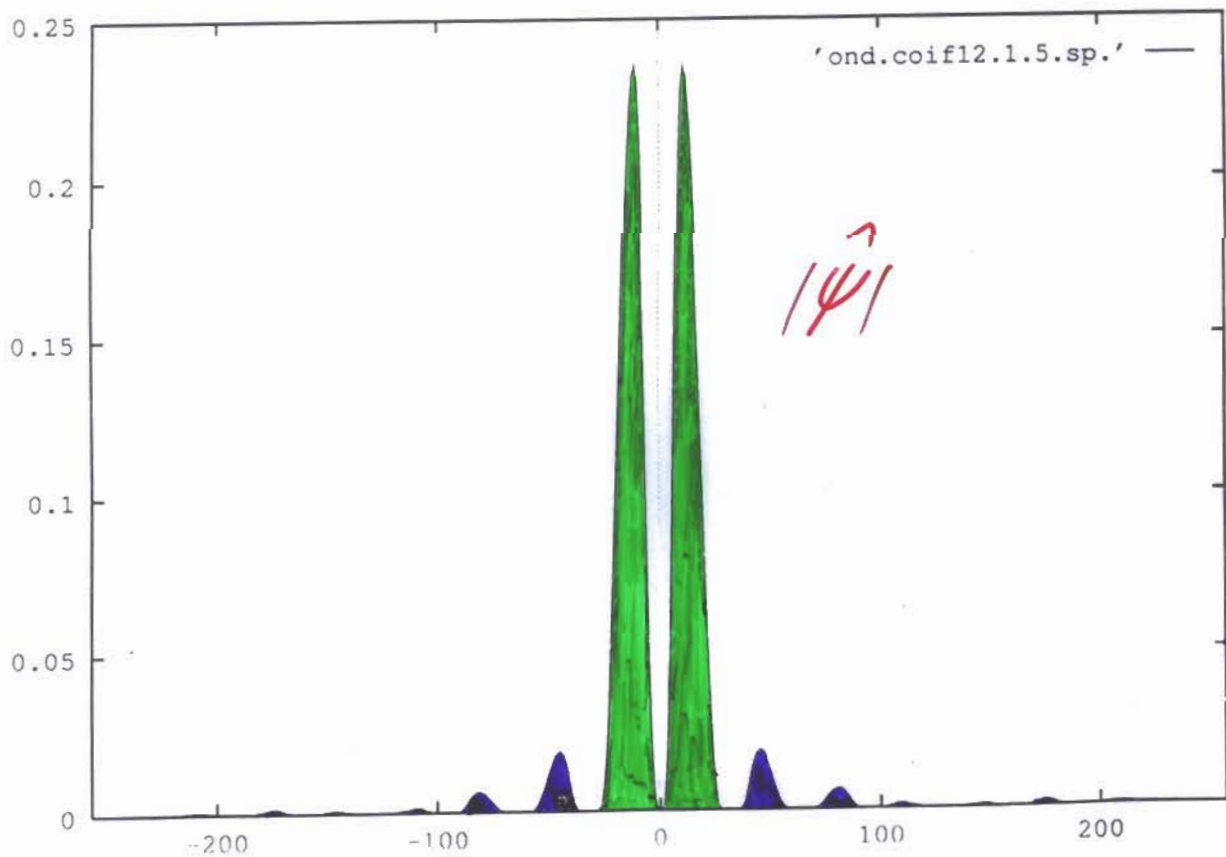
scale 0



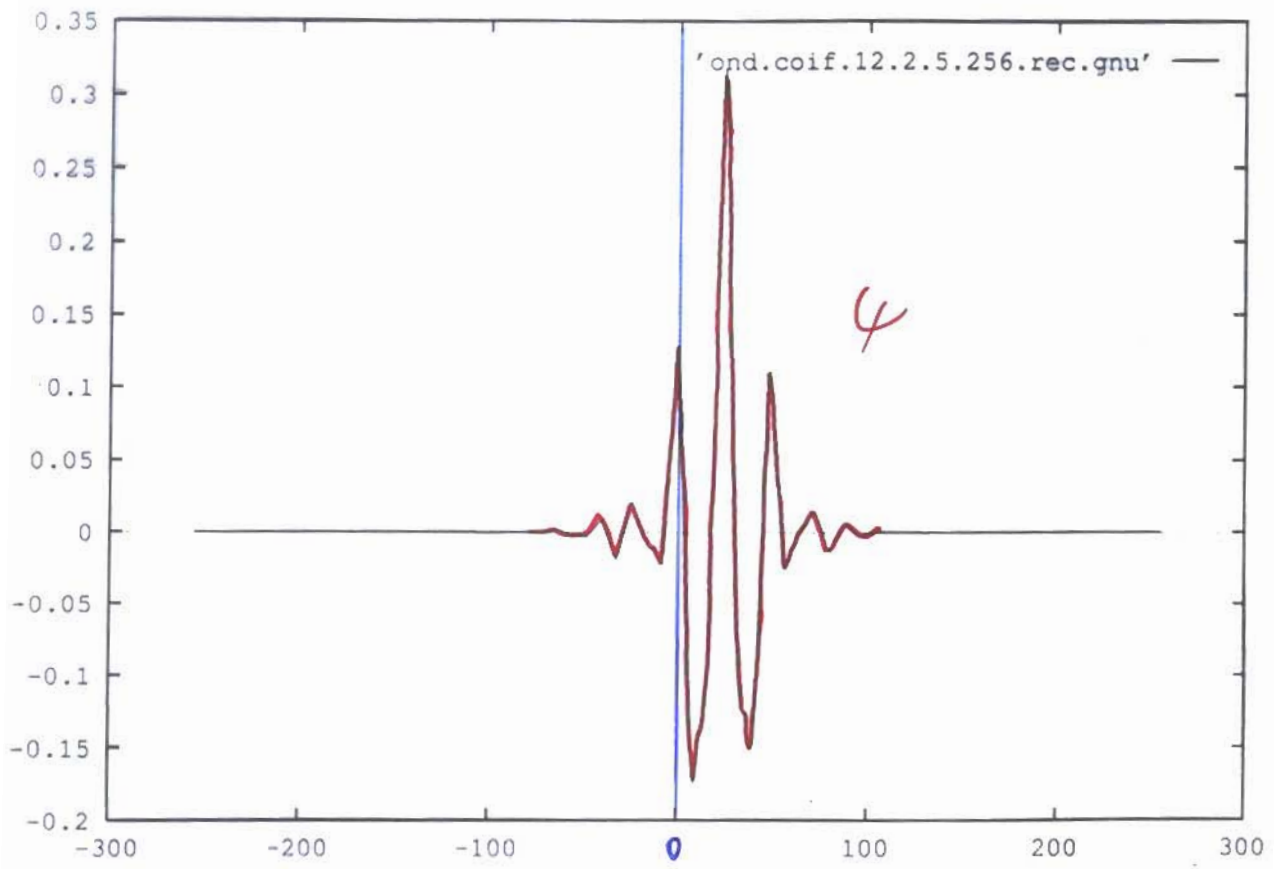
Coifman 12



Scale 1

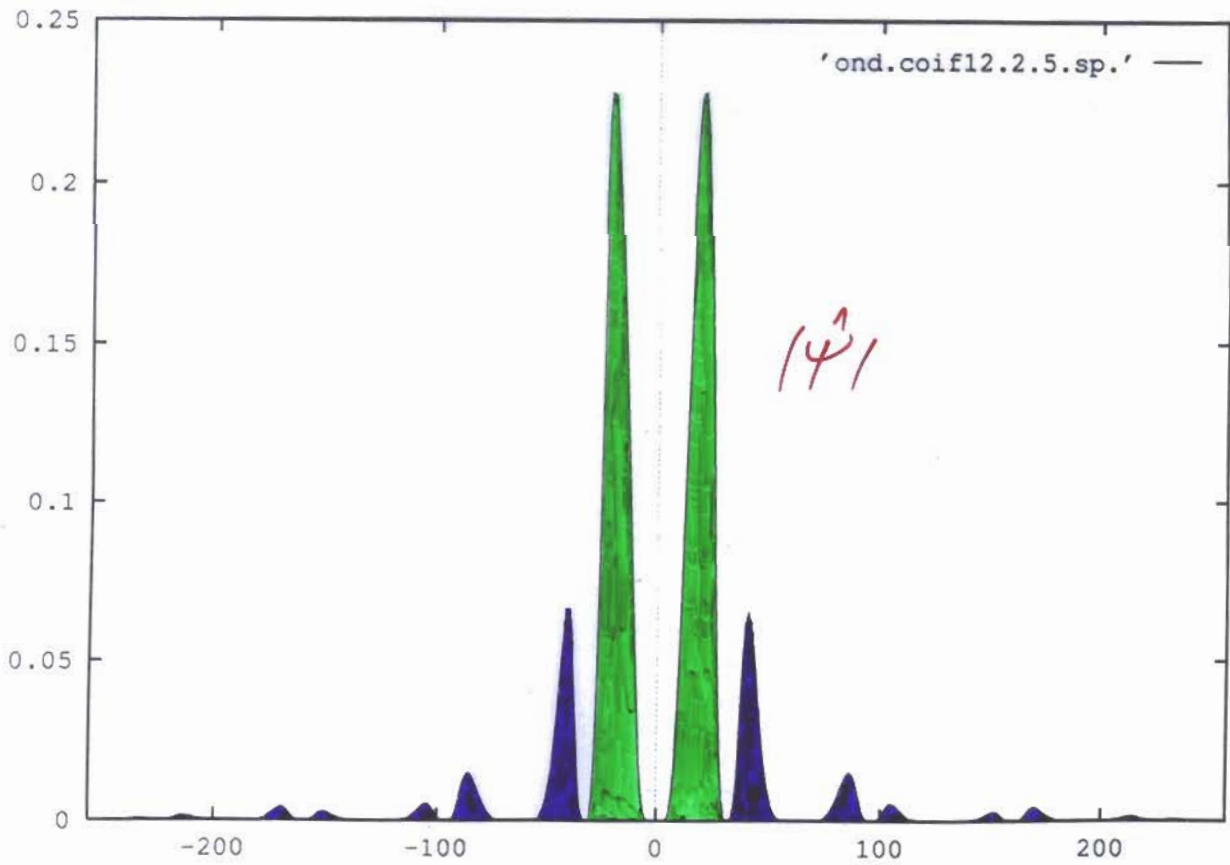


Coifman 12

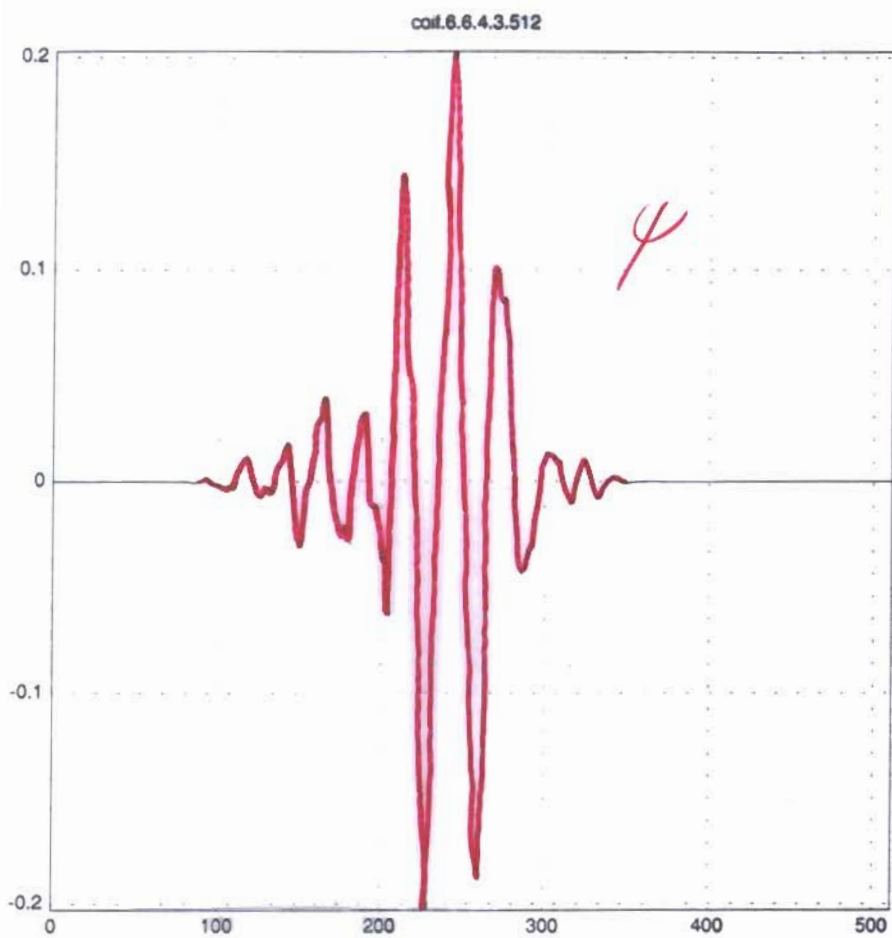


4

scal 2

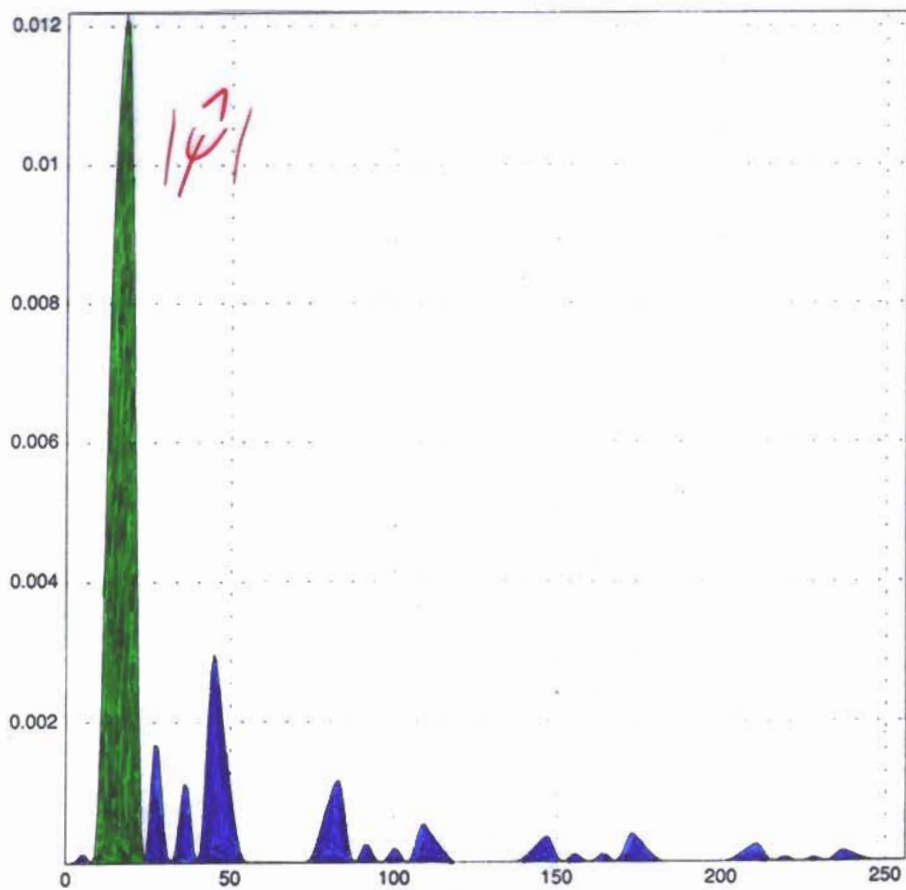


141



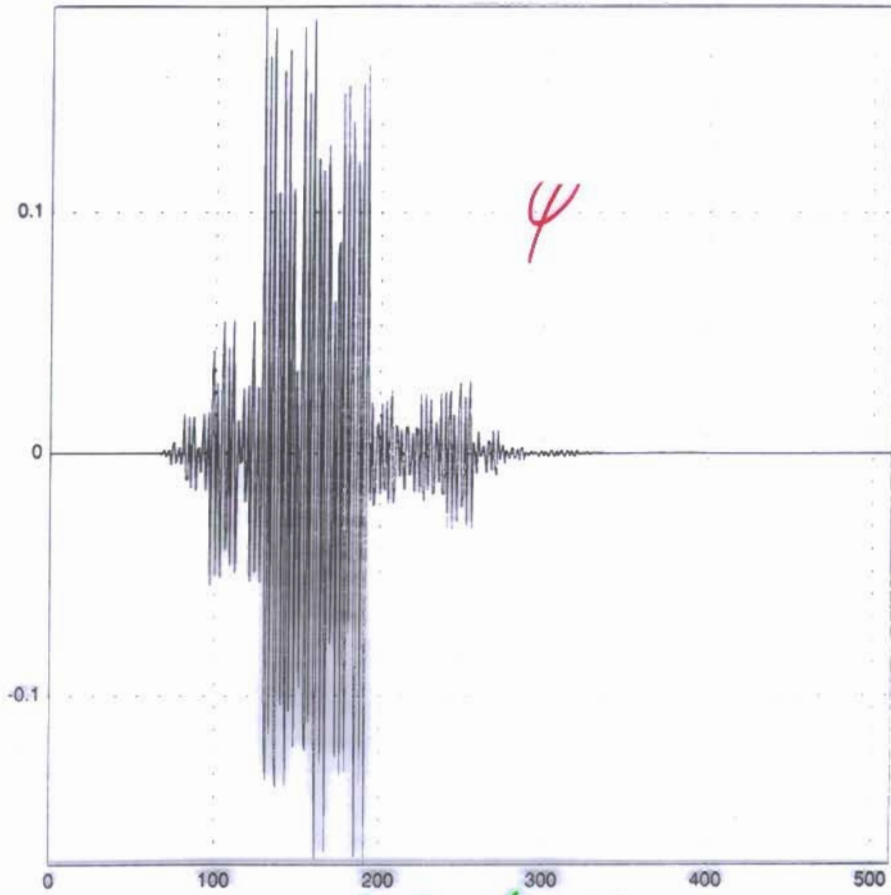
$j=6, k=4$

coil.6.6.4.3.512.fou



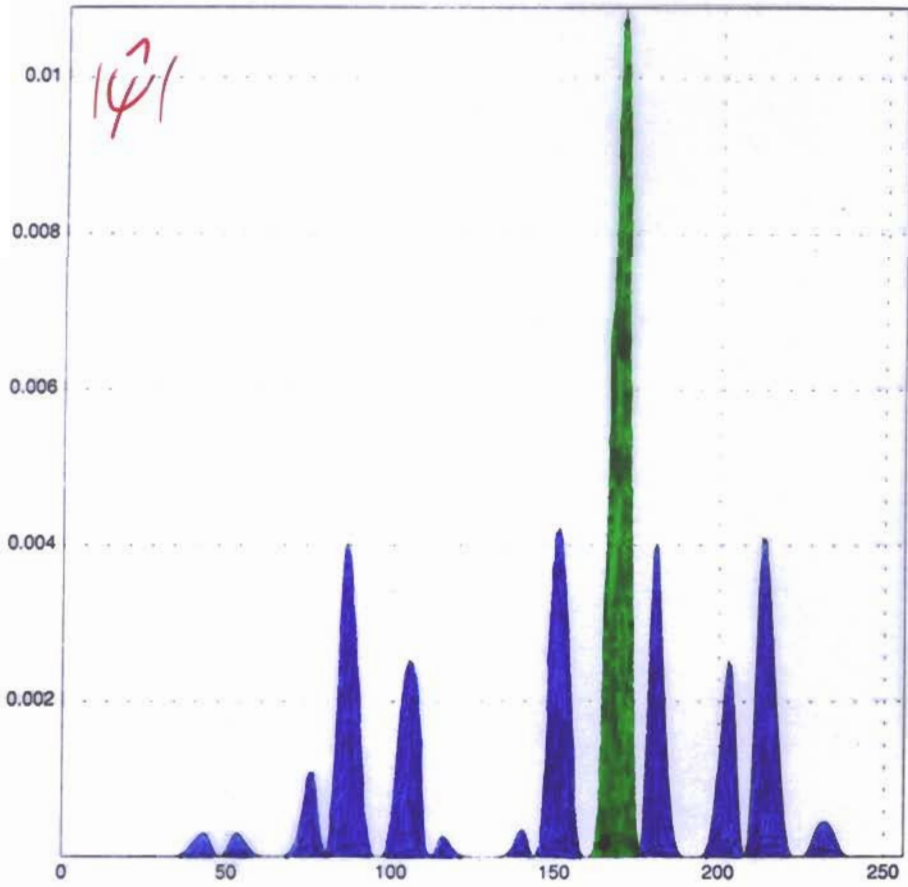
Coifman 6

coif.6.6.42.3.512



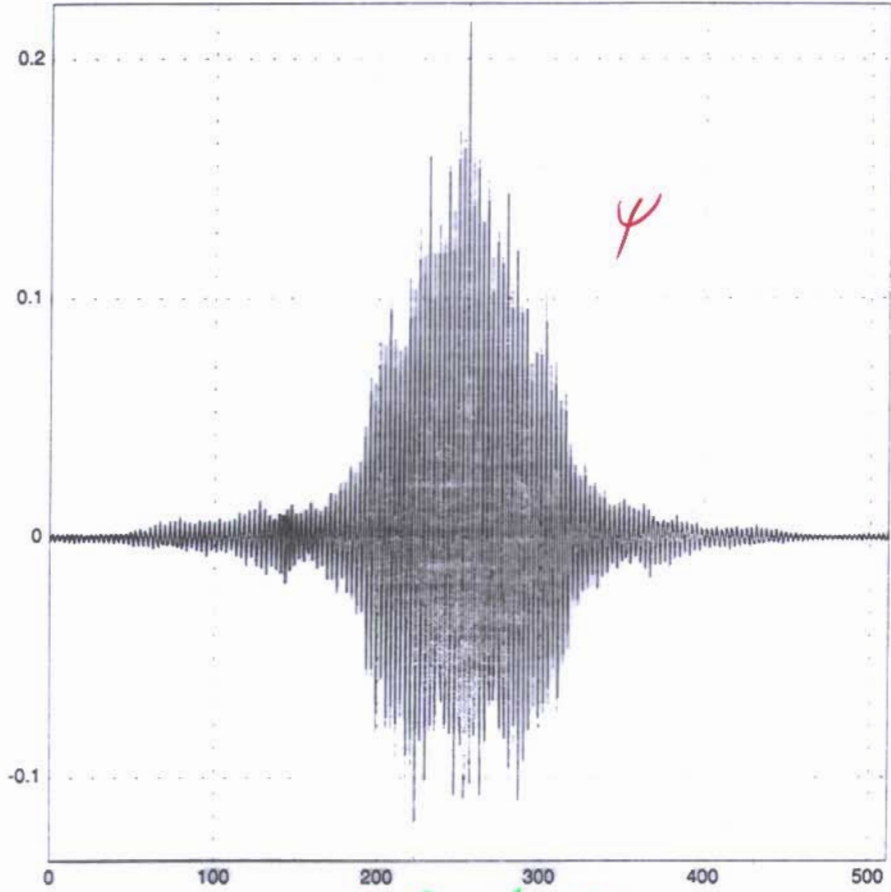
$j=6, k=42$

coif.6.6.42.3.512.fou



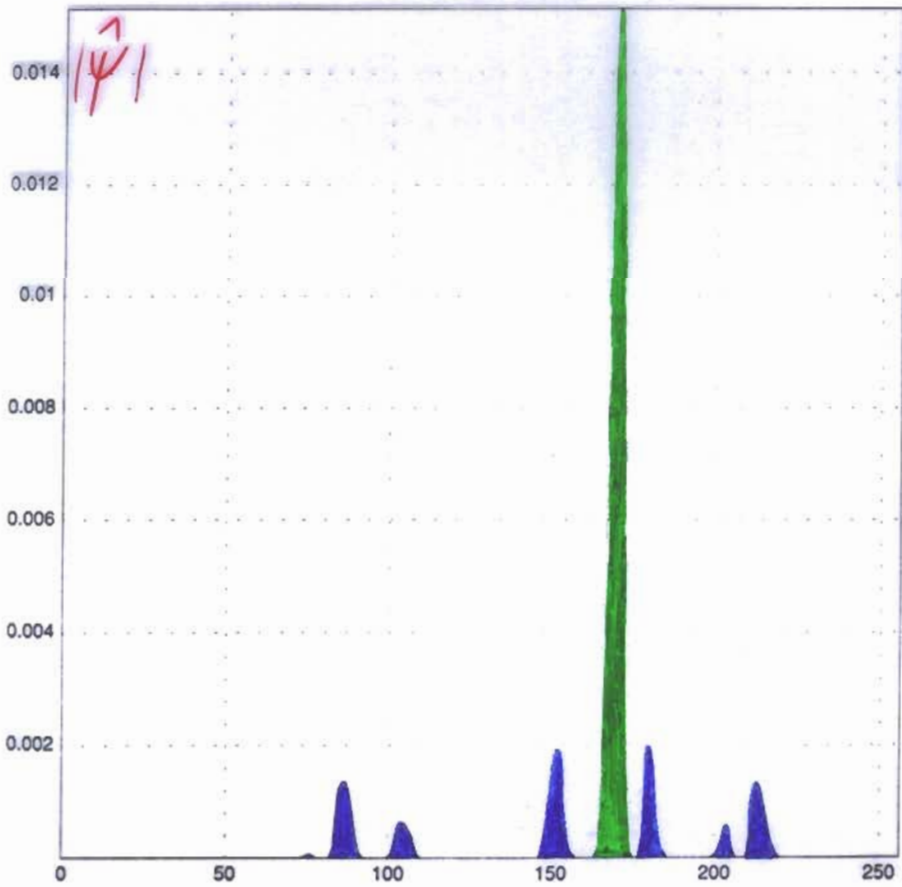
Coifman 6

coif.30.6.42.3.512



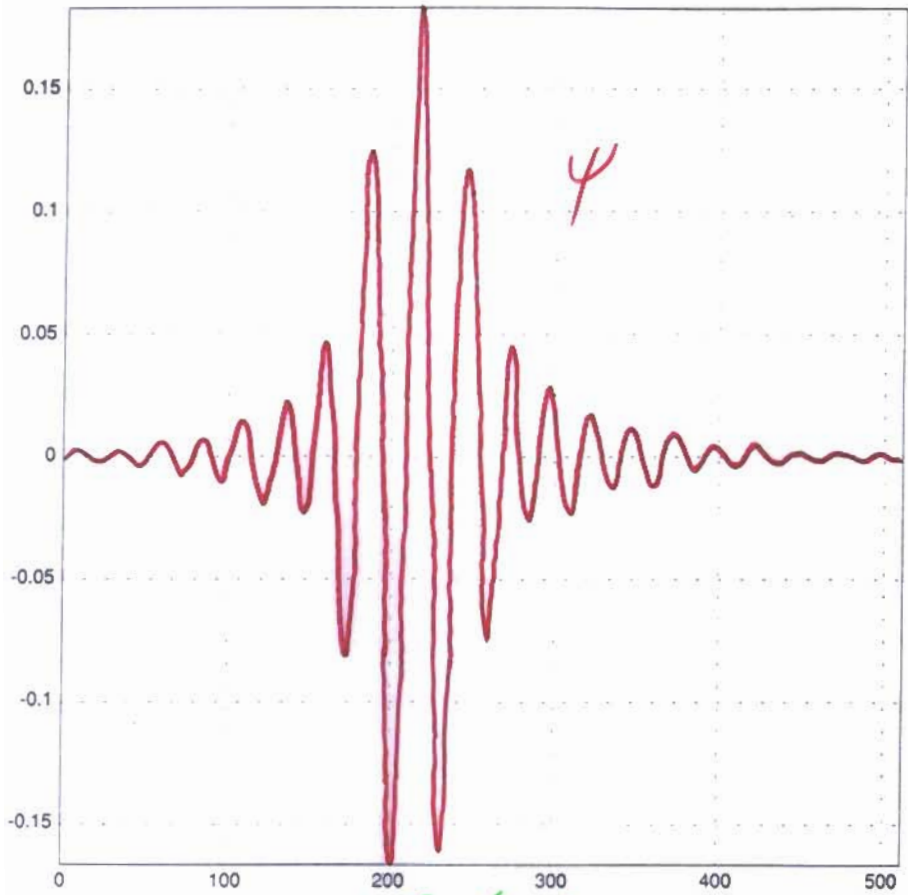
$j=6, k=42$

coif.30.6.42.3.512.fou



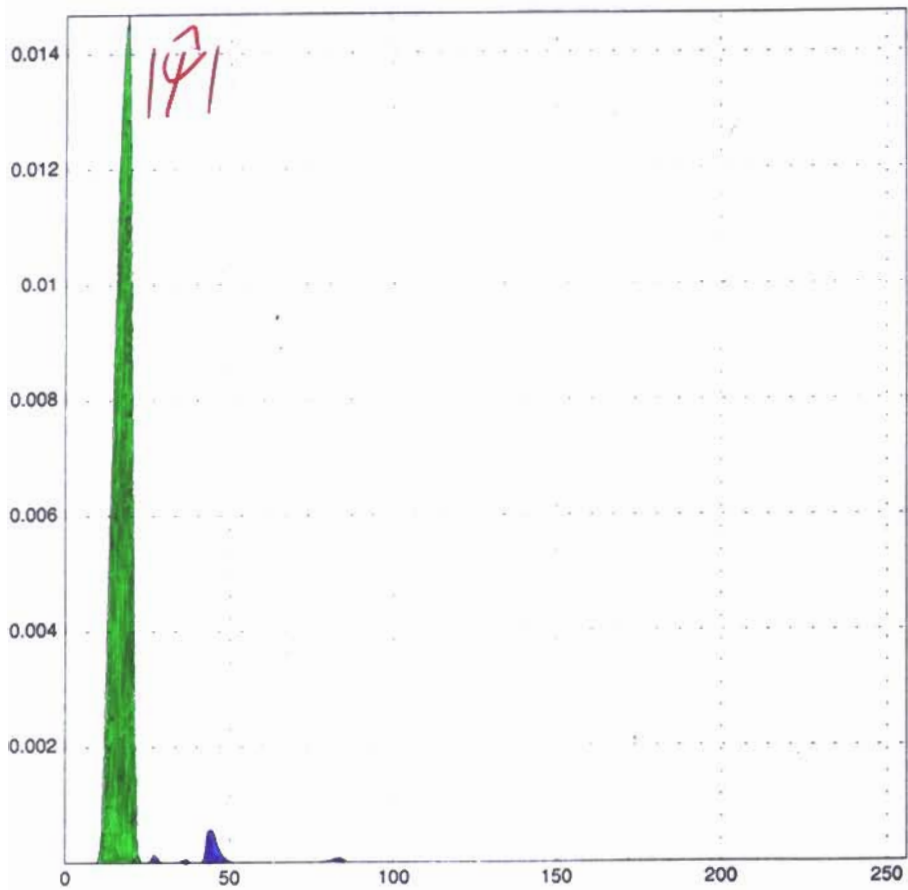
Coifman 30

coif.30.6.4.3.512



$j=6, k=4$

coif.30.6.4.3.512.fou



Coifman 30

ORTHOGONAL WAVELET DECOMPOSITION PACKET

$l = 2^0$
 $l = 2^J = 2^3 = \frac{1}{8}$ Sampling
 $l = 2^2 = \frac{1}{4}$ Smallest scale

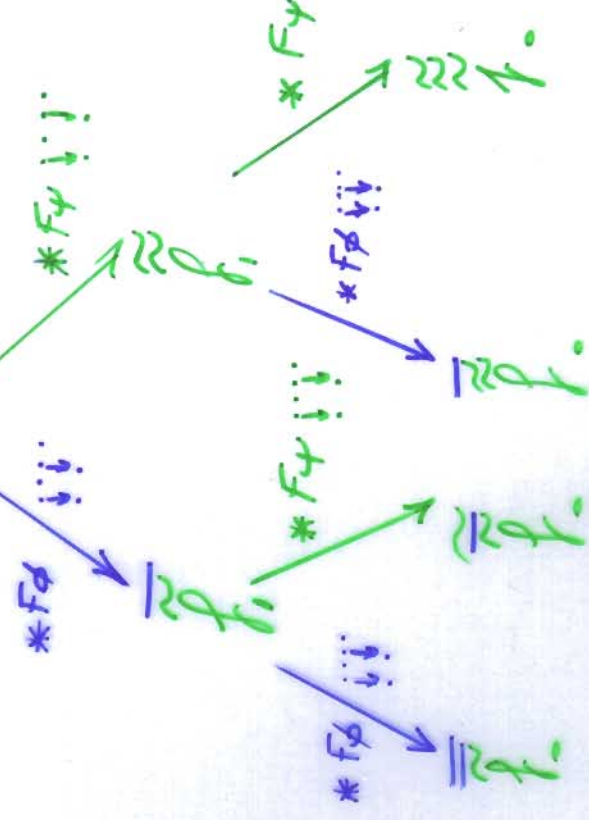
Signal f_3 $*F_4$

Trend f_2 $*F_4$

Fluctuation f_2 $*F_4$

$l = 2^1 = \frac{1}{2}$
 $*F_4$

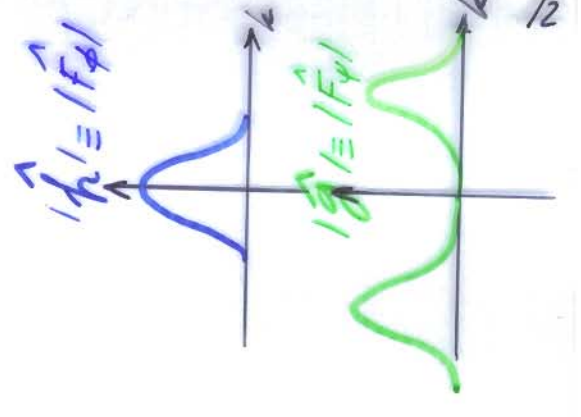
$l = 2^0 = 1$
 $*F_4$



Subsampling

QNF corresponding to the scaling function ϕ_j
 QNF corresponding to the wavelet ψ_j

Signal f_0
 $\{f_0, \tilde{f}_0, \tilde{f}_1, \tilde{f}_2\}$

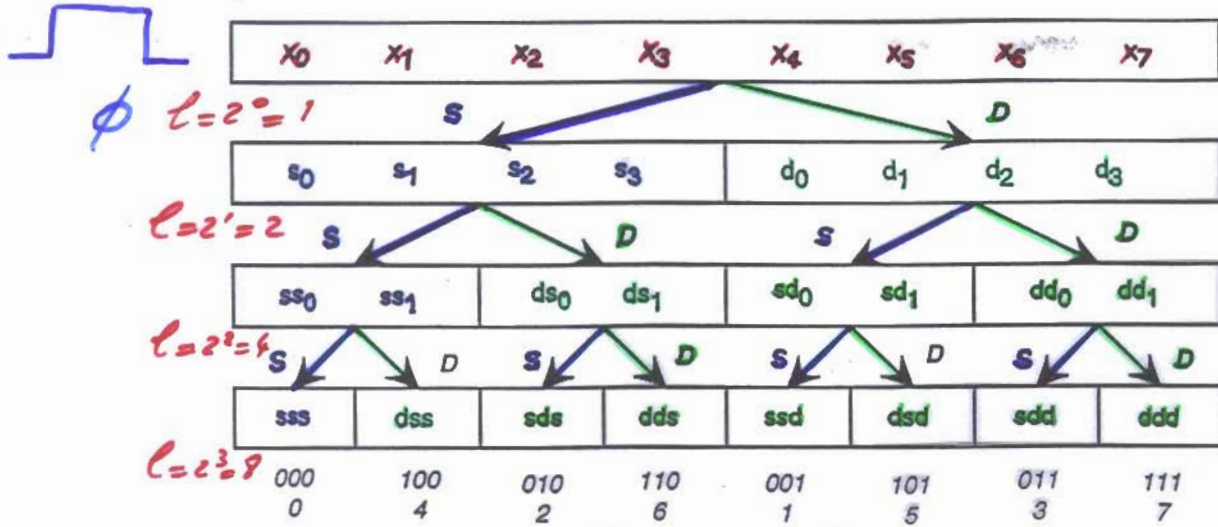


1D WAVELET PACKET BASES

Smoothing S

Naturally ordered wavelet packets on \mathbb{R}^8

Detailing D



Example:
Wavelet packet bases generated from a 8 grid-point sampling using Haar wavelet filter: $h = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}$
 $g = \left\{ \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\}$

Haar wavelet packets on \mathbb{R}^8 :

Small
Smallest scale = level 1;

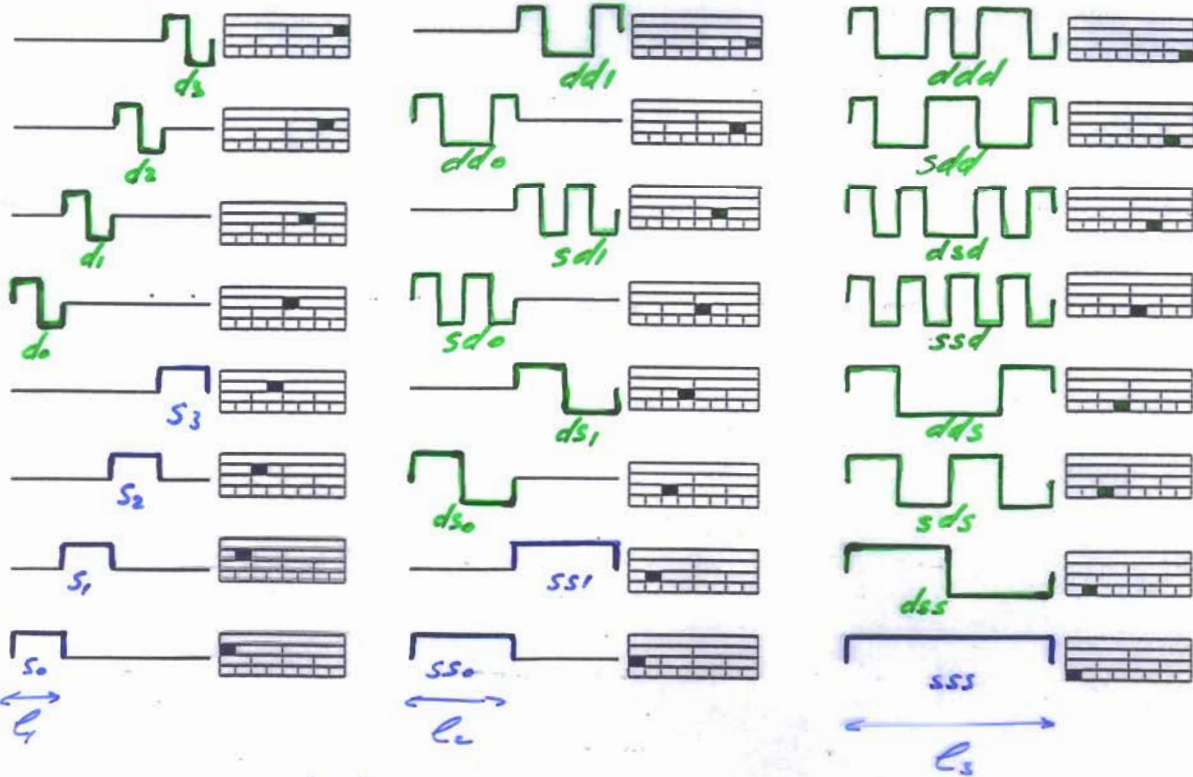
Intermediate
Intermediate scale = level 2;

Large
Largest scale = level 3.

$l=2^1=2$

$l=2^2=4$

$l=2^3=8$



$$ss_0 = \frac{1}{\sqrt{2}} (s_0 + s_1)$$

$$ss_1 = \frac{1}{\sqrt{2}} (s_2 + s_3)$$

$$ds_0 = \frac{1}{\sqrt{2}} (s_0 - s_1)$$

$$ds_1 = \frac{1}{\sqrt{2}} (s_2 - s_3)$$

A rectangle of wavelet packet coefficients.

$l=2^0=1$
 $l=2^1=2$
 $l=2^2=4$
 $l=2^3=8$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
s_1	s_2	s_3	s_4	d_1	d_2	d_3	d_4
ss_1	ss_2	ds_1	ds_2	sd_1	sd_2	dd_1	dd_2
sss_1	dss_1	sds_1	dds_1	ssd_1	dsd_1	sdd_1	ddd_1

Wavelet packet coefficients

A subband basis.

j							
0							
1							
2	ss_1	ss_2	ds_1	ds_2	sd_1	sd_2	dd_1 dd_2
3							

Windowed Walsh Sasis

The wavelet basis.

j							
0							
1					d_1	d_2	d_3 d_4
2			ds_1	ds_2			
3	sss_1	dss_1					

Wavelet Sasis

An orthonormal basis subset.

j							
0							
1	s_1	s_2	s_3	s_4			
2							dd_1 dd_2
3					ssd_1	dsd_1	

Other Sasis

BEST BASIS REPRESENTATION

It is the orthogonal basis made of wavelet packets which gives the smallest Shannon entropy expansion of the vorticity

$$S_B = \min_{\mathcal{B}} S = -\sum_{\mathcal{B}} |\langle w/\psi \rangle|^2 \log |\langle w/\psi \rangle|^2$$

It corresponds to the smallest measure of distance between w and its WP orthogonal decomposition $\langle w/\psi \rangle \Rightarrow$

Theoretical dimension of w : $d_B = e^{S_B}$

To compress information we then truncate in \mathcal{B} and

keep only $\langle w/\psi_B \rangle > \varepsilon$.

The threshold ε can be defined a priori if we assume that the incoherent components (those $< \varepsilon$) correspond to iid zero mean

Gaussian noise (cf. Donoho and Johnstone's Wavelet Shrinkage method)

BEST BASIS SELECTION

For a given signal f , made of N samples, we project it onto all orthogonal bases B_n ($n=1, 2^N$):

$$\tilde{f}_{ni} = \langle f | \psi_{ni} \rangle$$

For each projection n we compute the cost function:

$$c_n = \sum_i \tilde{f}_{ni}^2 \log \tilde{f}_{ni}^2$$

and we select the projection which minimizes c as the best basis.

The total operation cost of the best basis selection is $N \log_2 N$ operations.

INFORMATION COST FUNCTIONS

We define a real-valued functional $\mu_m(f)$ on sequences $\{\tilde{f}_m\}$, set of all projections of the signal f onto all possible orthogonal bases Ψ_m , describing the cost of representing f in the basis Ψ_m .

Then we search for the most economical representation, which gives the best basis for f .

POSSIBLE $\mu_m(f)$

- Number of coefficients \tilde{f}_m above a threshold value ε :

$$\mu_m(f) = \# \{ \tilde{f}_m < \varepsilon \}$$

- ℓ^p norm with $p < 2$:

$$\mu_m(f) = \sum_i |\tilde{f}_{mi}|^p$$

- Shannon-Weaver information entropy:

$$\mu_m(f) = - \sum_i |\tilde{f}_{mi}|^2 \log |\tilde{f}_{mi}|^2$$

- logarithm of energy:

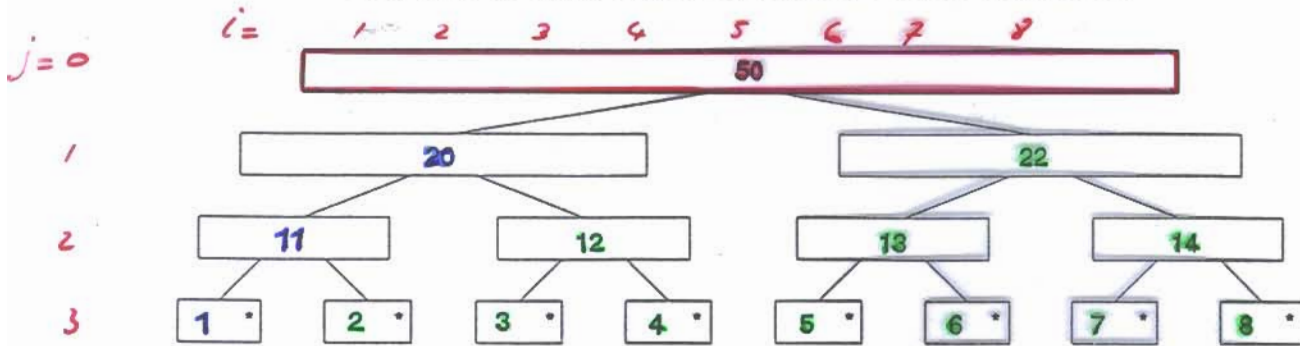
$$\mu_m(f) = \sum_i \log |\tilde{f}_{mi}|^2$$

- Bit counting:

$$\mu_m(f) = \sum_i \log_2 \left(1 + \frac{\tilde{f}_{mi}}{\varepsilon} \right)$$

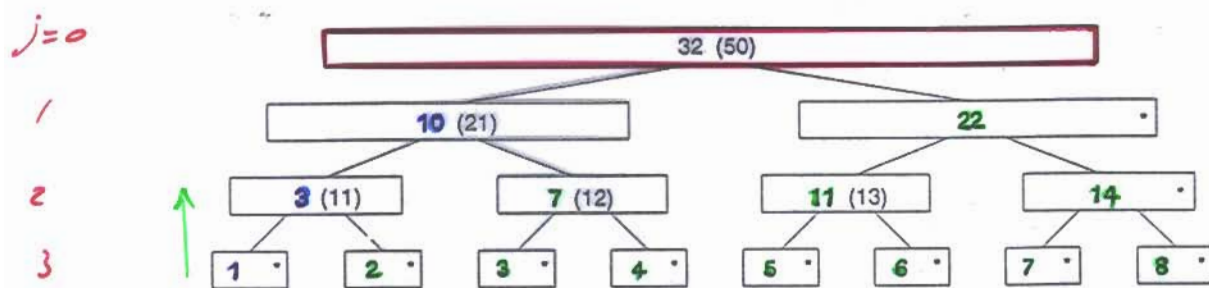
with ε machine precision.

Step 1 in the best basis search: mark all bottom nodes



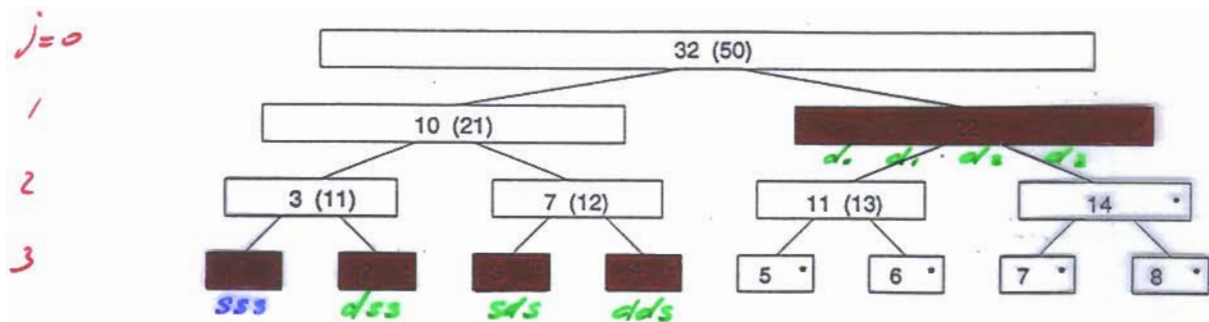
All possible waulet packets

Step 2 in the best basis search: mark all nodes of lower cost



Lower cost nodes

Step 3 in the best basis search: retain topmost marked nodes



Best Basis

THEORETICAL DIMENSION

The dimension of the representation in basis B_n is:

$$d_n = e^{-L_n}$$

It measures how many non negligible coefficients are present in the representation of f in basis B_n .

This dimension is minimal for the best basis representation of f which gives the theoretical dimension d_{th} of f . It is related to the dimension of the approximate inertial manifold characterizing the dynamical system which has generated f .

$$d_{th} \ll N \quad \text{low-order dynamics}$$

$$d_{th} = N \quad \text{white noise}$$

THEORETICAL DIMENSION

We can compare
sequences $\{\tilde{f}_m\}$ by
their rate of decay,
namely the rate at
which their elements
become negligible if they
are rearranged in
decreasing order.

The theoretical dimension
of $\{\tilde{f}_m\}$ is:

$$d = \exp \left\{ - \sum_i |\tilde{f}_{mi}|^2 \log |\tilde{f}_{mi}|^2 \right\}$$

\Rightarrow
the most efficient represen-
tation of f will correspond
to the lower dimension d .

BEST BASIS SELECTION ALGORITHM

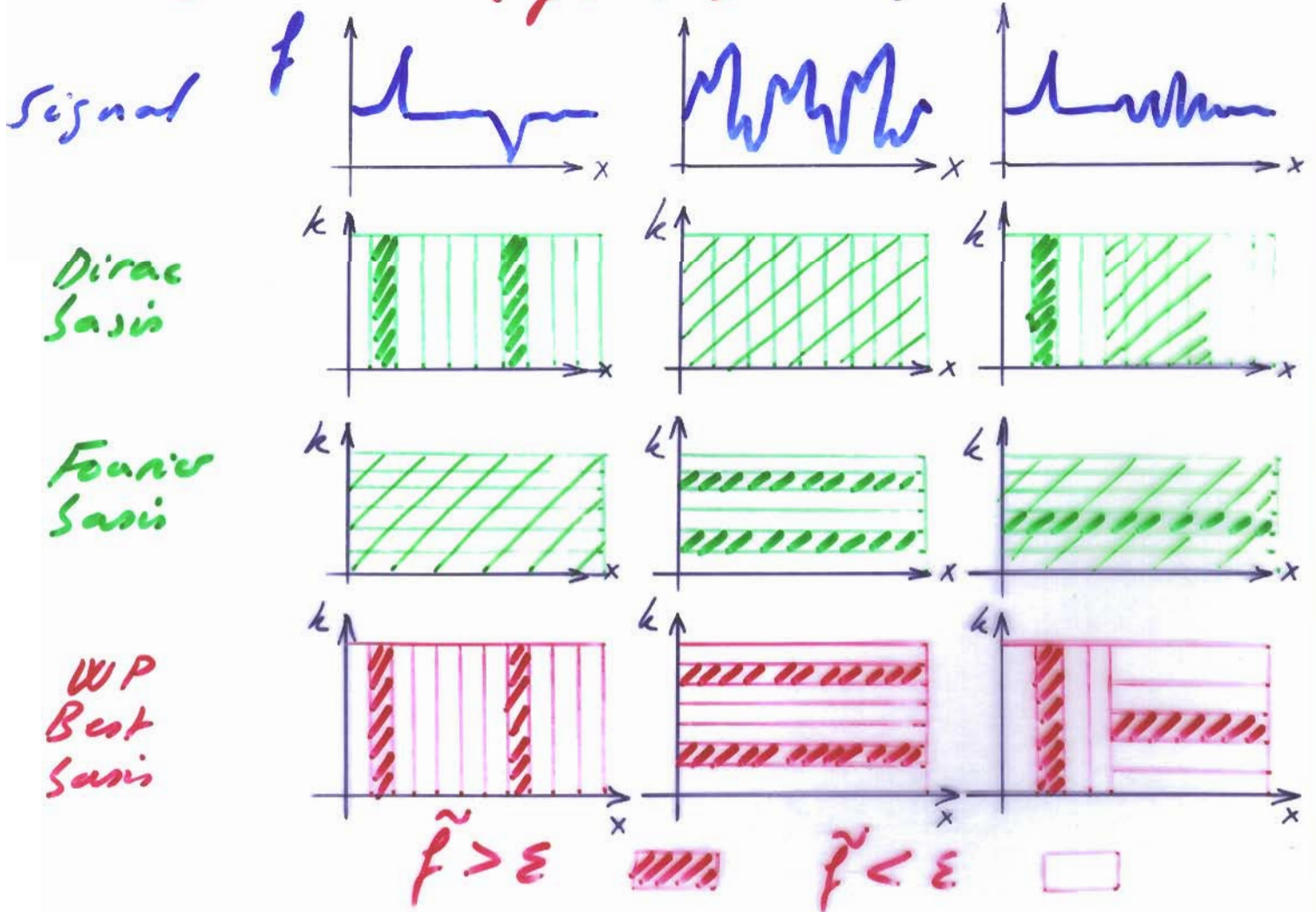
The field is defined on $N = 2^J$ grid-points

1. Then we generate the $2^{N/2}$ possible orthogonal bases constructed with $\psi_{i,j,k}^m$
2. We project w onto each possible orthogonal basis m
3. We compute the information entropy $S_m = -\sum_{i=1}^{2^{N/2}} w_{mi} \log w_{mi}$ for each projection m
4. We select the projection m which maximizes the entropy S_m
5. We project w onto the best basis m and get N coefficients \tilde{w}_n
6. We select the $N' \ll N$ strongest coefficients \tilde{w}_n
7. We reconstruct $w_>$ from the N' strongest coefficients
8. We reconstruct $w_<$ from the $N - N'$ weakest coefficients

The computation of the whole procedure requires $N \log_2 N$ operations.

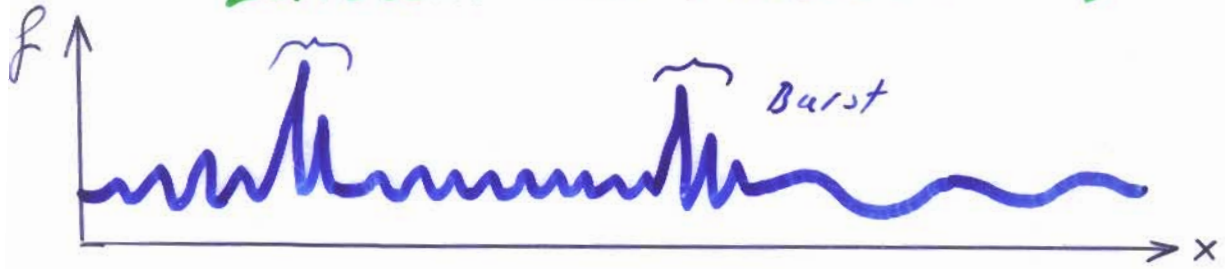
'BEST BASIS' SELECTION

Applied to a signal sampled on N points, the wavelet packet algorithm generates the 2^N possible orthogonal bases and then selects the 'best basis', i.e. the one which minimizes the number of significant coefficients ($\tilde{f} > \epsilon$). $C = \sum_i |f_i|^2 \log |f_i|^2$

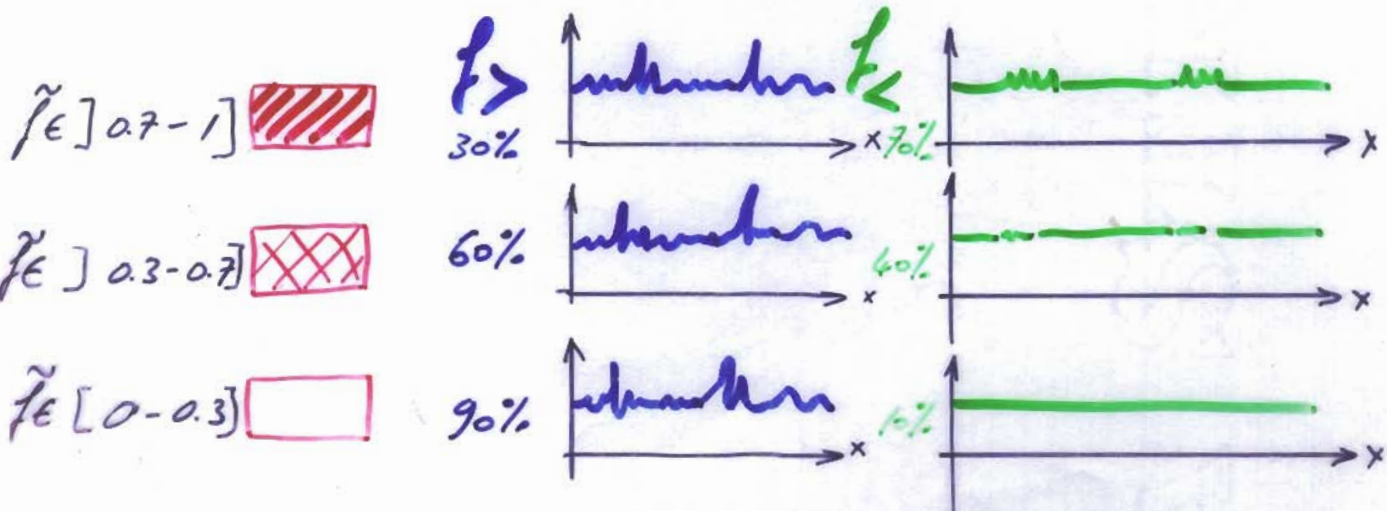
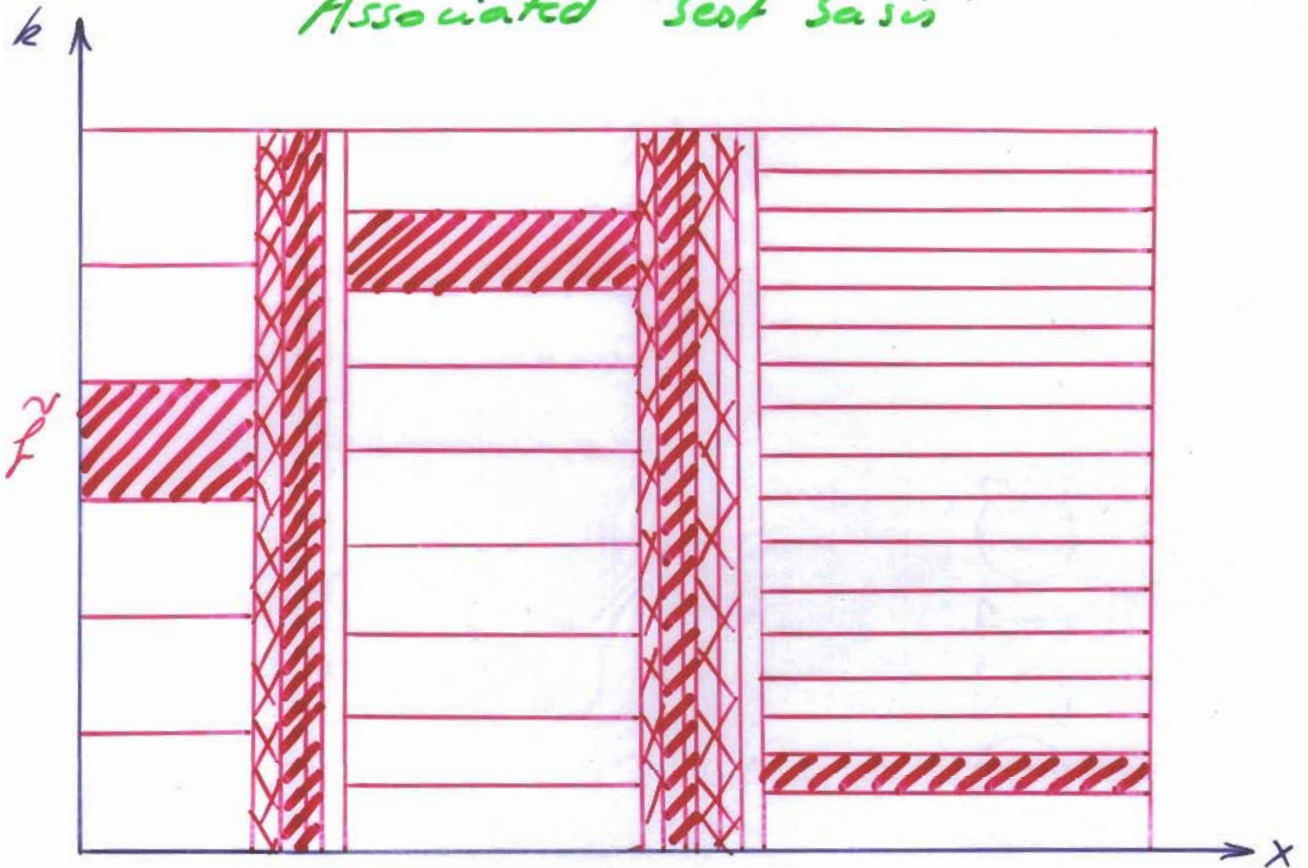


EXAMPLE

1D Intransient turbulent signal

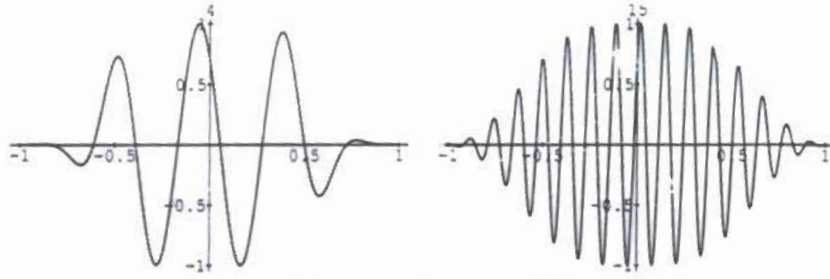


Associated 'Sect Basis'

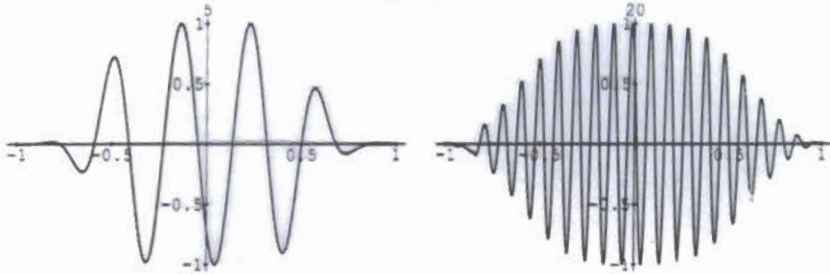


Reconstructions

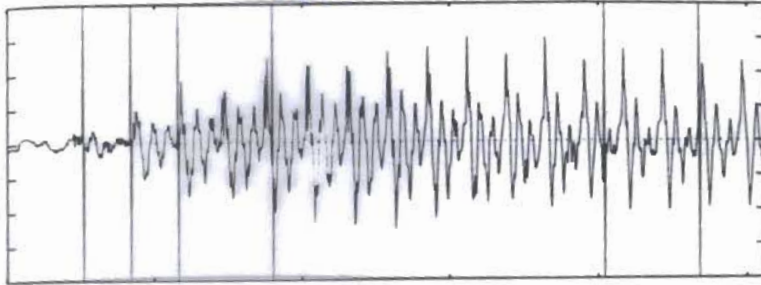
SPEECH SIGNAL COMPRESSION



*Local adapted
sines*



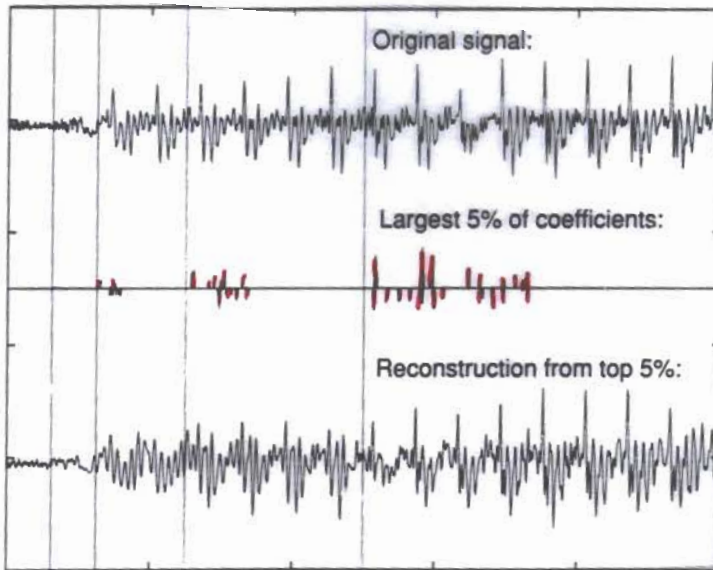
'Armadillo'



*Lowest
entropy
windows
≡
largest
efficiency
representation*

FIGURE 2. Automatic segmentation of a voice recording (armadillo) by using least entropy windowing in the local sine library. The windows are selected to obtain optimal efficiency in representing the signal. We see different patterns fall into distinct windows.

'Armadillo'

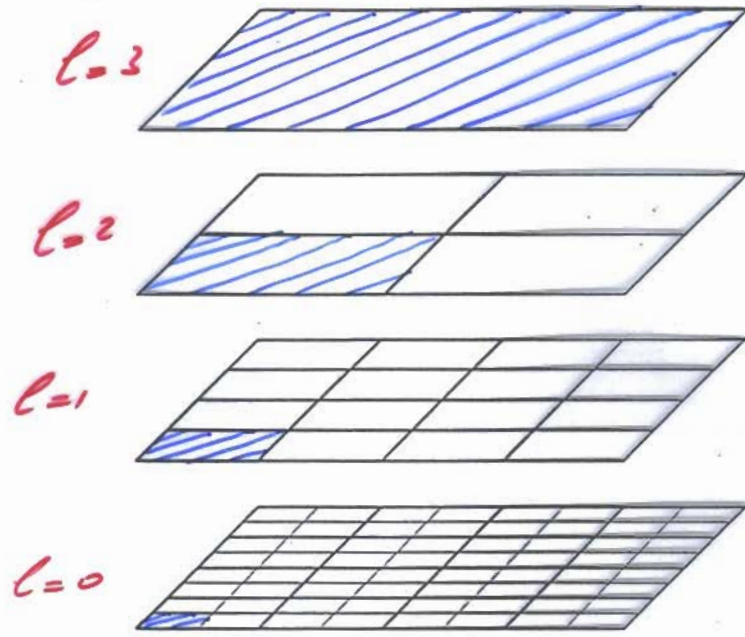


*Selection of
windows of
lowest
entropy
Sorting of the
coefficients
Reconstruction
from the top
5% coefficients*

FIGURE 1. The first 1024 samples ($\frac{1}{8}$ second) of the word armadillo, are plotted on the top part. The library of local sine waveforms is then used to select the combination of windows of highest efficiency (lowest entropy). Expansion coefficients are then ordered by window in decreasing order. The top 5% are plotted in the center and used to reconstruct a compressed form of the signal which is plotted below.

2D WAVELET PACKET BASES

2-dimensional wavelet decomposition to level 3



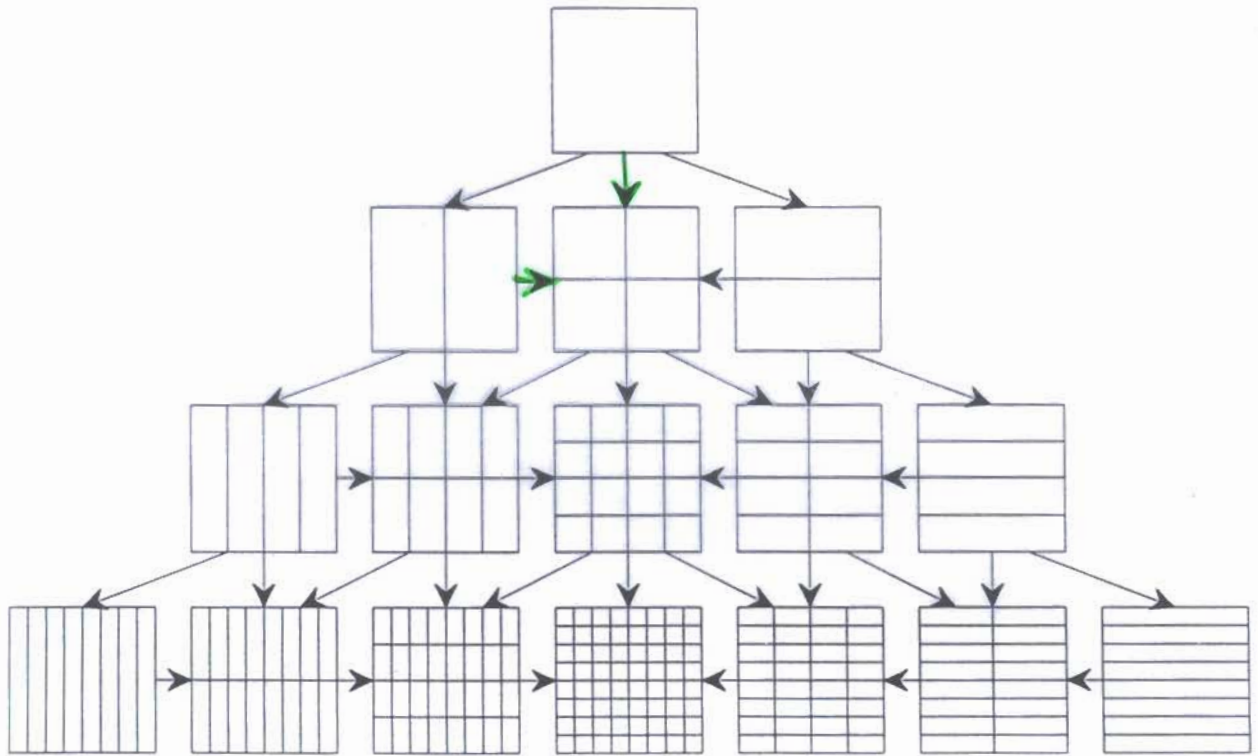
Example 2-dimensional wavelet packet bases



2D WAVELET PACKET ALGORITHM

→ Convolution-decimation
in the X direction

↓ Convolution-decimation
in the Y direction



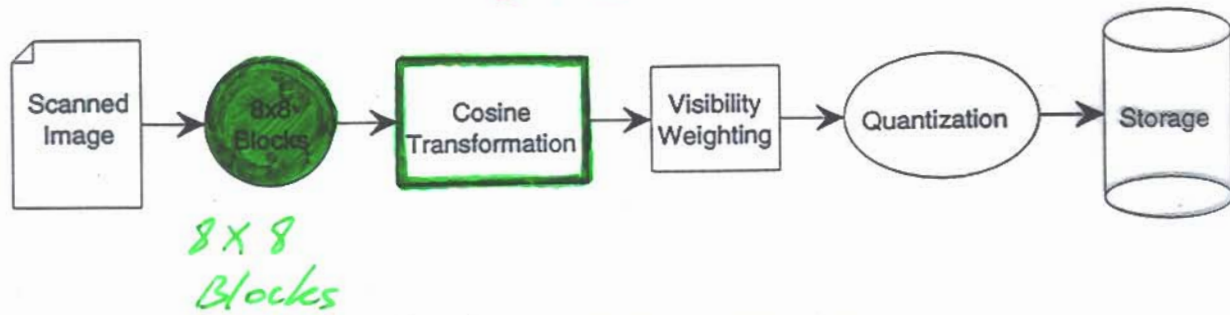
All tensor products
of Wavelet Packets
in 2D

There are $O[N(\log N)^2]$
coefficients

IMAGE COMPRESSION

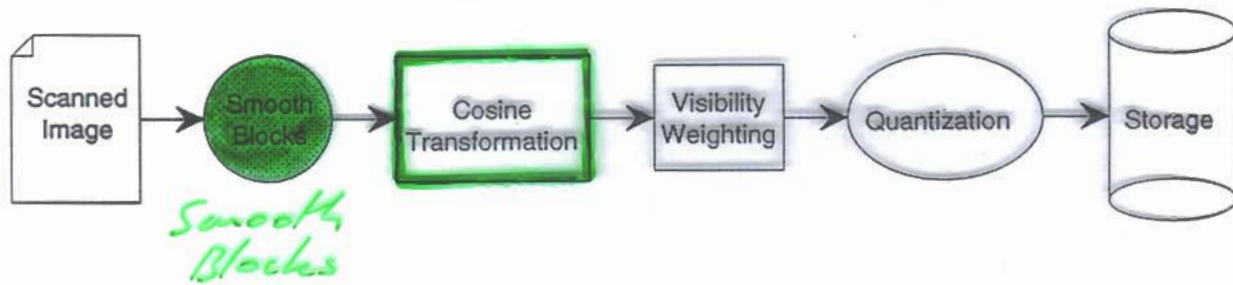
JPEG picture compression

JPEG



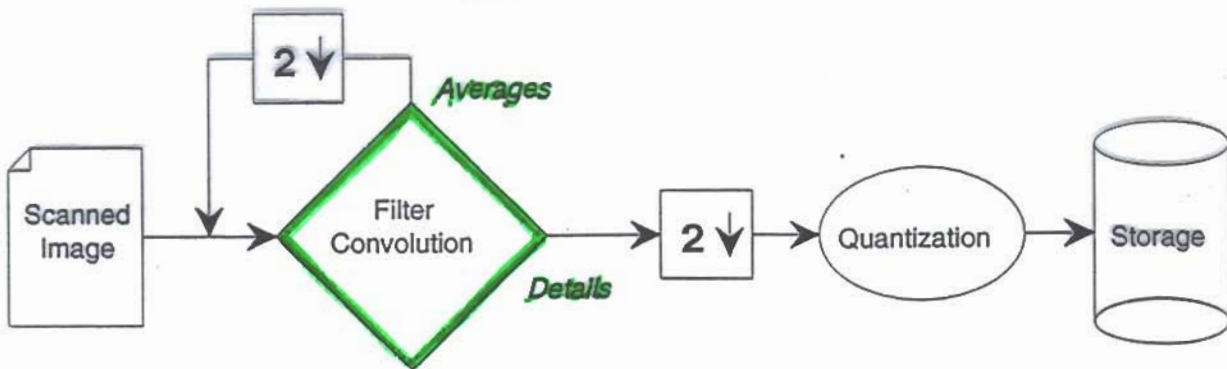
Local cosine picture compression

Local Cosines



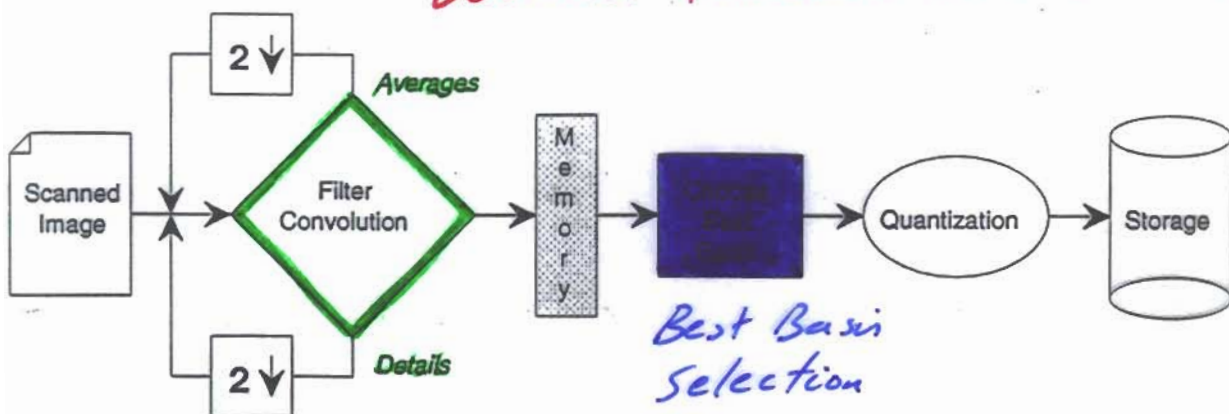
Wavelet or multiresolution picture compression

Wavelets



Wavelet packet best-basis picture compression

Wavelet Packets (JPEG 2000)



TURBULENT FLOW COMPRESSION

We are looking for the most concentrated representation of the vorticity field, in order to have the lowest dimensional system to compute and to store.



We want to find the orthogonal basis which will avoid unwanted complexity and therefore optimally concentrate information.



Karhunen-Loève
(POD or PCA)
approach

designed for
statistical ensembles

Considering a set of realizations and inverting the correlation matrix, it provides an orthonormal basis which is optimal in a L^2 average sense

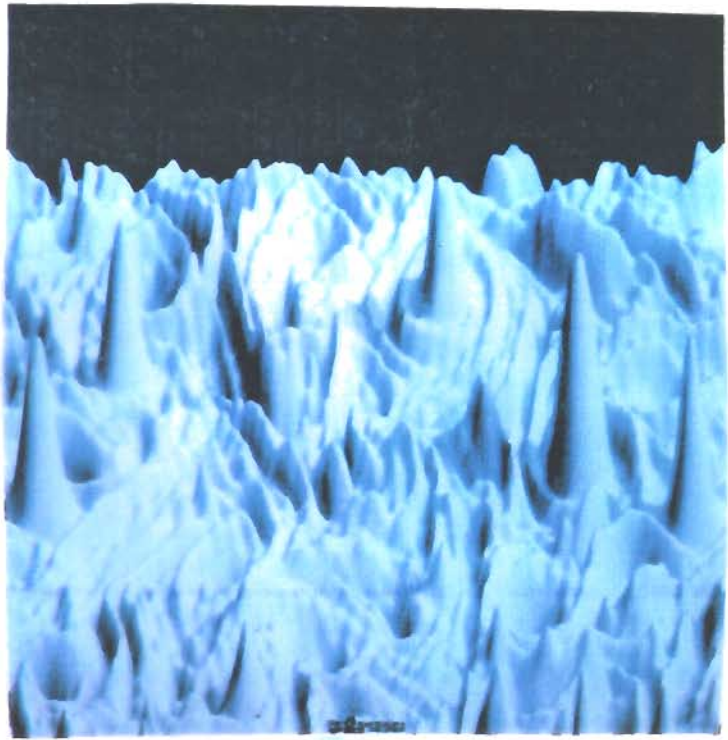
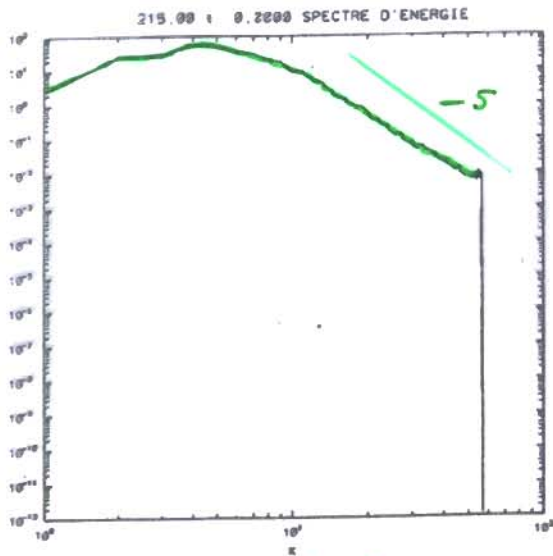
⇒ N^3 operations

'Best Basis'
approach

designed for
a given
realization

Search the basis which minimizes entropy out of a large library of orthogonal bases

⇒ $N \log N$ operations



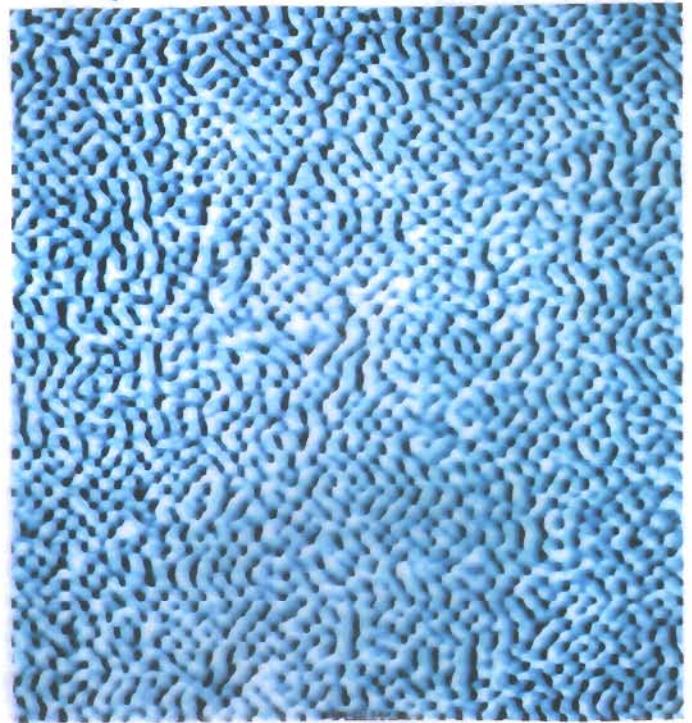
Vorticity field
 $N = 256^2$

Compression
 by 2

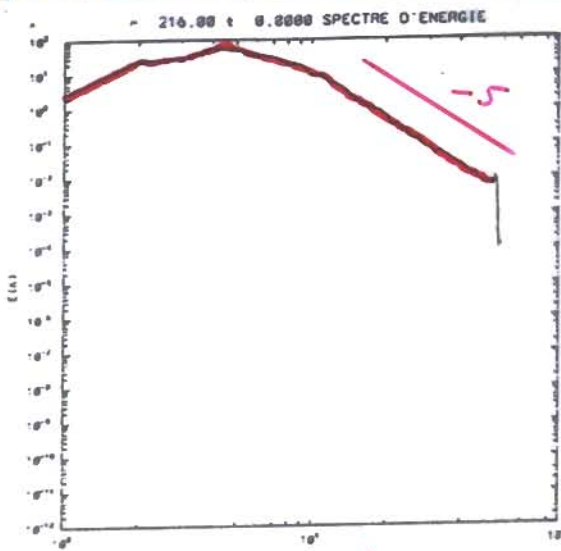
$\omega_1 \rightarrow 50\% N$

ω_1

$\omega_2 \rightarrow 50\% N$



+

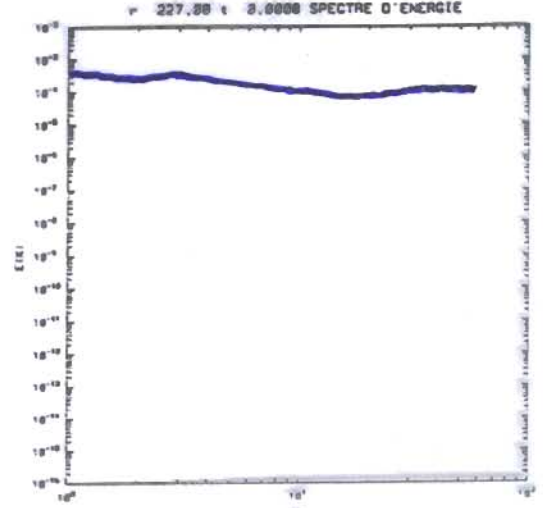


Splitting
 using
 orthogonal
 wavelet
 packets

99.96%

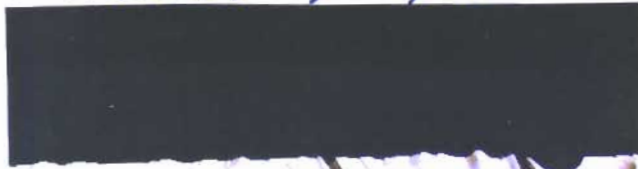
(Farr & al.
 Fluid Dyn. Res.
 10, 229, 1992)

'Active' vortices



'Passive' background

$N = 262144$ grid-points = 512^2



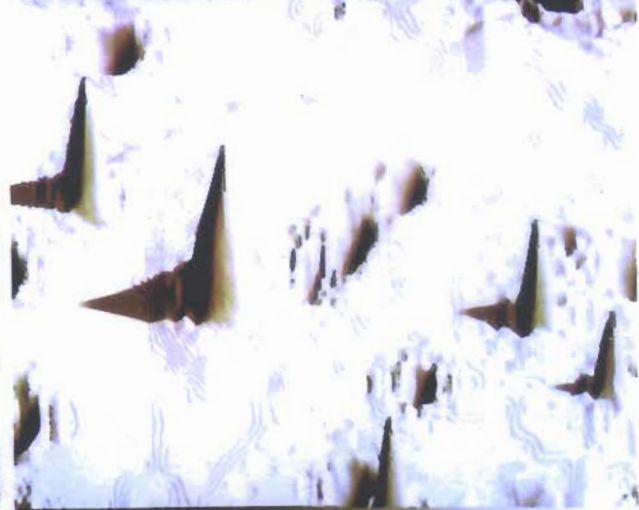
Vorticity
field
reconstructed
from the
Strongest
Wavelet
Packets



99.9% ϵ
3000 WP
1.14% N

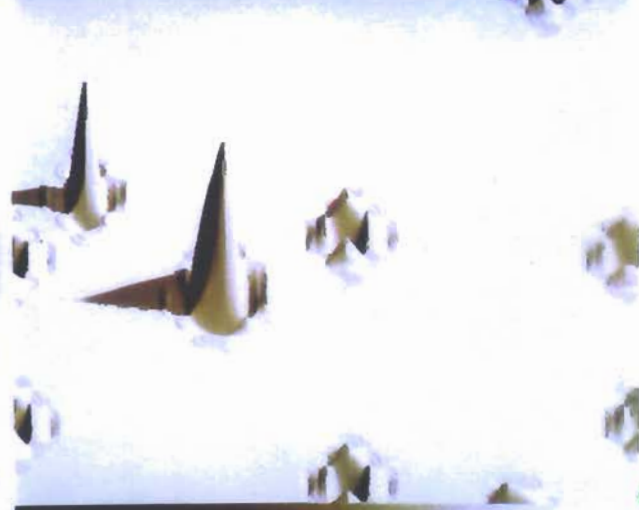
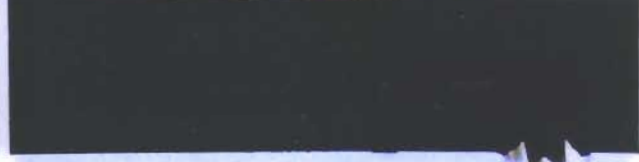
99%
739 WP
0.28% N

...TY.512.sec



...c12.ce989pc.rec

...c12.ce989pc.rec



...c12.ce989pc.rec

...c12.ce989pc.rec

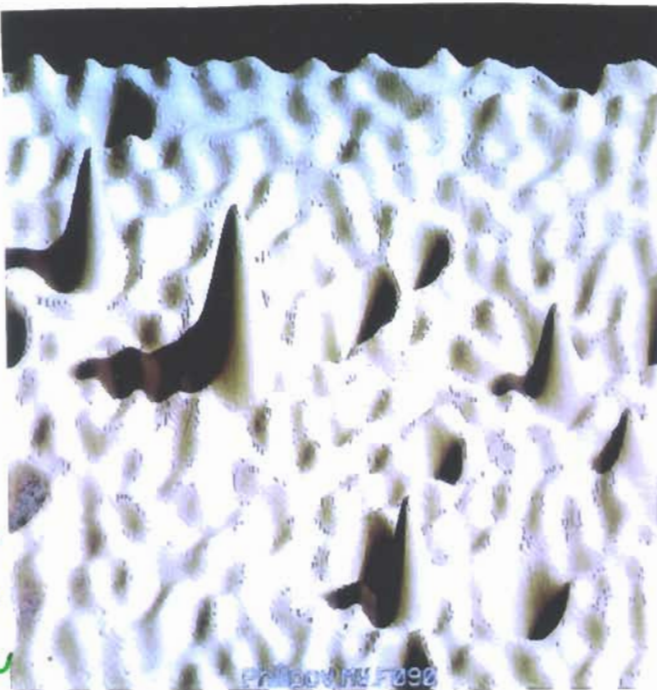
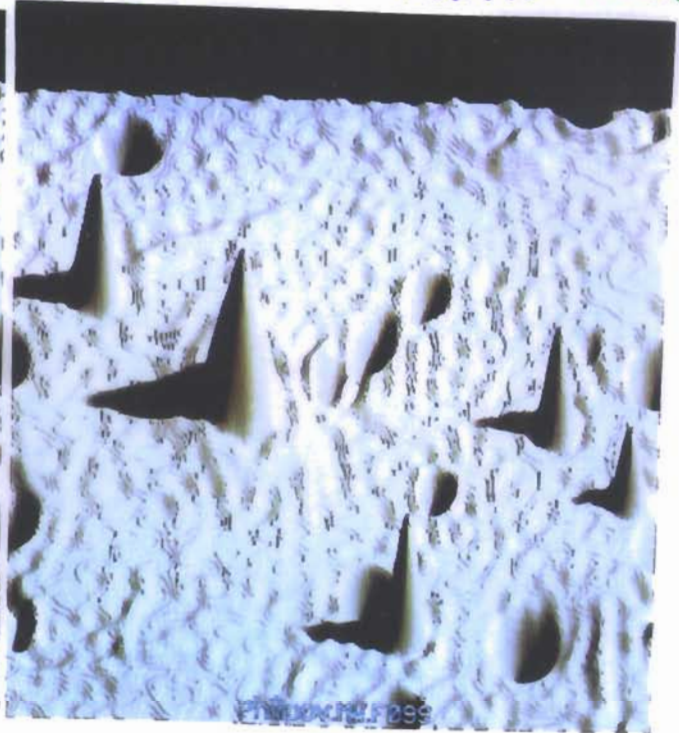
96% ϵ
73 WP
0.029% N

50% ϵ
14 WP
0.005% N



Vorticity
field
reconstructed
from the
strongest
Malar
wavelets
(locally
adapted
conius)
96% Z
739 MV 0.28% N

99.3% Z
3000 MW
1.14% N

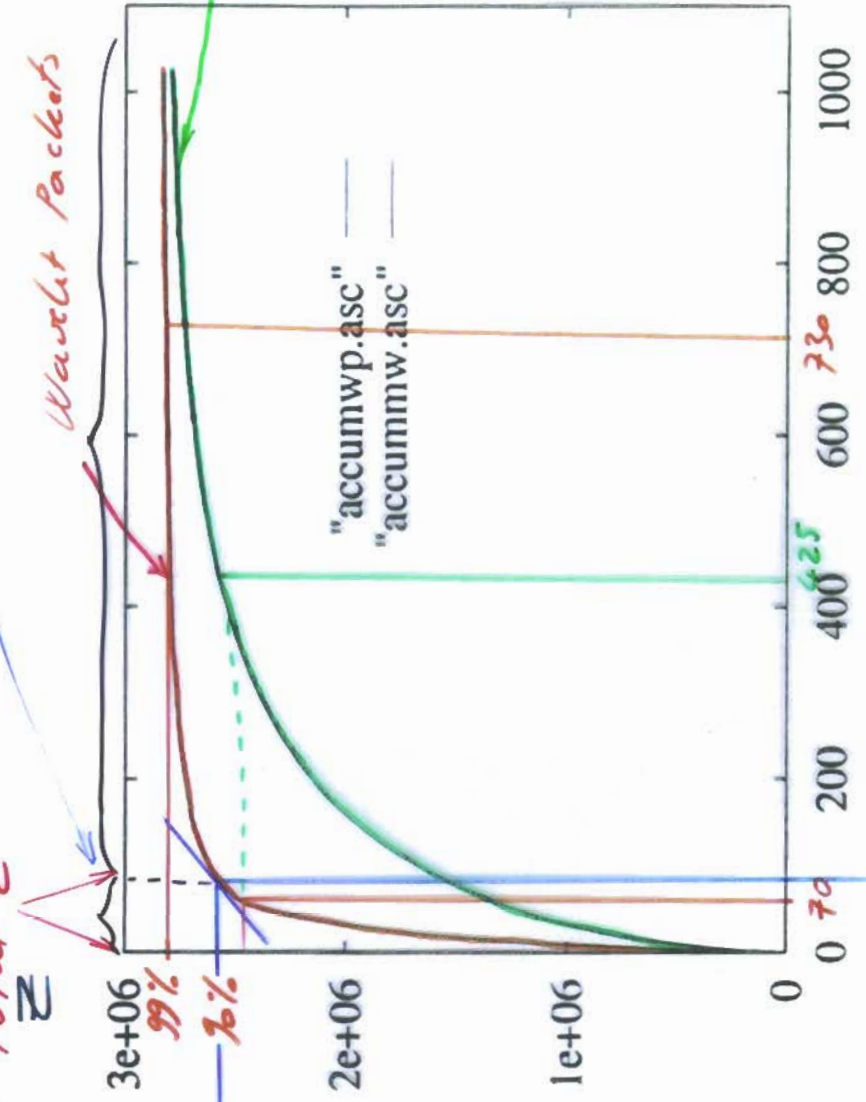


45% Z
73 MV
0.028% N

15% Z
14 MV
0.005% N

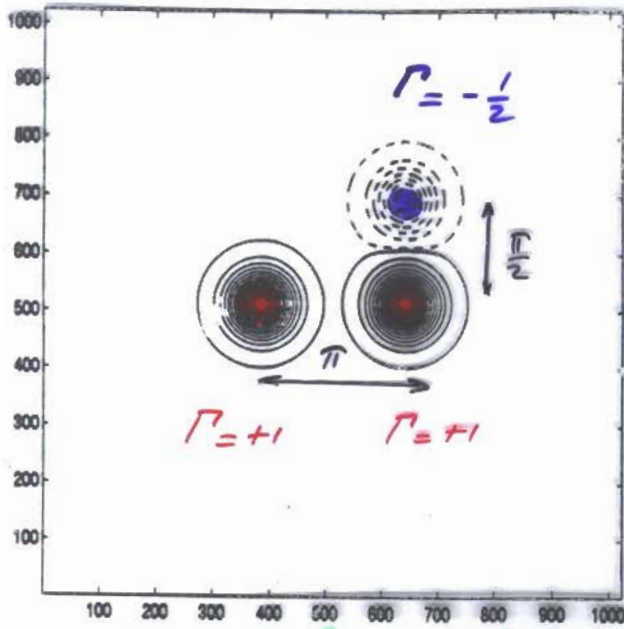
80 Coherent
 WP Components
 retain 93%
 of total Z

262064 Incoherent
 WP Components
 retain 7% of total Z



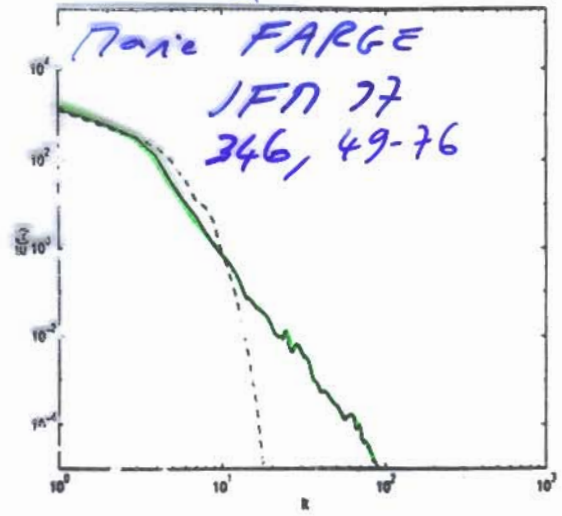
Number of coefficients = 262144 = 512²

80
 Optimal
 Compression
 (by 3277)

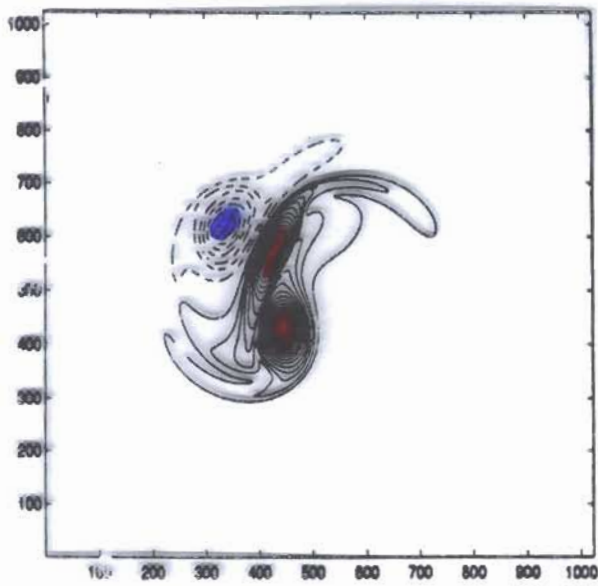


Inspired from Aref & Nosvikov
3 vortex configuration

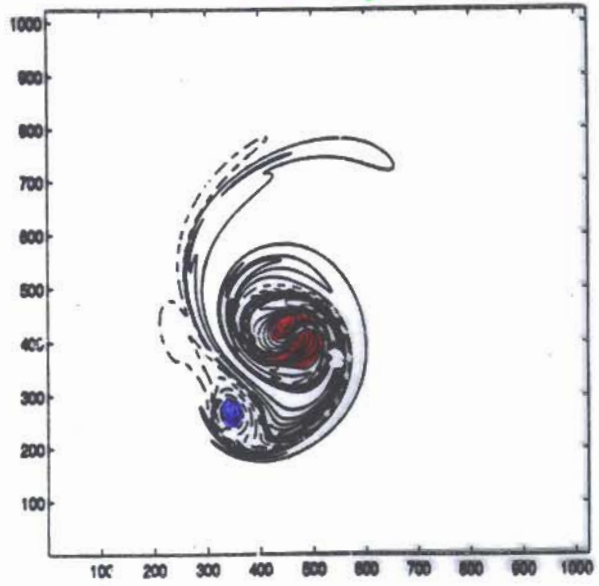
Nicholas KEULAHAN



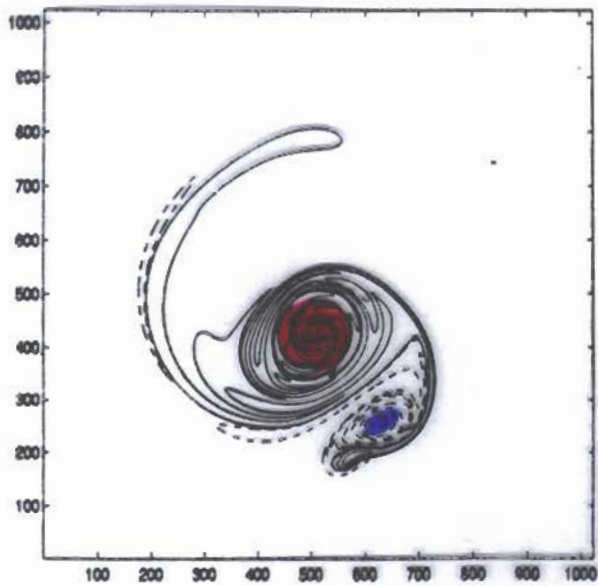
$N = 1024^2$
 $D = 5 \cdot 10^{-5} D^2$
 $Re = \frac{U}{\nu} = 2 \cdot 10^9$



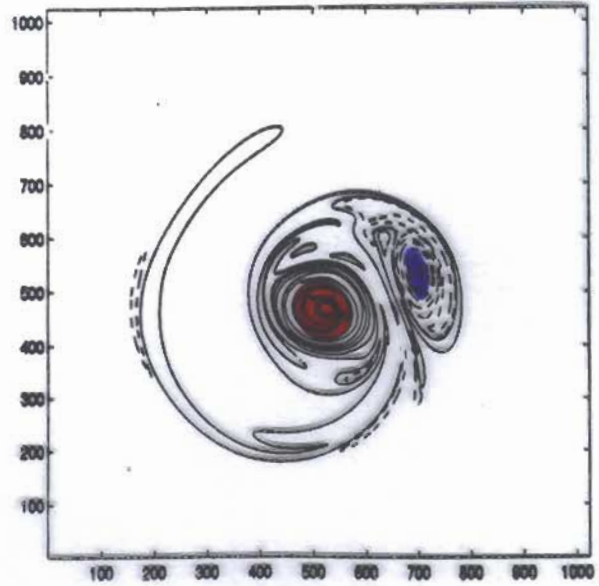
(a) $t = 10$



(b) $t = 20$

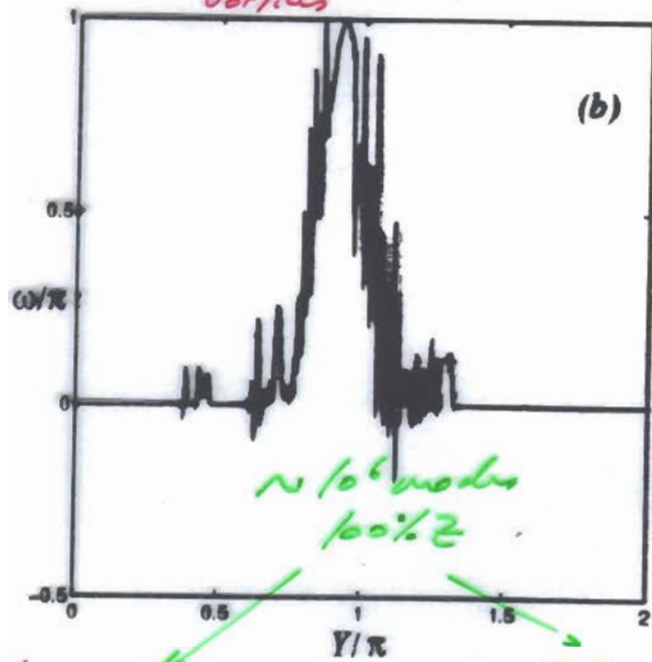
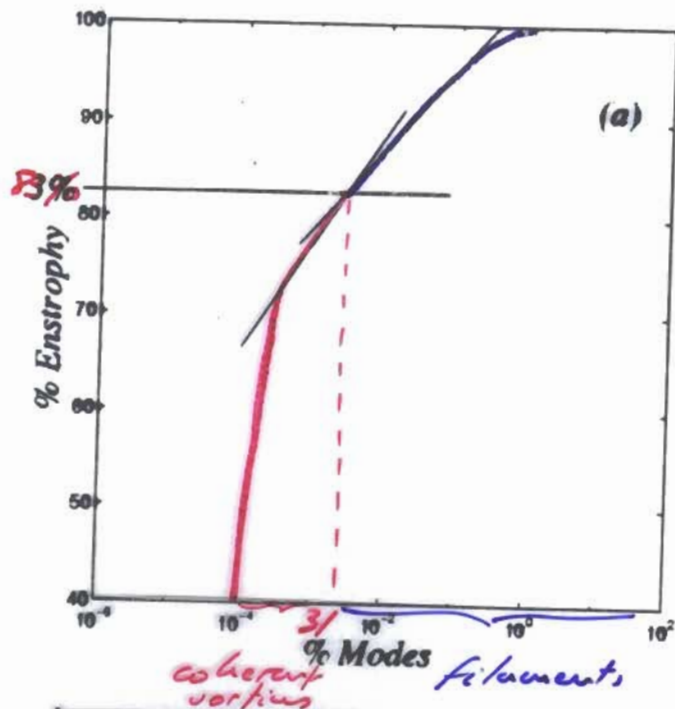


(c) $t = 30$



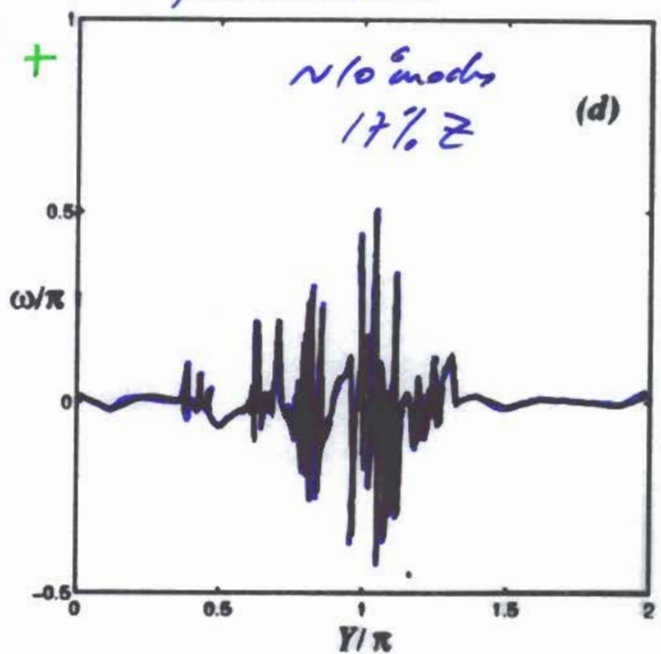
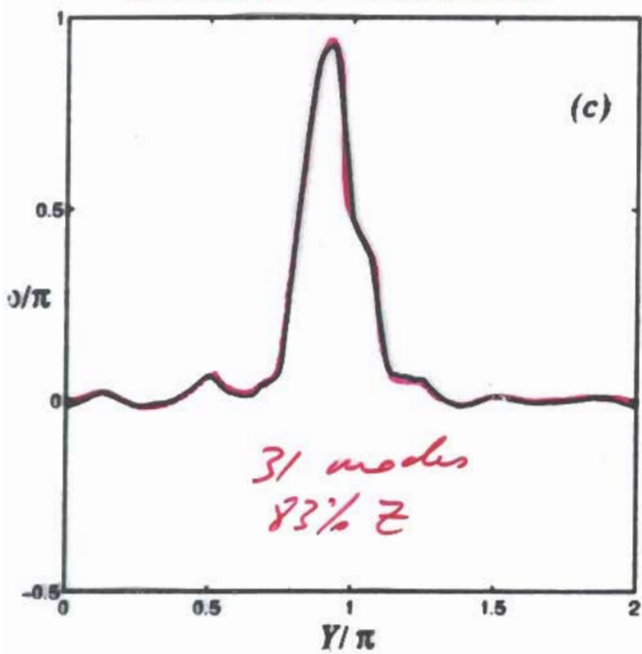
(d) $t = 40$

Pseudo-Fourier



Coherent vortices

filaments



+4.0e+00

$t=29$

With

Hyperdissipation

$D=40(-\nu^2)^8$

Resolution

1024^2

+0.0e+00

-2.0e+00



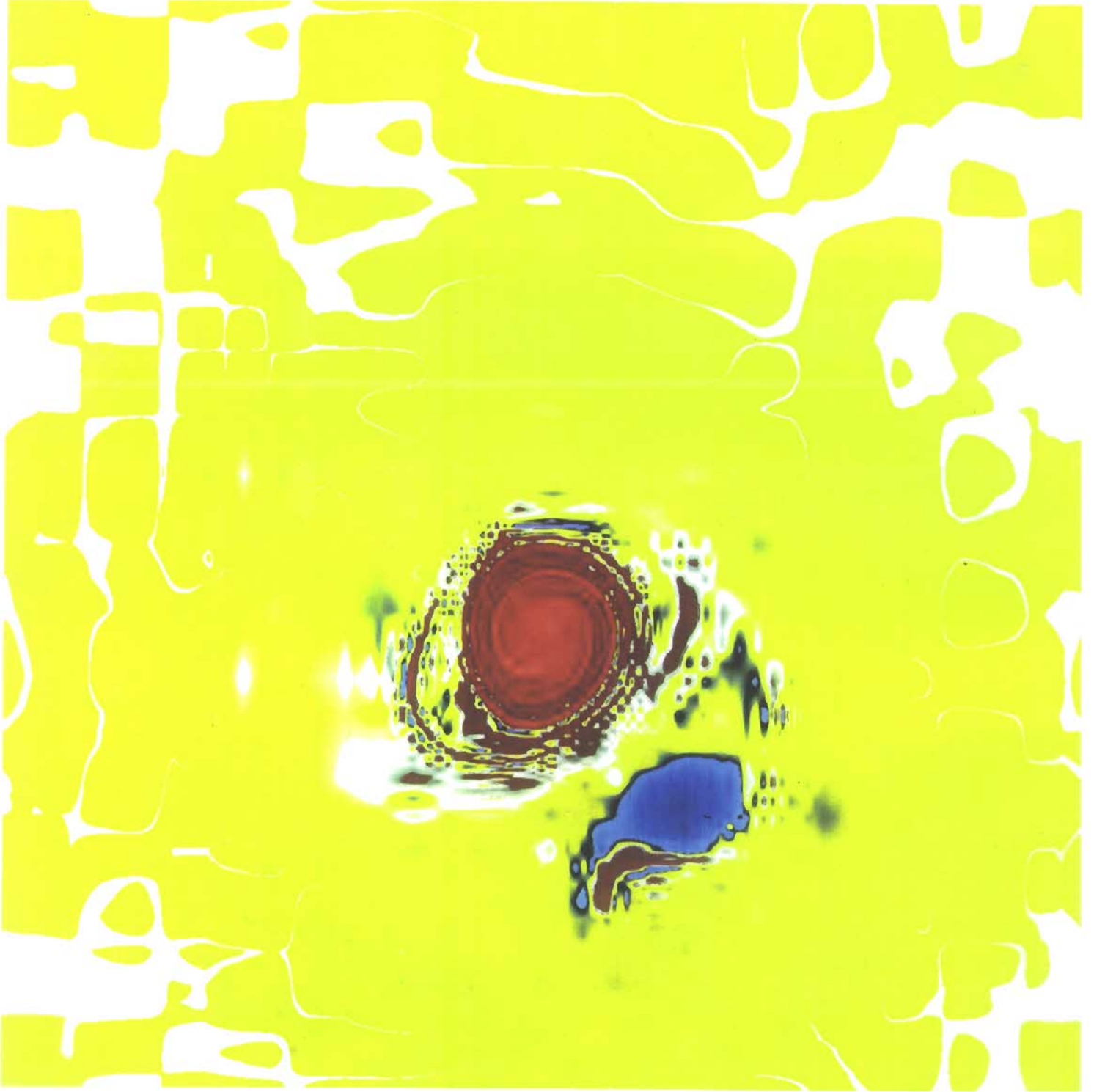
$+4.0e+00$

30%
catastrophic

292
Strong WP
coefficients

Compression
by 1200

$+0.0e+00$

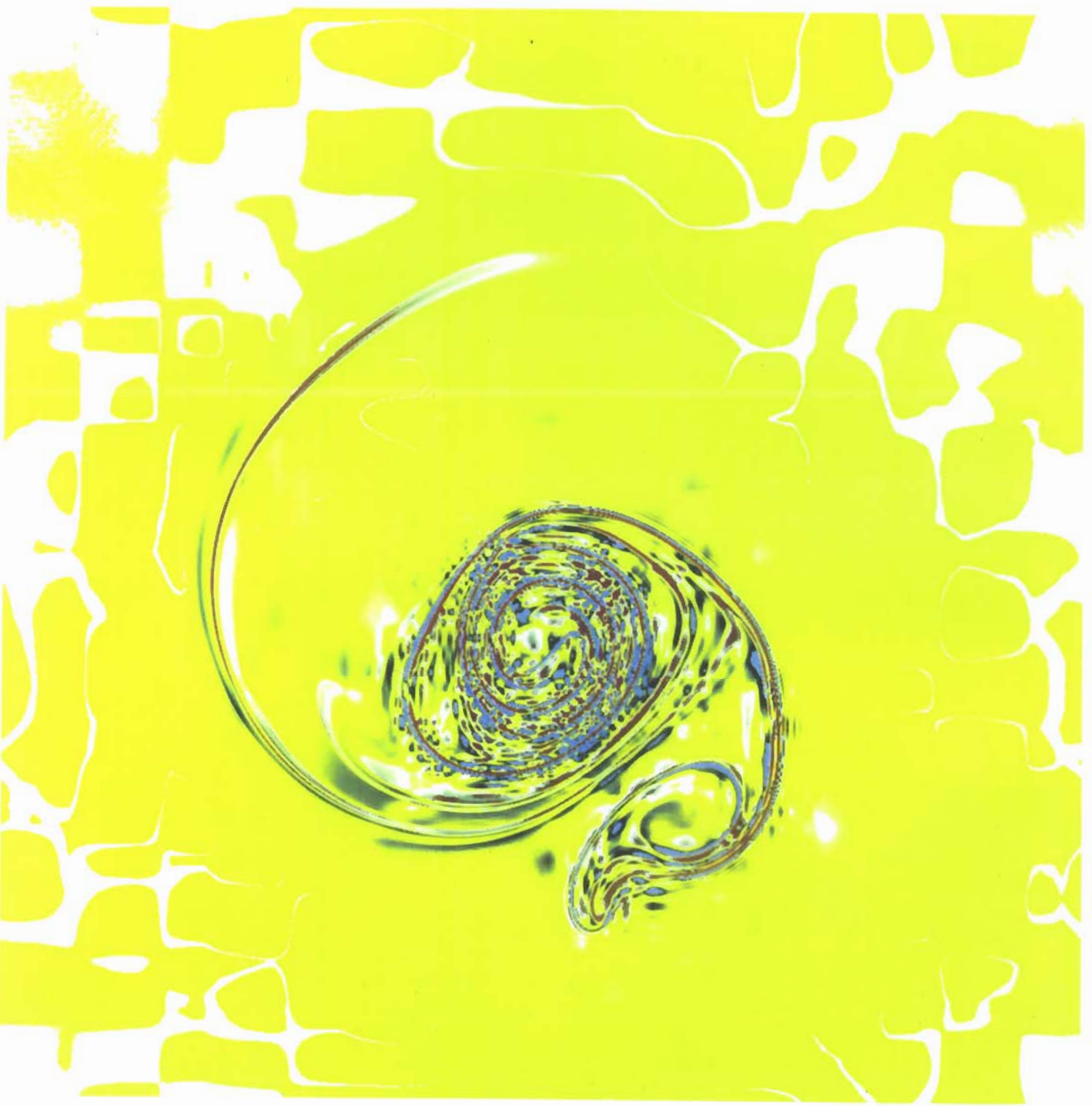


+4.0e+00

10%
catastrophy

933 209
weak WP
coefficients

+0.0e+00

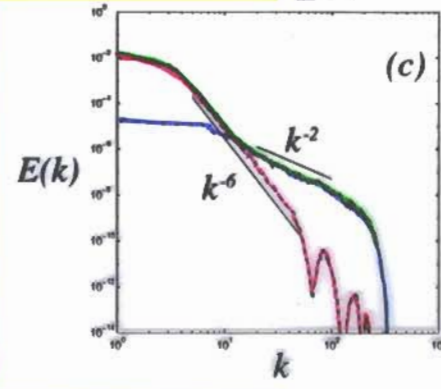
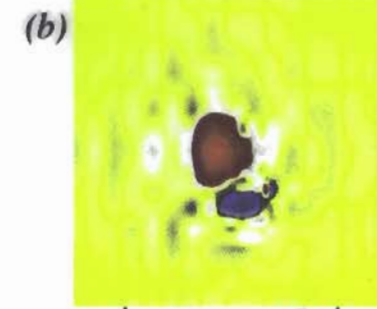
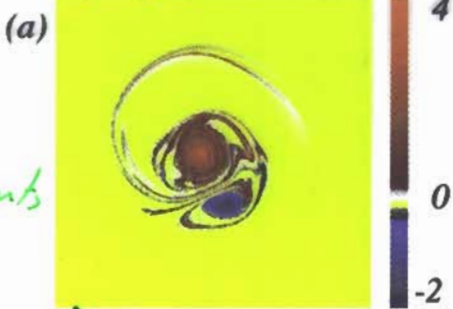


*Vortices
& filaments*

Vortices

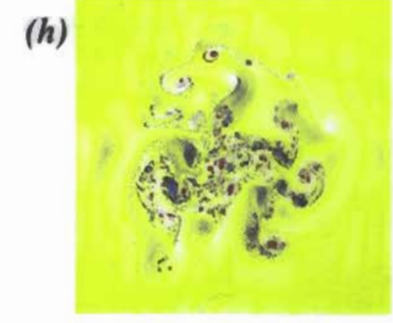
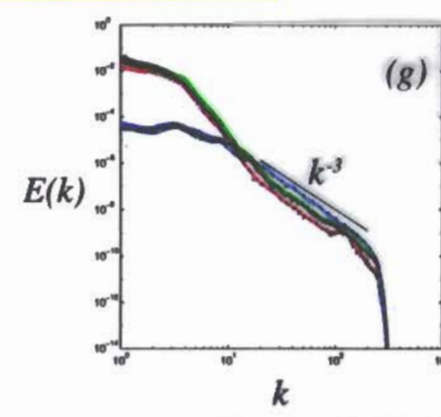
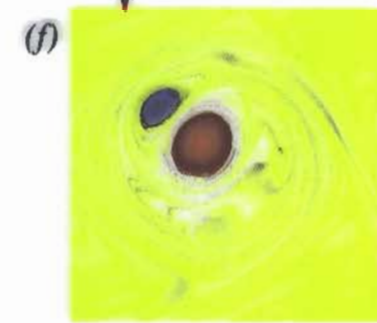
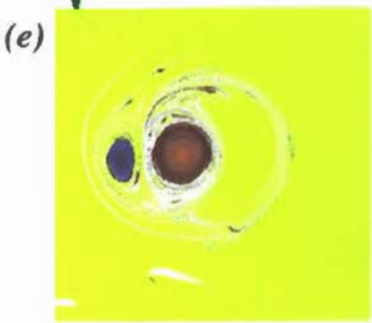
*~ 10⁶
coefficients
100% Σ*

*31 coefficients
83% Σ*



Filaments

*~ 10⁶
coefficients
17% Σ*



CONJECTURE

We have observed that the strain imposed by the coherent vortices inhibits the nonlinear instabilities to grow in the background



The background flow is stable to the coherent vortices and there is no backscatter from the background to the coherent vortices

Wavelet
Gap



We propose to model the background flow by a Gaussian (with a k^{-1} scaling) stochastic process.