NEED FOR A TIME-FREQUENCY REPRESENTATION

Jean VILLE, 1948
'Théorie et application de la notion de signal analytique',
Cables et transmissions.

'If we consider a musical piece and that a note, for instance an 'A', appears at least once in it, the harmonic analysis will represent the corresponding frequency with a certain amplitude and a certain phase, but without localizing the 'A' in time. However, it is obvious that during this musical piece there are instants for which we do not hear the 'A' note.

Although the Fourier representation is mathematically correct, because the phases of nearby notes are organized in such a way that they destroy by interference the 'A' when we do not hear it or reinforce it, also by interference when we hear it; but, if there is in this conception a skill which honours mathematical analysis, we should not hide the fact that there is a distormation of reality: indeed, when we do not hear the 'A', the genuine reason is that it has not been emitted.

To analyze a signal we have to look for a mixed definition, comparable to the one proposed.

by Gabor: at each instant we have a certain number of frequencies, which give the pitch and the timbre of the sound emitted such as we hear it; to each frequency is associated a certain time distribution which corresponds to the instants when the note is emitted.

We are then led to define an instantaneous spectrum, function of time, which gives the structure of the signal at a given instant; the signal spectrum, in the usual Fourier sense, which gives the frequency structure of the signal for its entire duration, is then obtained by adding all the instantaneous spectra'.

BRIEF HISTORY

Theory:

1984 Continuous wavelet transform, (CWT)
Alex Grossmann & Jean Morlet

1986 orthogonal wavelet transform,
Yves Meyer & Pierre-Gilles Lemarié

1987 Compactly supported wavelets,
Ingrid Daubechies

1988 Fast wavelet transform,
Stéphane Mallat

1989 CWT in n dimensions,
Romain Murenzi

Applications:

1984 Geophysics, Morlet
1986 Music, Risset & Kronland-Martinet
1986 quantum mechanics, Paul & Daubechies
1987 Fractals, Holschneider & Jaffard
1988 Image processing, Mallat
1988 Turbulence, Farge
1990 PDE's, Liandrat & Tchamitchian
1991 Numerical analysis, Coifman & Beylkin
1996 Navier-Stokes, Schneider, Perrier

Reference book
Barbara Burke
The world according to wavelets
A. K. Peters, Wellesley, 1996

INTEGRAL TRANSFORM PRINCIPLE

Signal $f(x)$ to be analyzed

Analyzing functions $\psi_k(x)$

Analysis

$$\hat{f}(k) = \int f(x)\, \psi_k(x)\, dx$$

Synthesis

$$f(x) = \frac{1}{c} \int \hat{f}(k)\, \psi_k(x)\, d\mu(k)$$

Energy conservation (Parseval)

$$\int |f(x)|^2 dx = \frac{1}{c} \int |\hat{f}(k)|^2 dk$$

DIFFERENT FOURIER TRANSFORMS

Used for Signal Processing

$$\hat{f}(k) = \int_{-\infty}^{+\infty} f(x)\, e^{-2i\pi k x}\, dx$$

$$f(x) = \int_{-\infty}^{+\infty} \hat{f}(k)\, e^{+2i\pi k x}\, dk$$

$$e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi k^2}$$

$$\int |f(x)|^2\, dx = \int |\hat{f}(k)|^2\, dk$$

$$\widehat{f(x) * g(x)} = \hat{f}(k) \cdot \hat{g}(k)$$

$$\widehat{\frac{\partial^n f(x)}{\partial x^n}} = (2i\pi k)^n\, \hat{f}(k)$$

Used for Harmonic Analysis

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(k)\, e^{ikx}\, dk$$

$$e^{-\frac{x^2}{2}} \xrightarrow{\mathcal{F}} e^{-\frac{k^2}{2}}$$

$$\int |f(x)|^2\, dx = \int |\hat{f}(k)|^2\, dk$$

$$\widehat{f(x) * g(x)} = \frac{1}{2\pi} \hat{f}(k) \cdot \hat{g}(k)$$

$$\widehat{\frac{\partial^n f(x)}{\partial x^n}} = (ik)^n\, \hat{f}(k)$$

Used for Group Theory

$$\hat{f}(k) = \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(k)\, e^{ikx}\, dk$$

$$e^{-\pi x^2} \longrightarrow e^{-\frac{k^2}{4\pi}}$$

$$\int |f(x)|^2\, dx = \frac{1}{2\pi} \int |\hat{f}(k)|^2\, dk$$

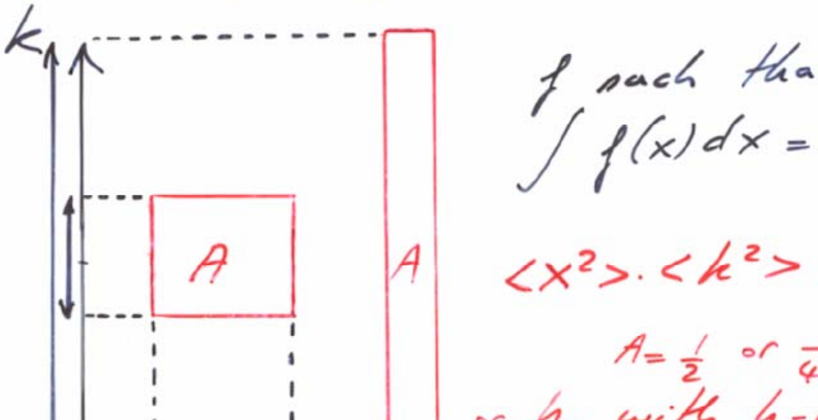
$$\widehat{f(x) * g(x)} = \hat{f}(k) \cdot \hat{g}(k)$$

$$\widehat{\frac{\partial^n f(x)}{\partial x^n}} = (ik)^n\, \hat{f}(k)$$

HEISENBERG'S UNCERTAINTY PRINCIPLE

A signal cannot be concentrated in both time and frequency, or position and wavenumber.

This is not a limitation of our perception of reality.
This is just by definition of the frequency or wavenumber.



f such that $\int f(x)\,dx = 1$

$\langle x^2\rangle \cdot \langle k^2\rangle \geqslant A$

$A = \frac{1}{2}$ or $\frac{1}{4\pi}$ or $\frac{1}{16\pi^2}$ or $\frac{h}{2\pi}$ with $h = 6.6\ 10^{-27}\ gcm^2/s$

$$\langle k^2\rangle = \int k^2 |\hat{f}(k)|^2 dk$$

$$\widehat{f(\lambda x)} = \frac{1}{\lambda}\hat{f}\left(\frac{k}{\lambda}\right)$$

Space dilation $\Longleftrightarrow$ Wavenumber contraction

$$\langle x^2\rangle = \int x^2 |f(x)|^2 dx$$

Time-frequency plane,
or position-wavenumber plane,
or position-impulsion ($p = \frac{hk}{2\pi}$, h Planck's constant),
or phase space,
or information plane.

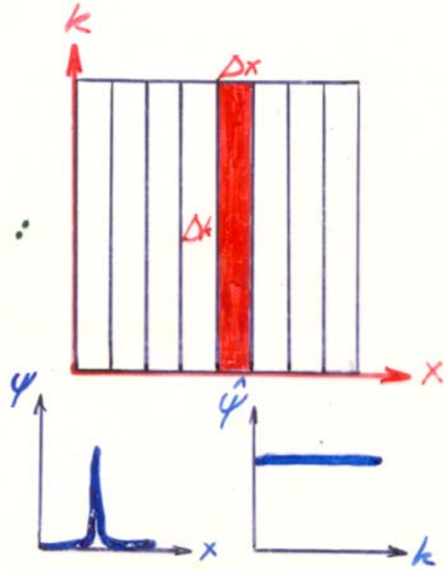
INTEGRAL TRANSFORM PRINCIPLE

Analysis $\quad T_f(k) = \int f(\vec{x})\, \psi_k(\vec{x})\, d\vec{x}$

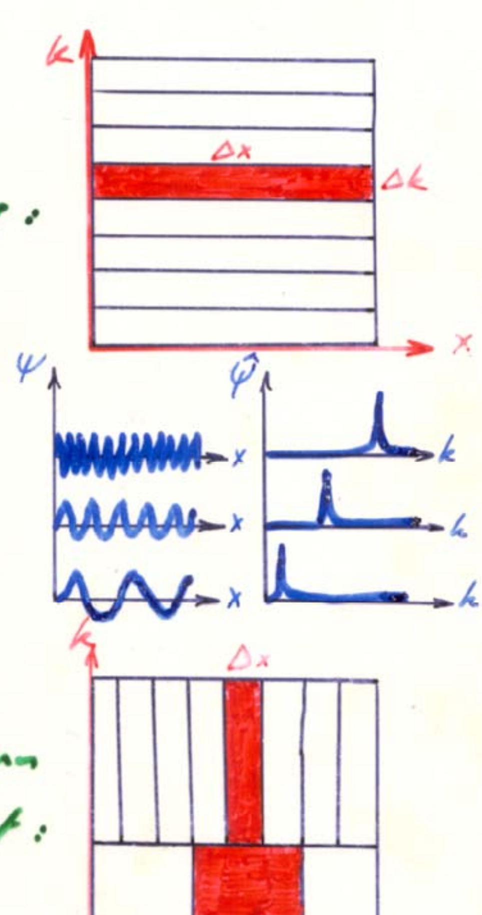
Synthesis $\quad f(\vec{x}) = \frac{1}{c} \int T_f(k)\, \psi_x(\vec{k})\, d\mu(k)$

$$\Delta x \cdot \Delta k \geqslant Cst$$

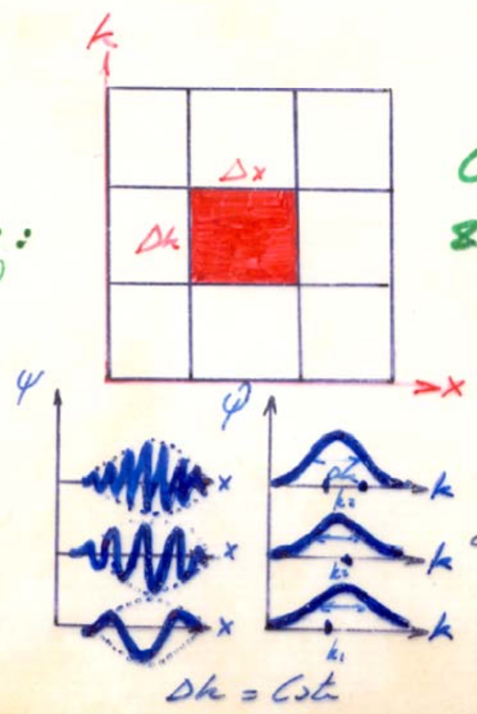
Shannon:



Fourier:
(1807)

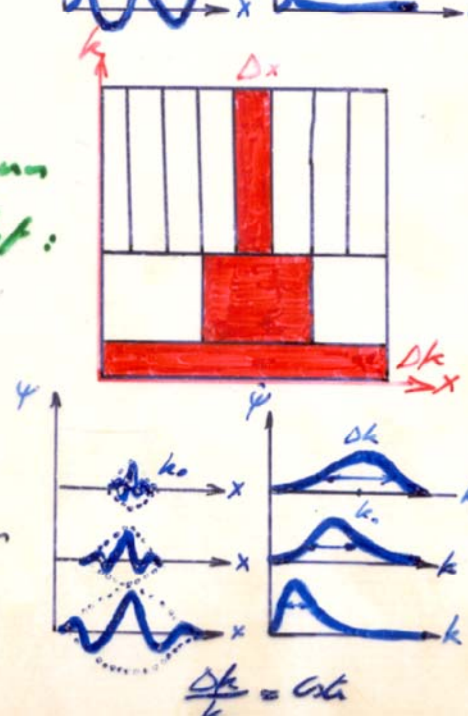


Gabor:
(1946)



Baliau's construction (1981)

Grossmann & Morlet:
(1981)



CONTINUOUS WAVELET TRANSFORM

Choice of the 'mother wavelet'

Admissibility condition: $\psi \in \{L^1 \cap L^2\}$

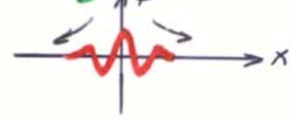
$$C = \int |\hat{\psi}(k)|^2 \frac{dk}{|k|} < \infty \Rightarrow \hat{\psi}(0) = 0 \quad \text{i.e.} \int \psi(x)\,dx = 0$$

in order to have a finite energy reproducing kernel

And, if possible:

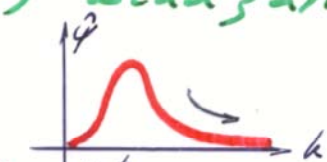
Good space (/time) localization,

$$|\psi(x)| < \frac{1}{1+|x|^n}$$



Good scale (/frequency) localization,

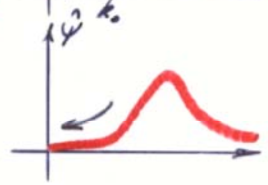
$$|\hat{\psi}(k)| < \frac{1}{1+|k-k_0|^n}$$



Zero high order moments,

$$\int \psi(x)\, x^m\, dx = 0$$

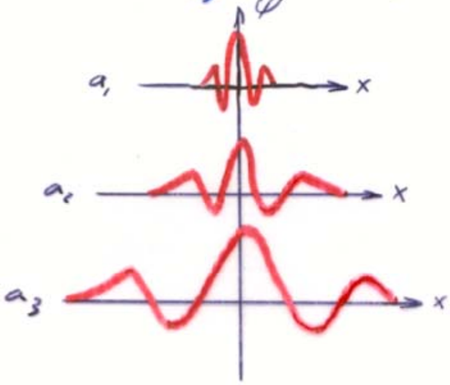
$$= \left(\frac{d}{dk}\right)^m \hat{\psi}(k)\Big|_{k=0}$$

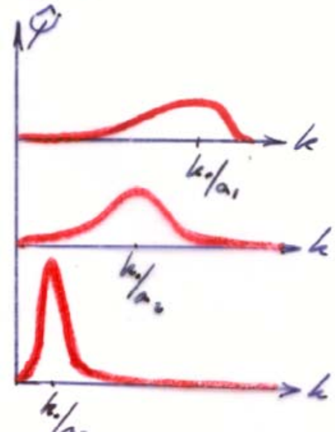


Generation of the 'wavelet family'

Affine Group $\Big\{$ by translation (parameter s) and dilation (parameter a):

$$\psi_{s,a}(x) = \frac{1}{\sqrt{a}}\, \psi\left(\frac{x-s}{a}\right)$$



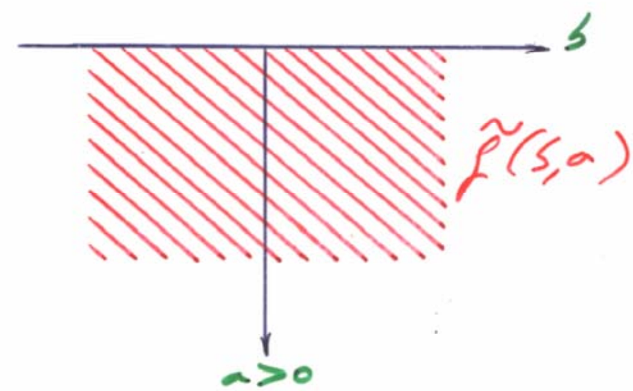


ANALYSIS/SYNTHESIS

Analysis: $\overline{\psi}$ complex conjugate

$$\tilde{f}(b,a) = \int \overline{\psi_{a,b}}(x)\, f(x)\, dx$$
$$= \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} \bar{\psi}\left(\frac{x-b}{a}\right) f(x)\, dx$$
$$= a^{+1/2} \int_{-\infty}^{+\infty} \underbrace{\hat{\bar{\psi}}(ak)}_{\text{Filter with } \frac{\Delta k}{|k|} = \text{Cste}} e^{ibk} \hat{f}(k)\, dk$$

The wavelet coefficients are defined on the open half-plane $(b,a) \in \mathbb{R} \times \mathbb{R}^+$



Synthesis:

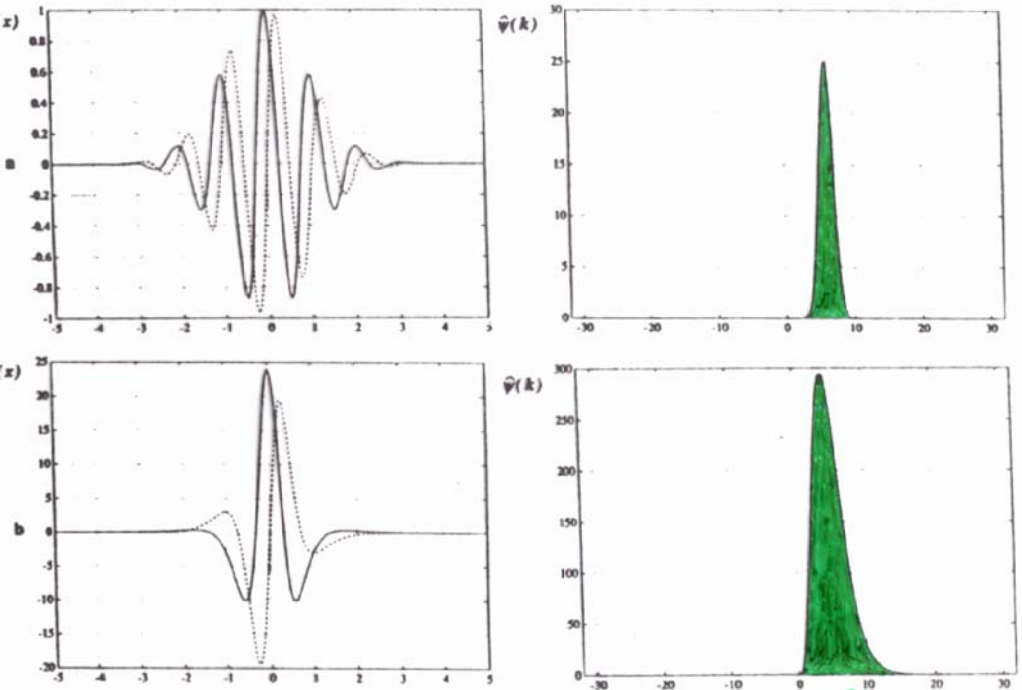
$$f(x) = \frac{1}{C} \iint \psi_{a,b}(x)\, \tilde{f}(b,a)\, \frac{da\, db}{a^2}$$
$$= \frac{1}{C} \int_{0^+}^{+\infty} \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} \psi\left(\frac{x-b}{a}\right) \tilde{f}(b,a)\, \frac{da\, db}{a^2}$$

avec $C = 2\pi \int_{-\infty}^{+\infty} |\hat{\psi}(k)|^2 \frac{dk}{|k|}$

Energy conservation (Parseval):

$$\int |f(x)|^2 dx = \frac{1}{C} \iint |\tilde{f}(b,a)|^2 \underbrace{\frac{da\, db}{a^2}}_{\text{Haar Measure}}$$

Morlet

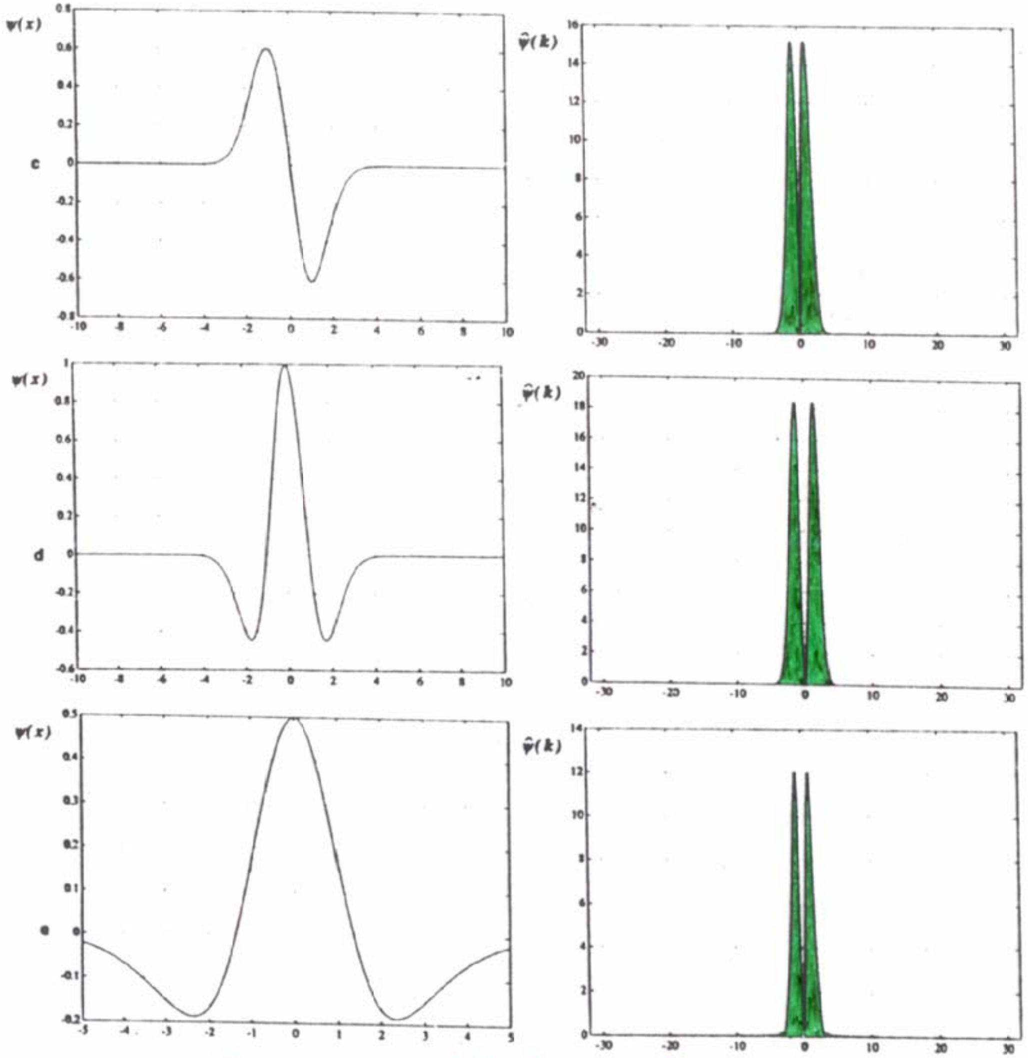


ψ $\hat{\psi}$

Paul

Complex-valued wavelets

Derivatives of a Gaussian

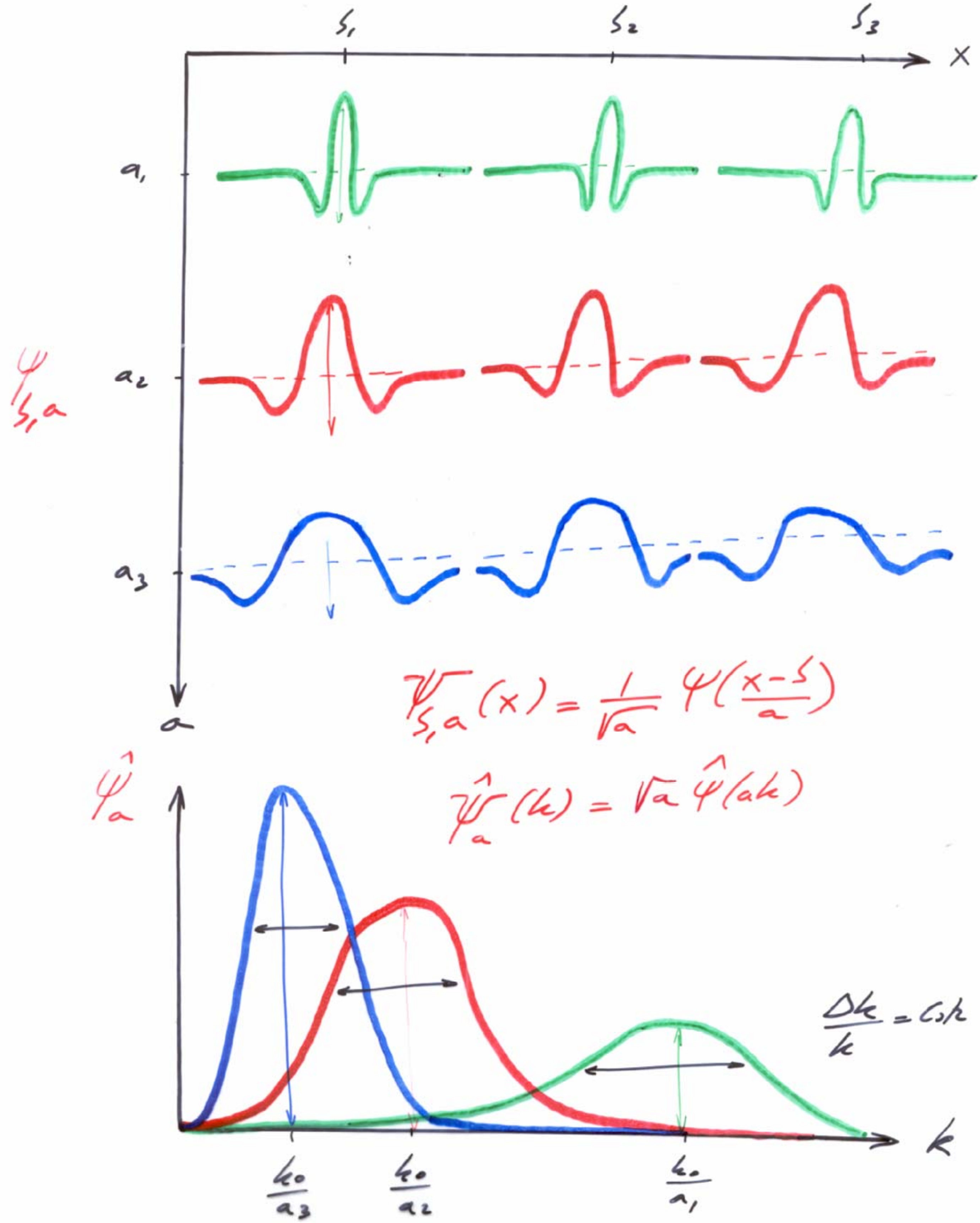


Difference of two Gaussians

ψ $\hat{\psi}$

Real-valued wavelets

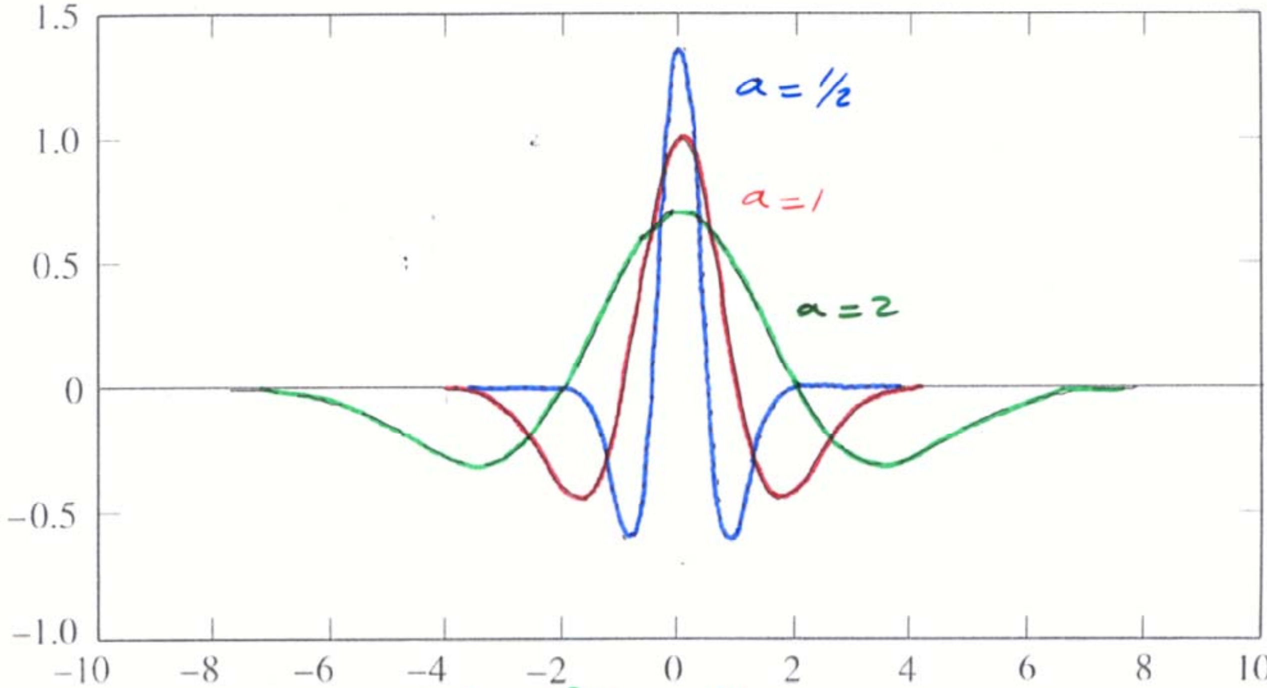
WAVELET FAMILY IN L^2-NORM



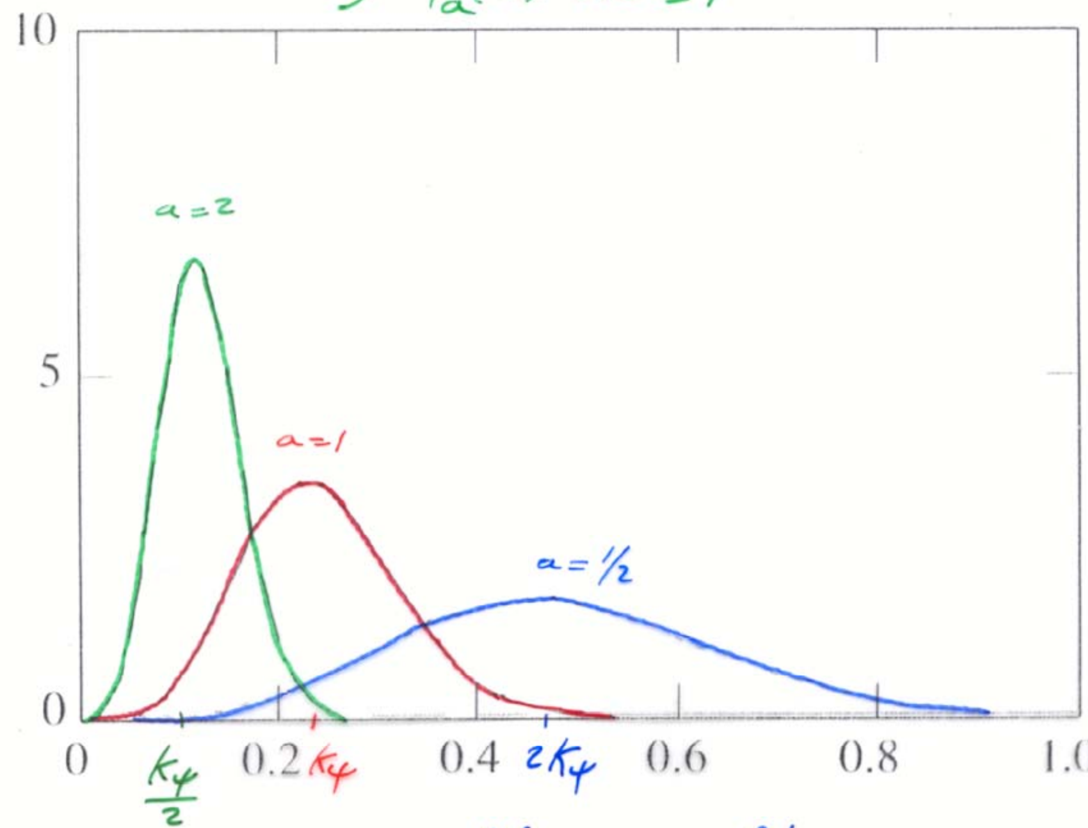
$$\psi_{s,a}(x) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{x-s}{a}\right)$$

$$\hat{\psi}_a(k) = \sqrt{a}\,\hat{\psi}(ak)$$

WAVELET FAMILY WITH L2-NORM

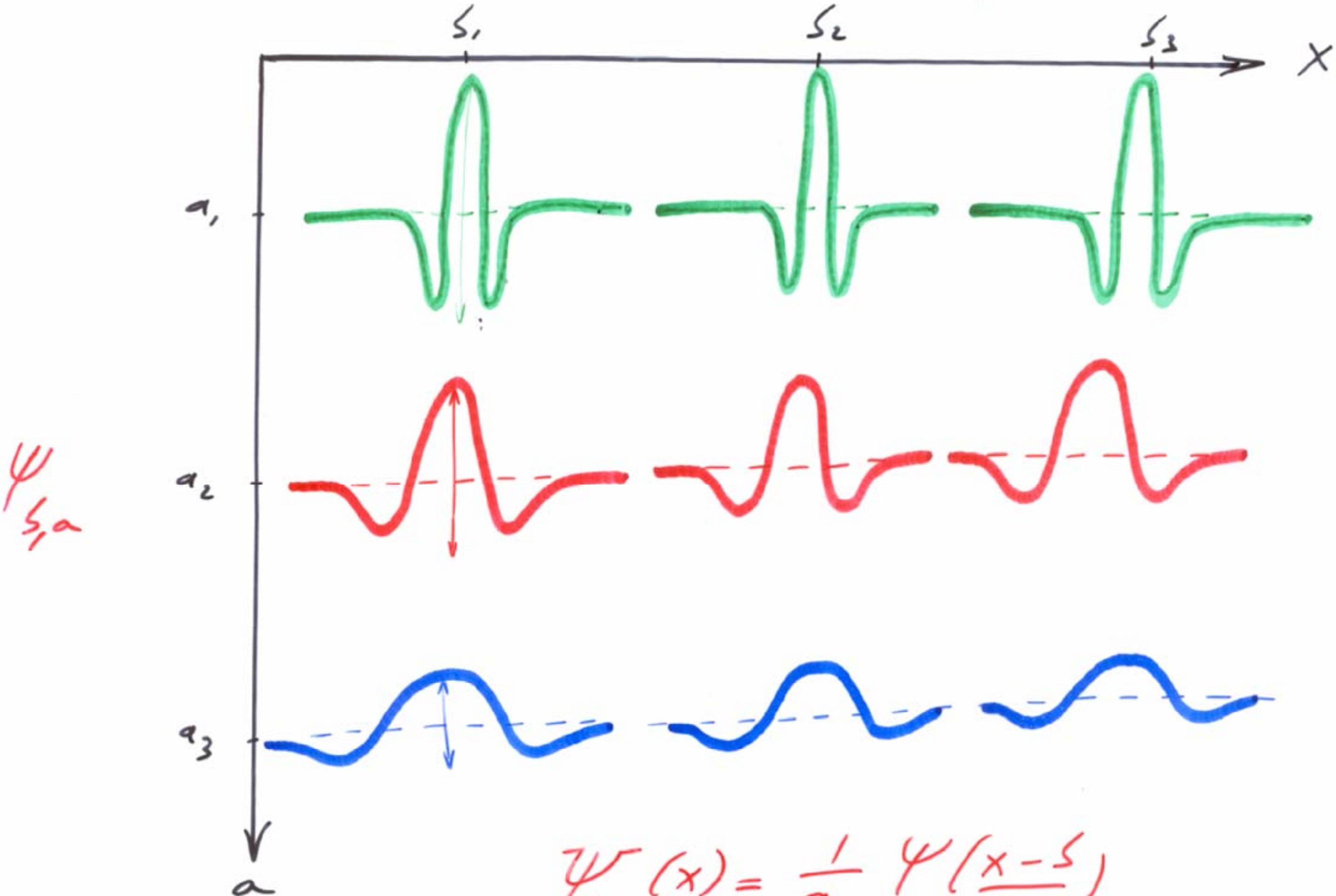


$$\int \psi_a^2(t)\,dt = 1$$



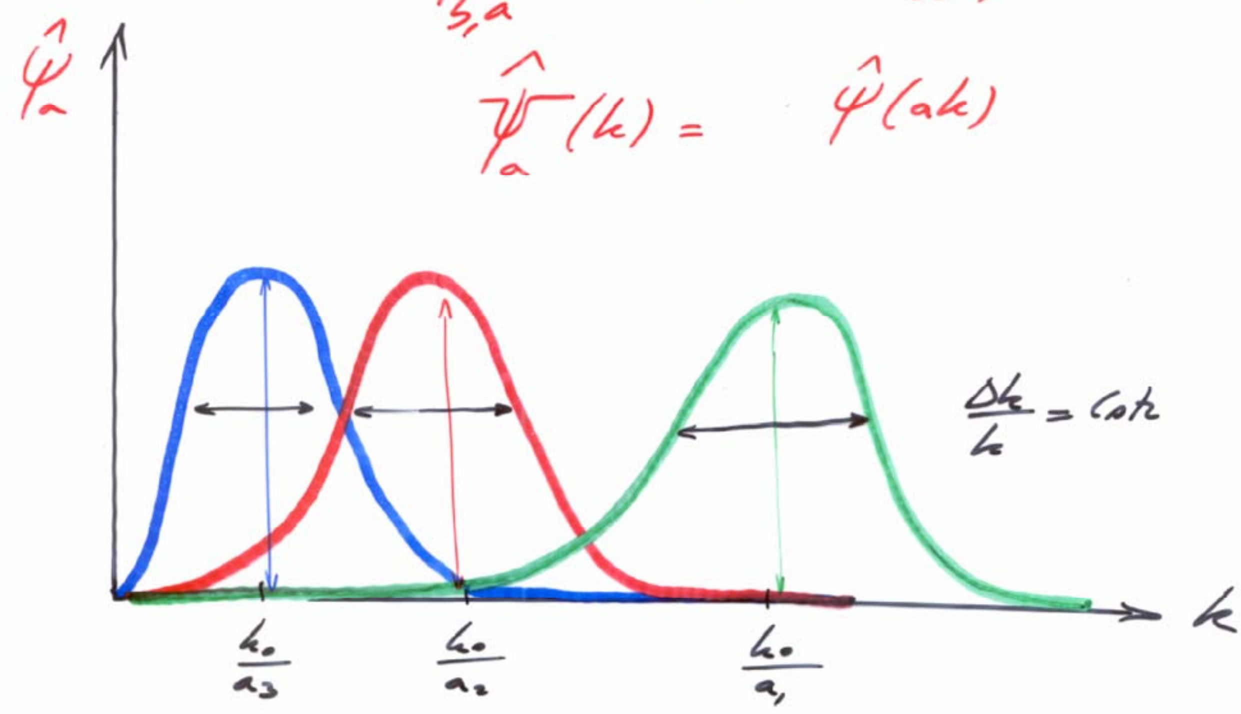
$$\Rightarrow \tilde{f}_{L^2} = \sqrt{a}\,\tilde{f}_{L^1}$$

WAVELET FAMILY IN L^1-NORM

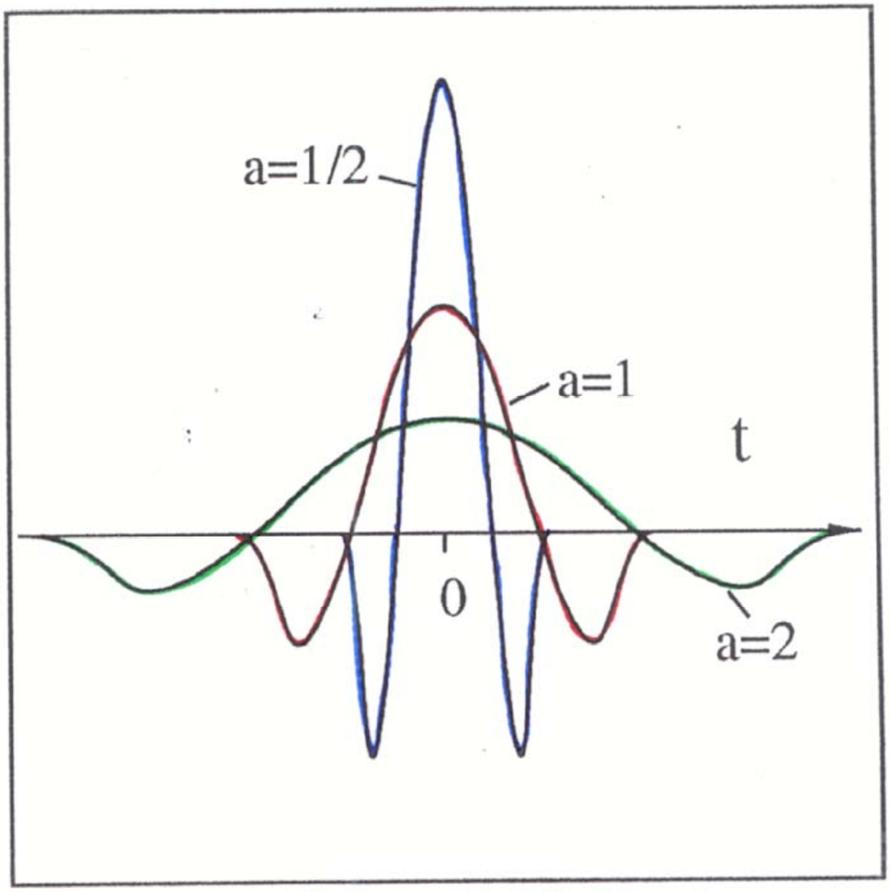


$$\psi_{s,a}(x) = \frac{1}{a}\,\psi\!\left(\frac{x-s}{a}\right)$$

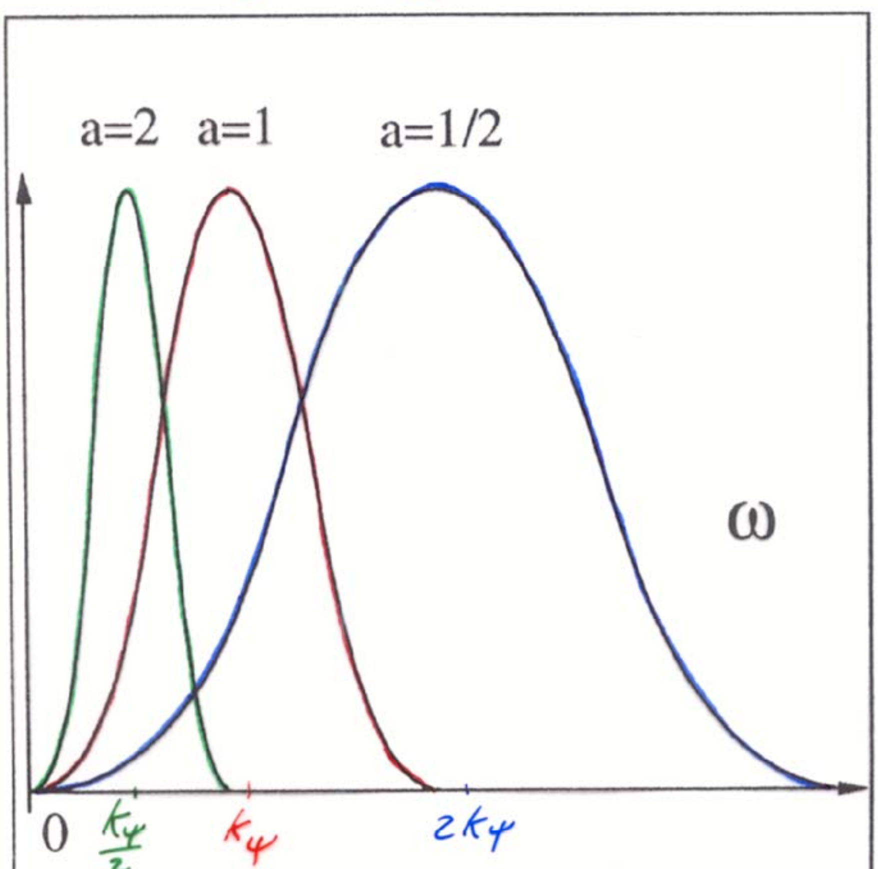
$$\hat{\psi}_a(k) = \hat{\psi}(ak)$$



WAVELET FAMILY WITH L1-NORM



$\int |\psi_a(t)|\, dt = 1$



$\Longrightarrow \tilde{f}_{L^1} = \frac{1}{\sqrt{a}} \tilde{f}_{L^2}$

Real part

Parameters:

Sampling frequency: 44.1 kHz

Number of voice per octave = 1

Highest voice: index = 1 frequency = 6000 Hz

Lowest voice: index = 7 frequency = 93.7 Hz

Time window = 18 ms Signal de parole (1983)

Jean Morlet

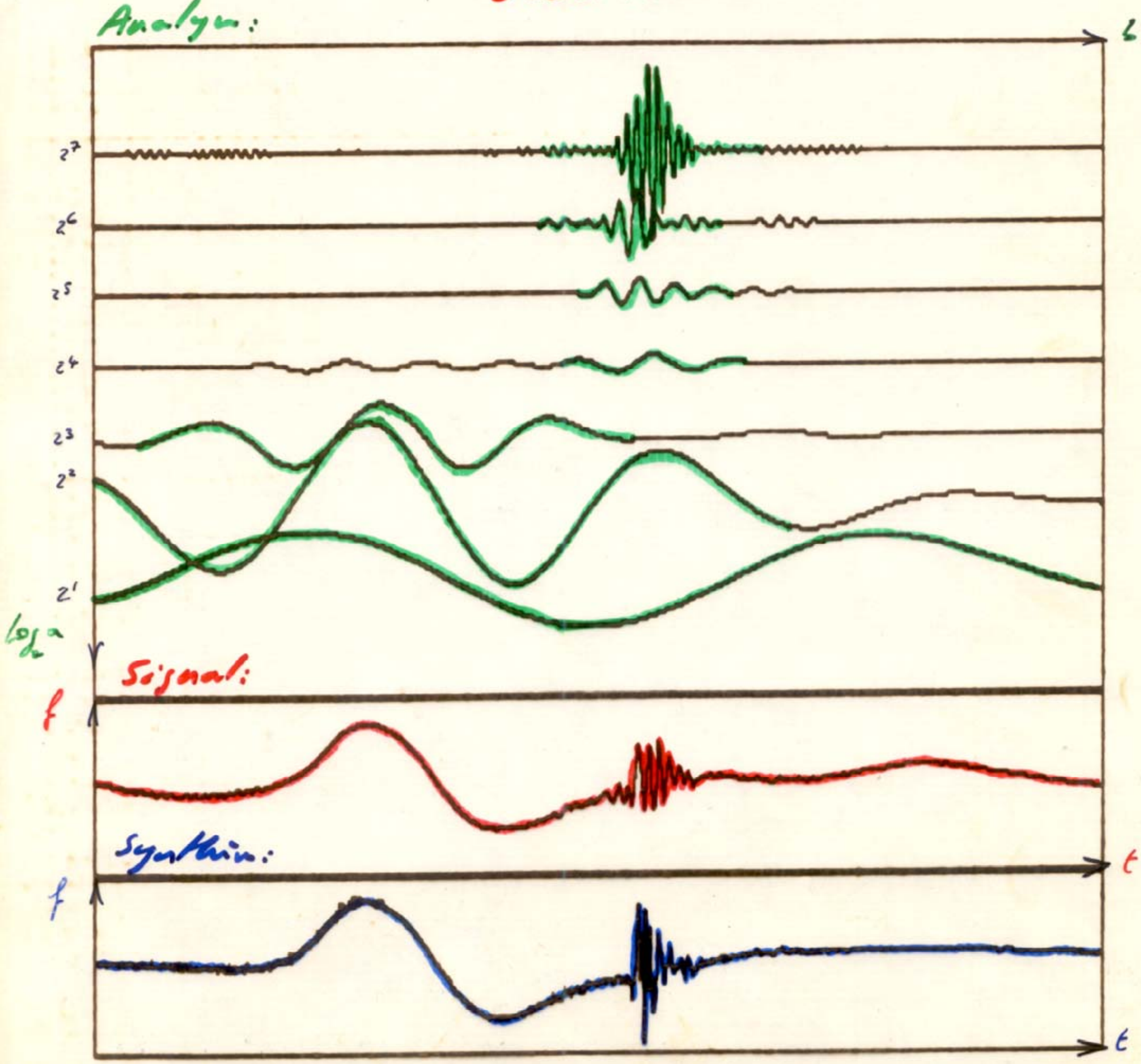


Figure 5

TWO WAVELET FORMULA

We can use different wavelets for analysis ψ_A and synthesis ψ_S.

Admissibility condition:

$$C_\psi = \int_0^\infty \overline{\hat{\psi}_A}(k)\,\hat{\psi}_S(k)\,\frac{dk}{k} = \int_0^\infty \overline{\hat{\psi}_A}(-k)\,\hat{\psi}_S(-k)\,\frac{dk}{k} < \infty$$

Then isometry:

$$f(x) = \frac{1}{C_\psi}\int_{0^+}^{\infty}\frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty}\tilde{f}(b,a)\,\psi_S(b,a)\,\frac{da\,db}{a^2}$$

with $\tilde{f}(b,a) = \frac{1}{\sqrt{a}}\int f(x)\,\overline{\psi_A}(b,a)\,dx$

Morlet's reconstruction formula:

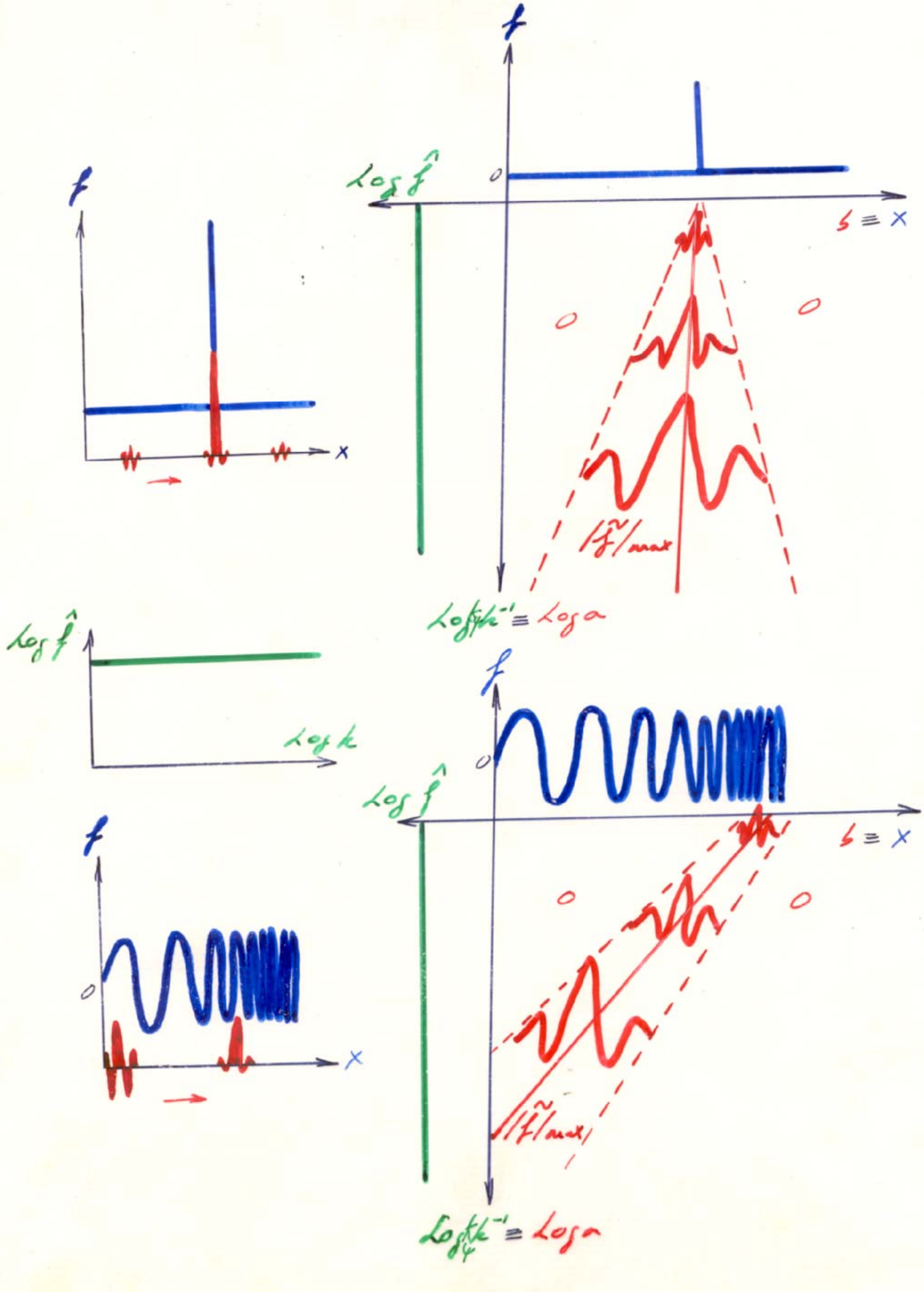
$$\psi_S(x) = \delta(x)$$

then $f(x) = \frac{1}{C_\psi}\int_{0^+}^{+\infty}\frac{1}{\sqrt{a}}\,\tilde{f}(b,a)\,\frac{da}{a}$

with $C_\psi = \int_0^\infty \overline{\hat{\psi}_A}(k)\,\frac{dk}{k} = \int_0^\infty \overline{\hat{\psi}_A}(-k)\,\frac{dk}{k} < \infty$

We need only one integration to reconstruct.

TeO : EXEMPLES



WAVELET TRANSFORM OF HARMONIC SIGNALS

Progressive wavelet $\psi(x) \in \mathcal{H}$ Hardy space

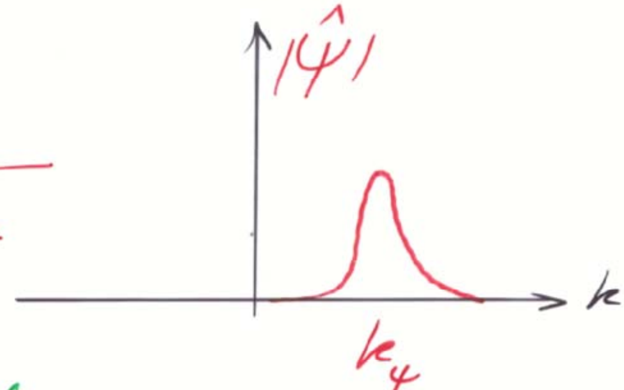
$\Longleftrightarrow$ $\psi(x) \in \mathbb{C} \;/\; \hat{\psi}(k \leq 0) = 0$,

i.e. $\mathcal{R}(\psi) \xleftrightarrow{H} \mathcal{I}(\psi)$

H Hilbert transform,

with maximum of $\psi(x)$ at

$$k_\psi = \frac{\int_{-\infty}^{+\infty} k^2 \psi^2(k)\, dk}{\int_{-\infty}^{+\infty} \psi^2(k)\, dk}$$



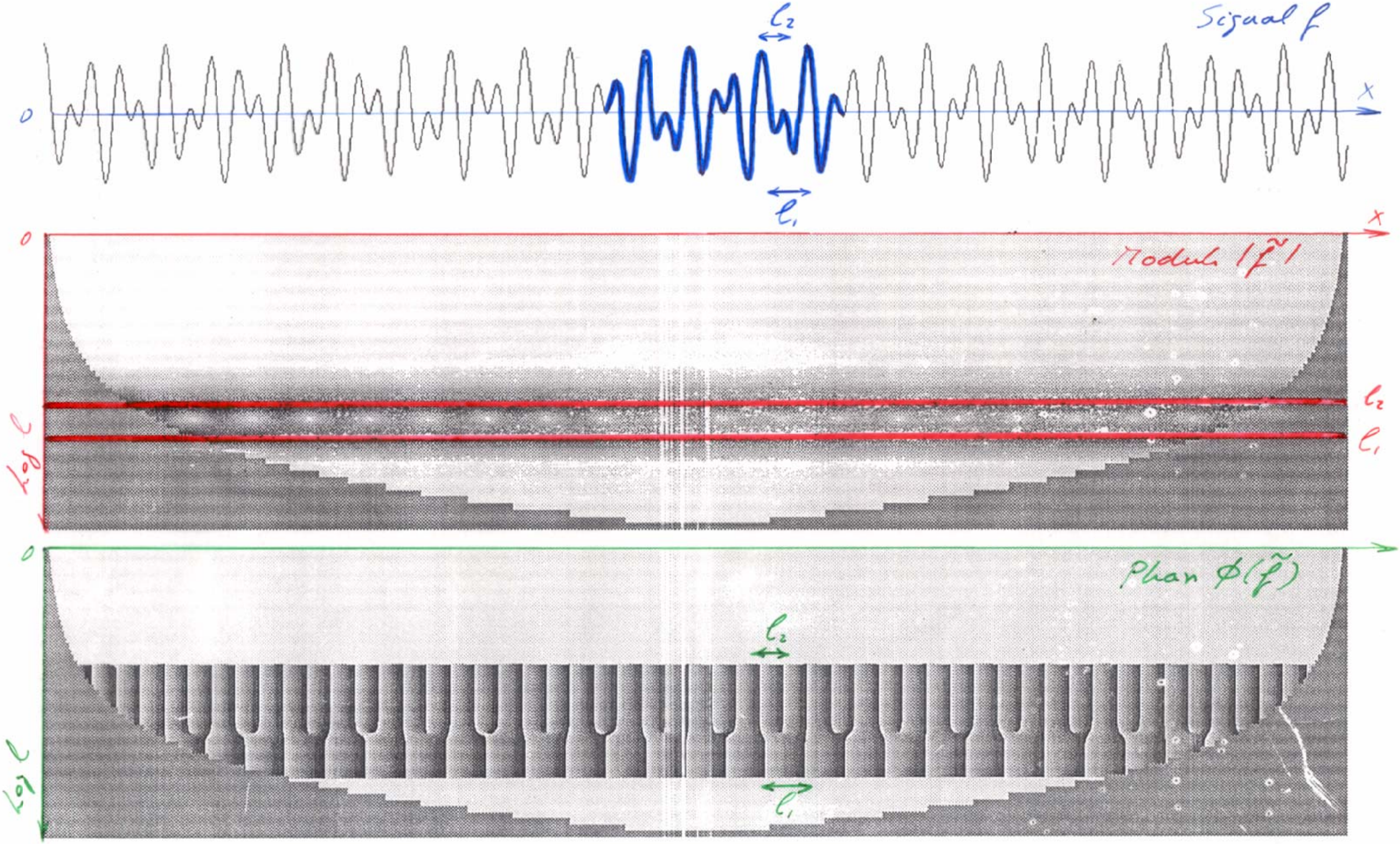
Harmonic signal

$$f(x) = \cos k_0 x = \frac{e^{ikx} + e^{-ikx}}{2}$$

$$\tilde{f}(s, a) = \sqrt{a}\, e^{isk_0} \bar{\hat{\psi}}(a\nu_0)$$

- Modulus behaves as $\bar{\hat{\psi}}(a\nu_0)$ which is maximal for $\hat{\psi}(\nu_\psi) \Rightarrow a = \frac{k_\psi}{k_0}$
- Phase varies linearly with s and therefore unfolds the signal phase in space $\Rightarrow \frac{\partial \varphi}{\partial s} = \frac{k_0}{k_\psi}$

cos(t)+cos(1.68t)



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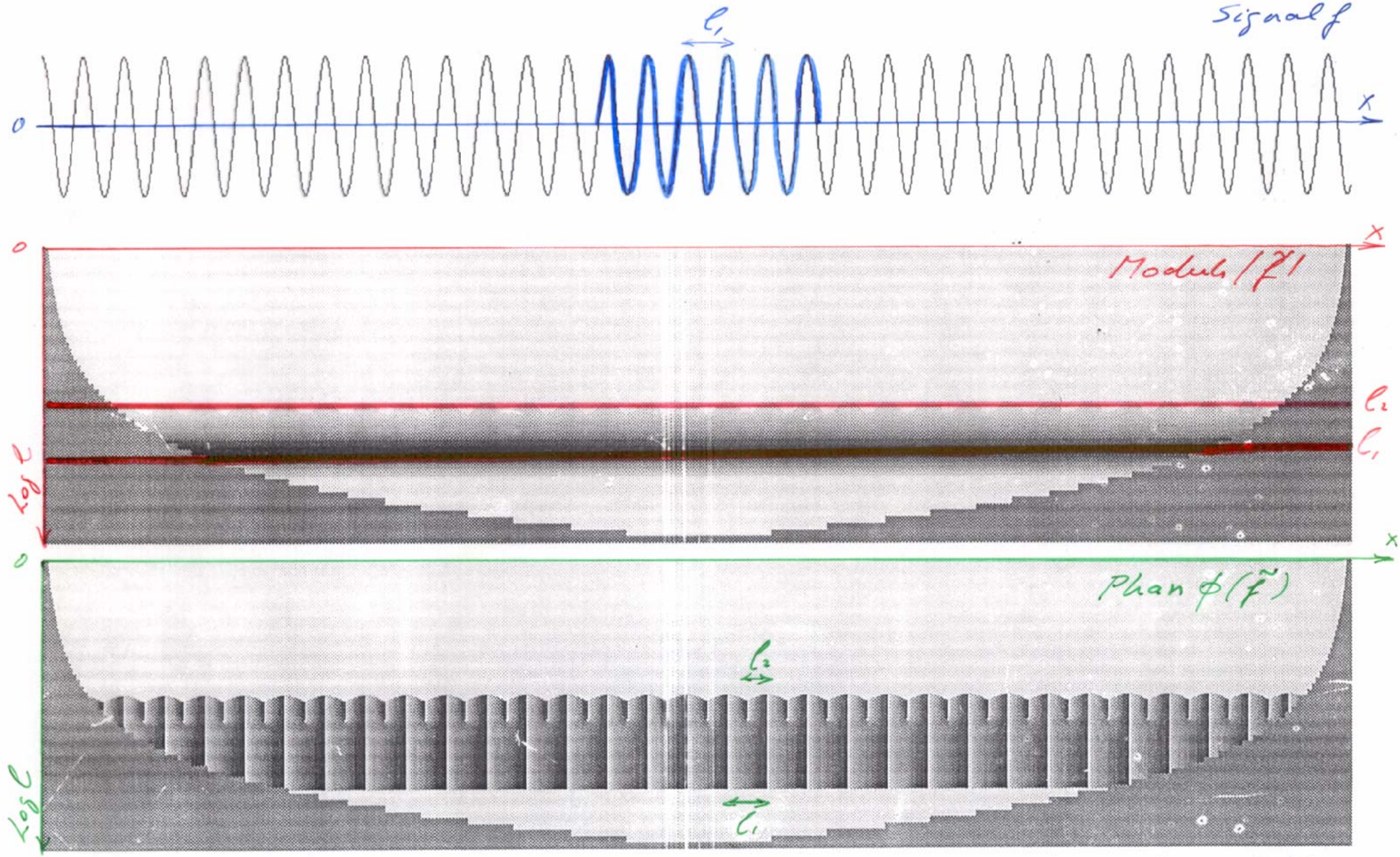
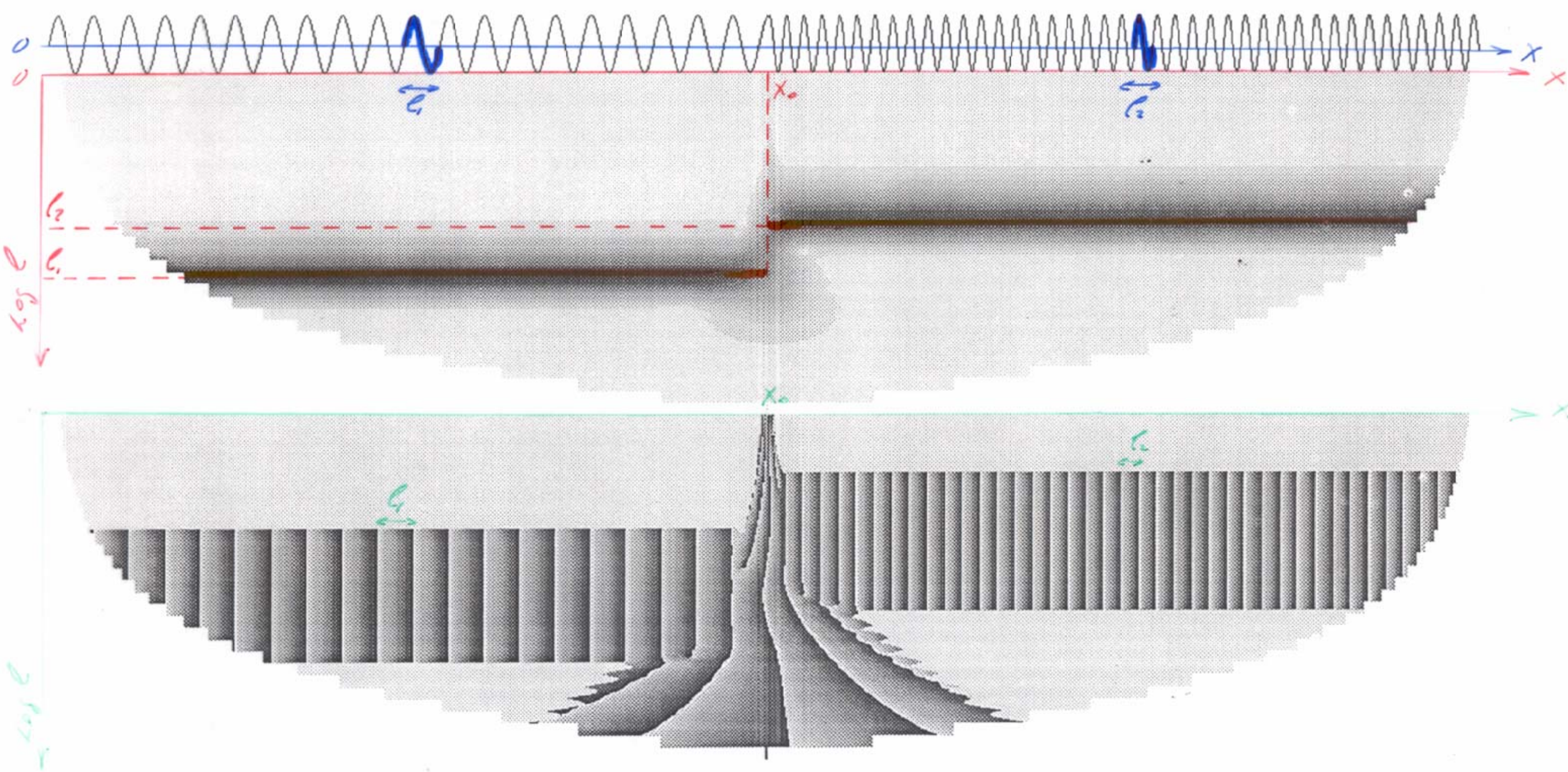
cos(t)+0.02cos(2t)

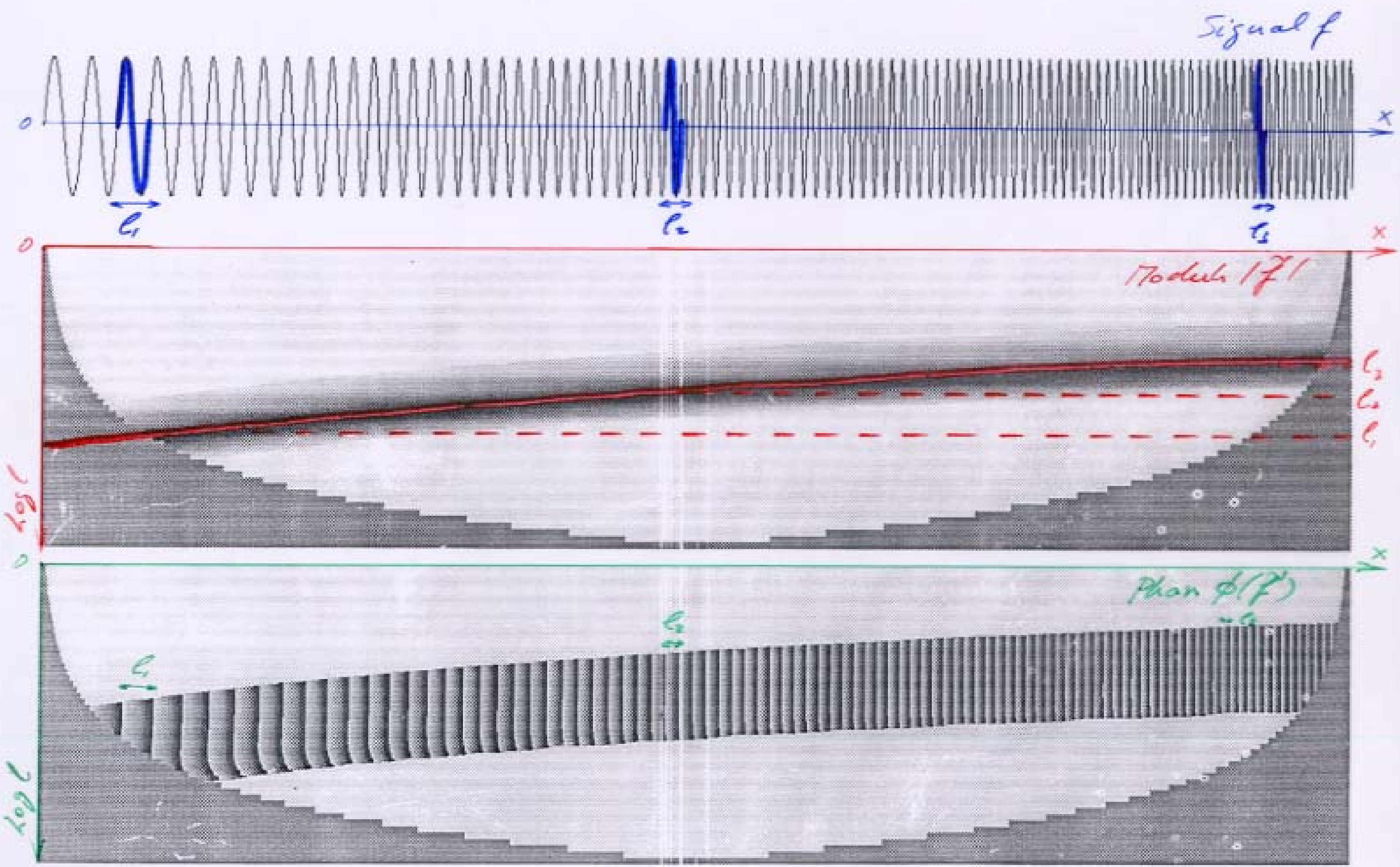


SIN(t) si t<512 SIN(2t) sinon

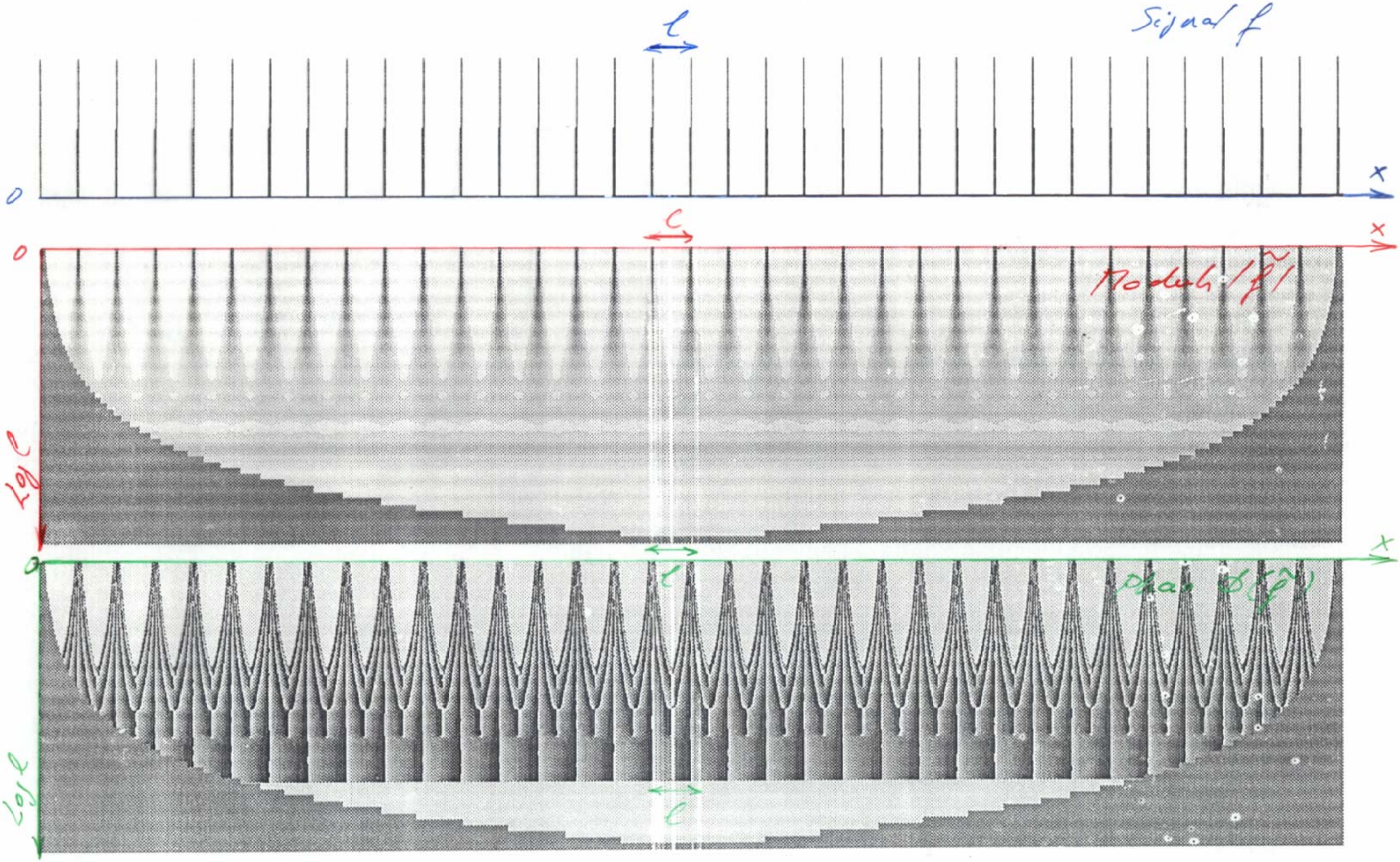


M

$\sin(t^2)$

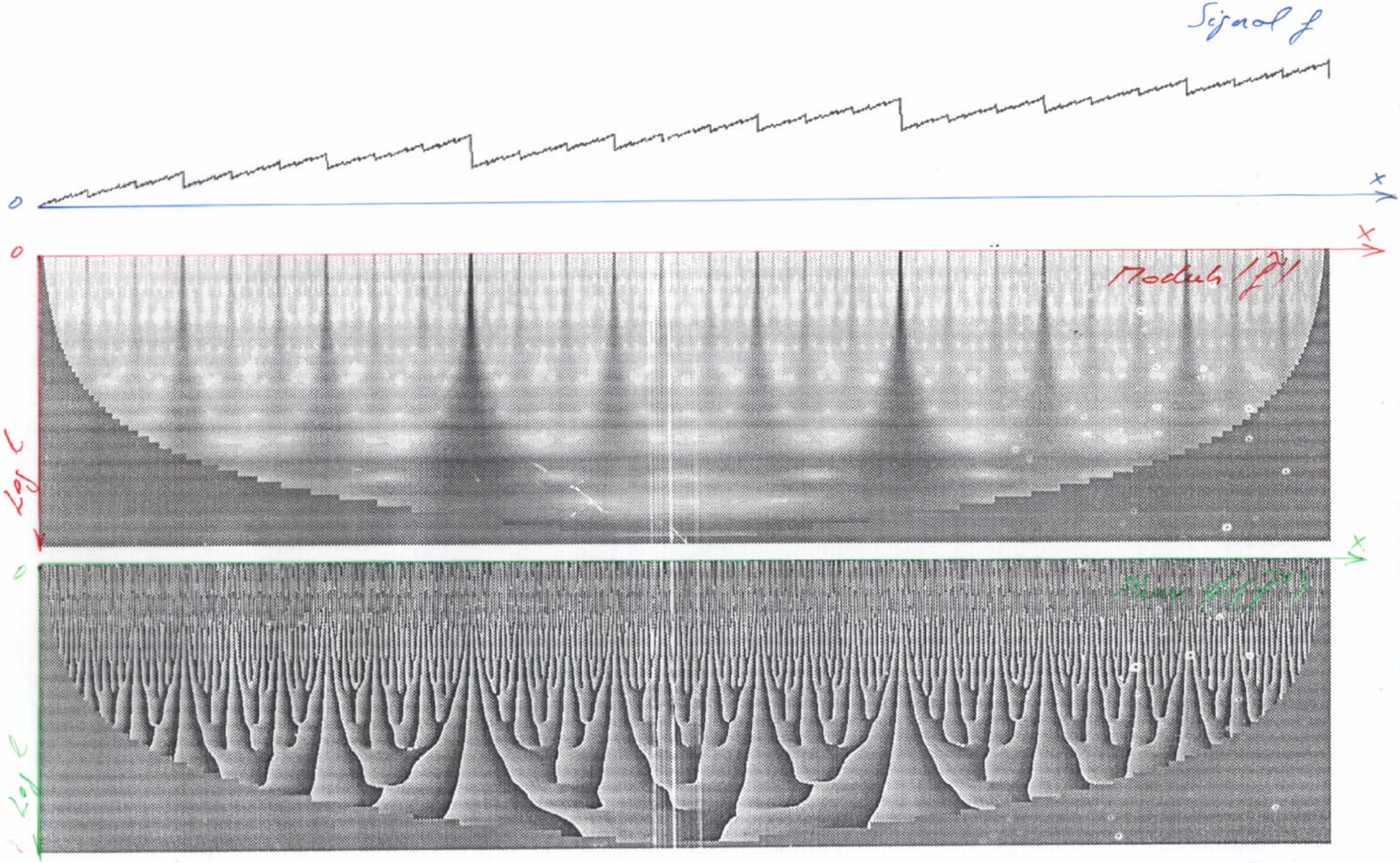


DELTA

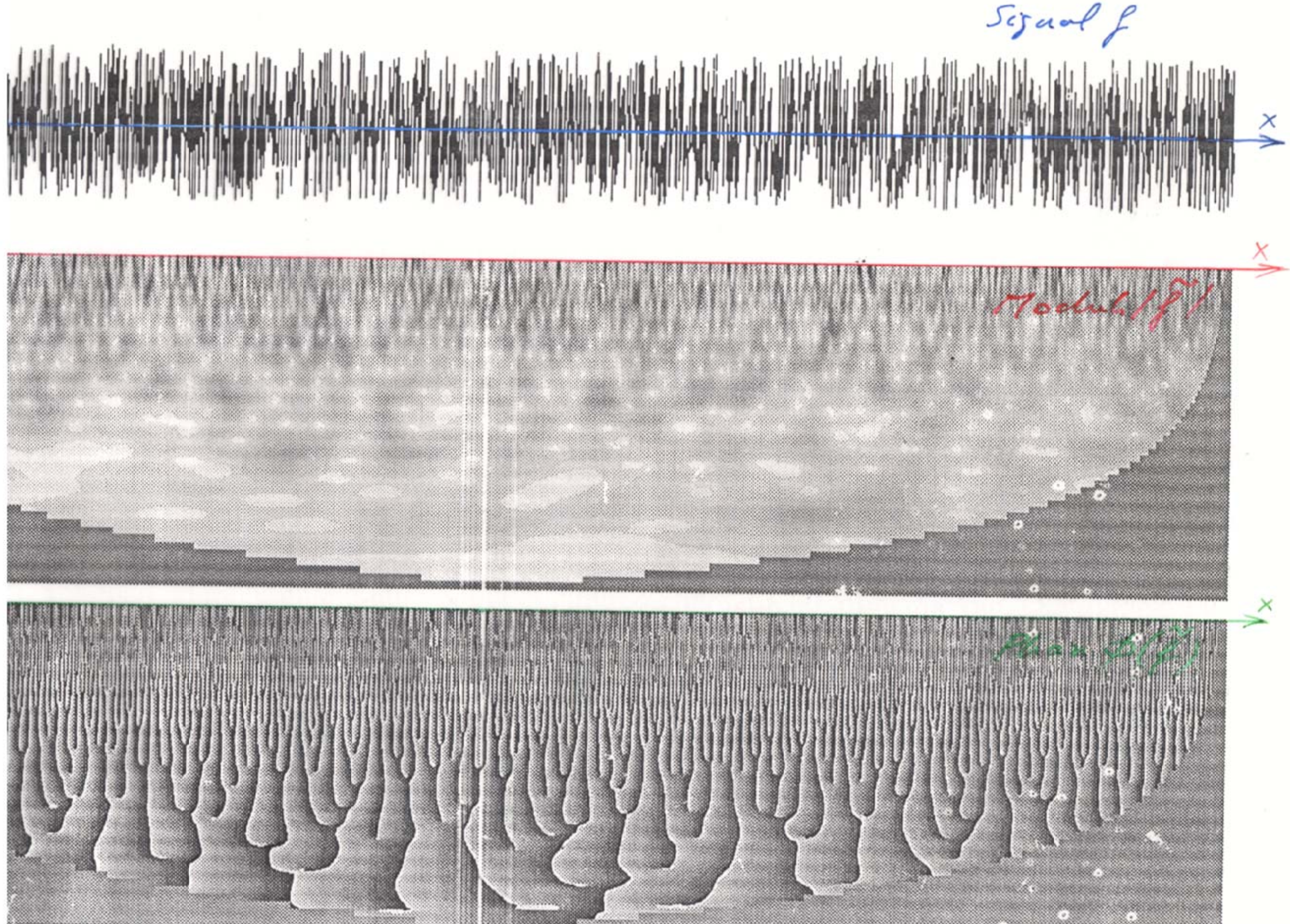


Signal f
ℓ
0
x
Module $|\tilde{f}|$
ℓ
0
x
$\log a$
Phase $\Phi(\tilde{f})$
ℓ
0
x
$\log a$
ℓ

SAUTS



RANDOM



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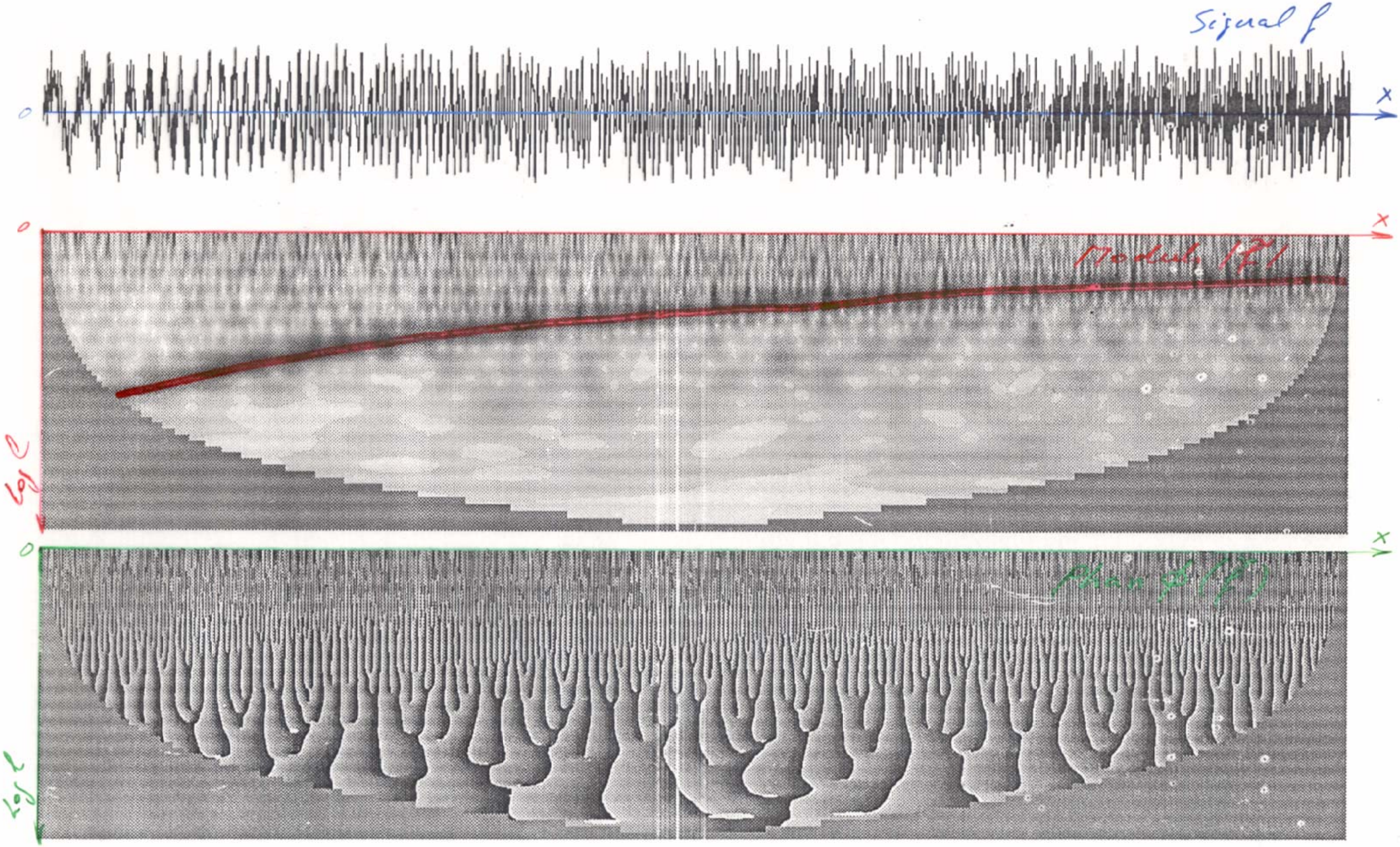
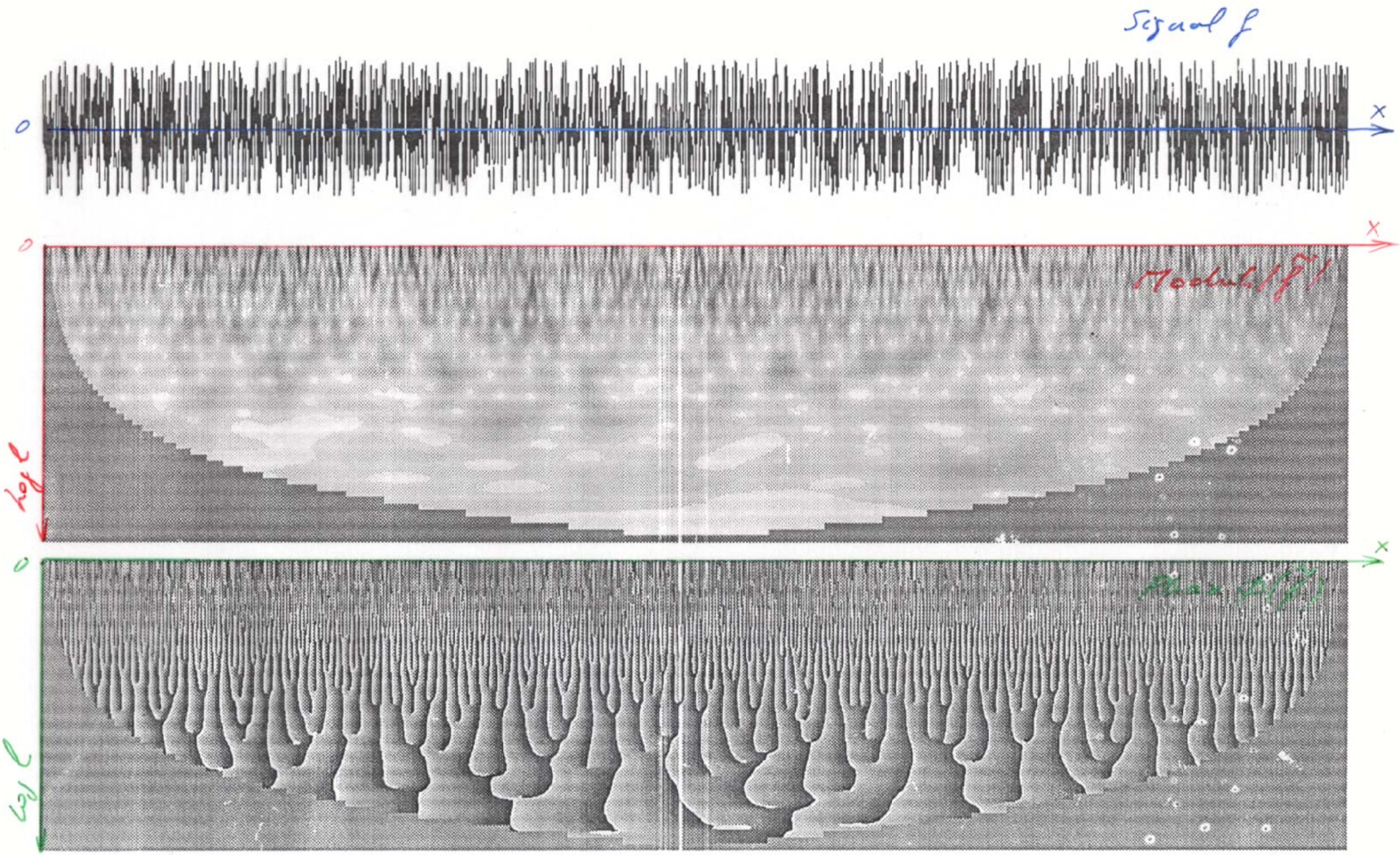
sinus(t^2)+random



RANDOM



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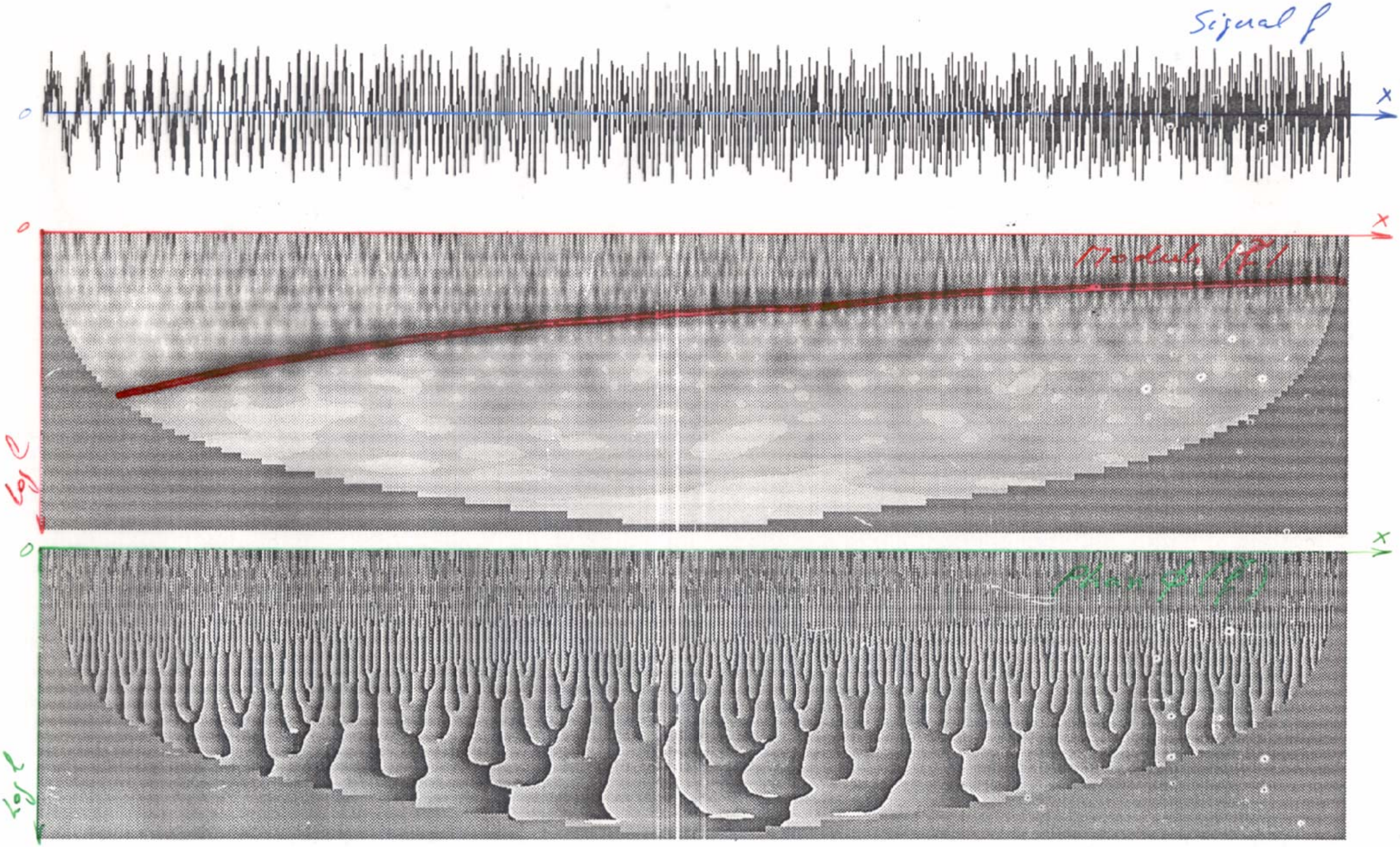
sinus(t^2)+random



LINEARITY

W is the continuous wavelet transform operator

$$W[f(x)+g(x)] = \tilde{f}(\zeta,a) + \tilde{g}(\zeta,a)$$

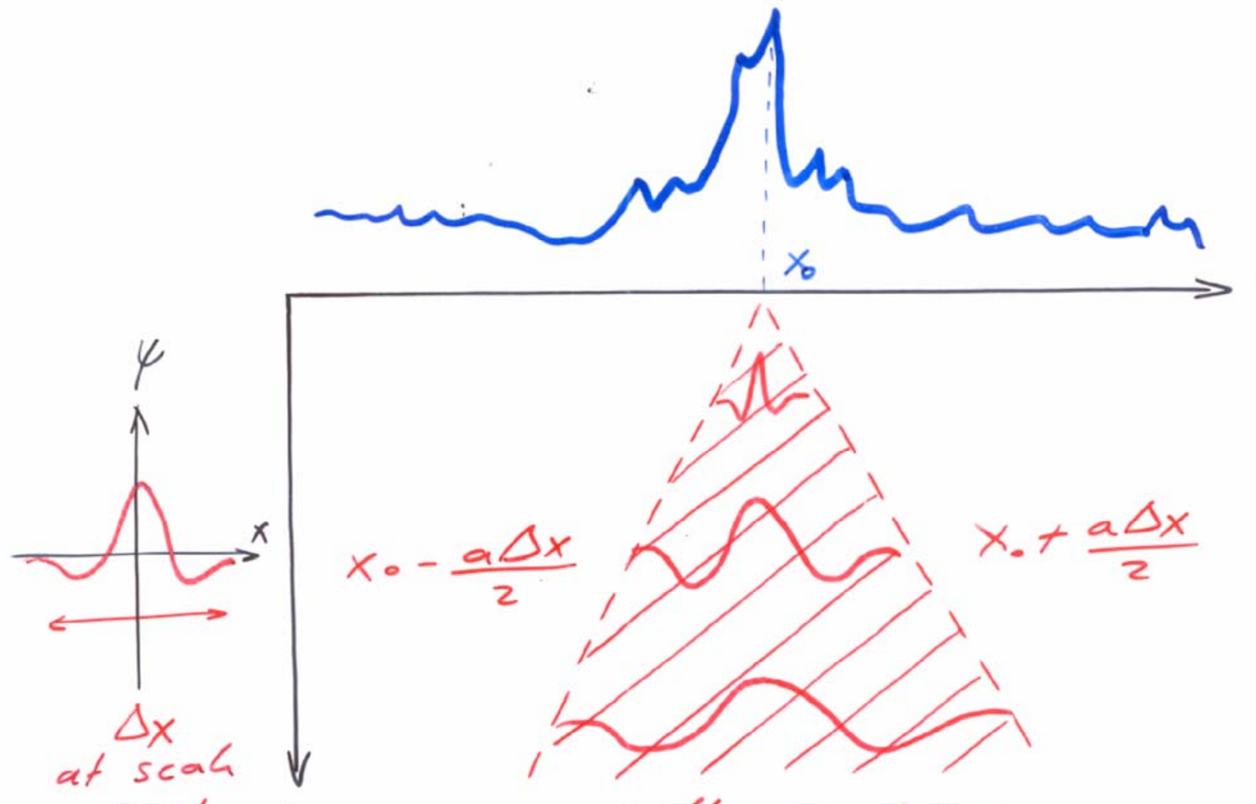
$$\Rightarrow \quad W[\vec{v}\,] = \tilde{\vec{v}} \left| \begin{array}{l} \tilde{f}(x) \\ \tilde{f}(y) \\ \tilde{f}(z) \end{array} \right.$$

with $\vec{v} \left| \begin{array}{l} f(x) \\ f(y) \\ f(z) \end{array} \right.$

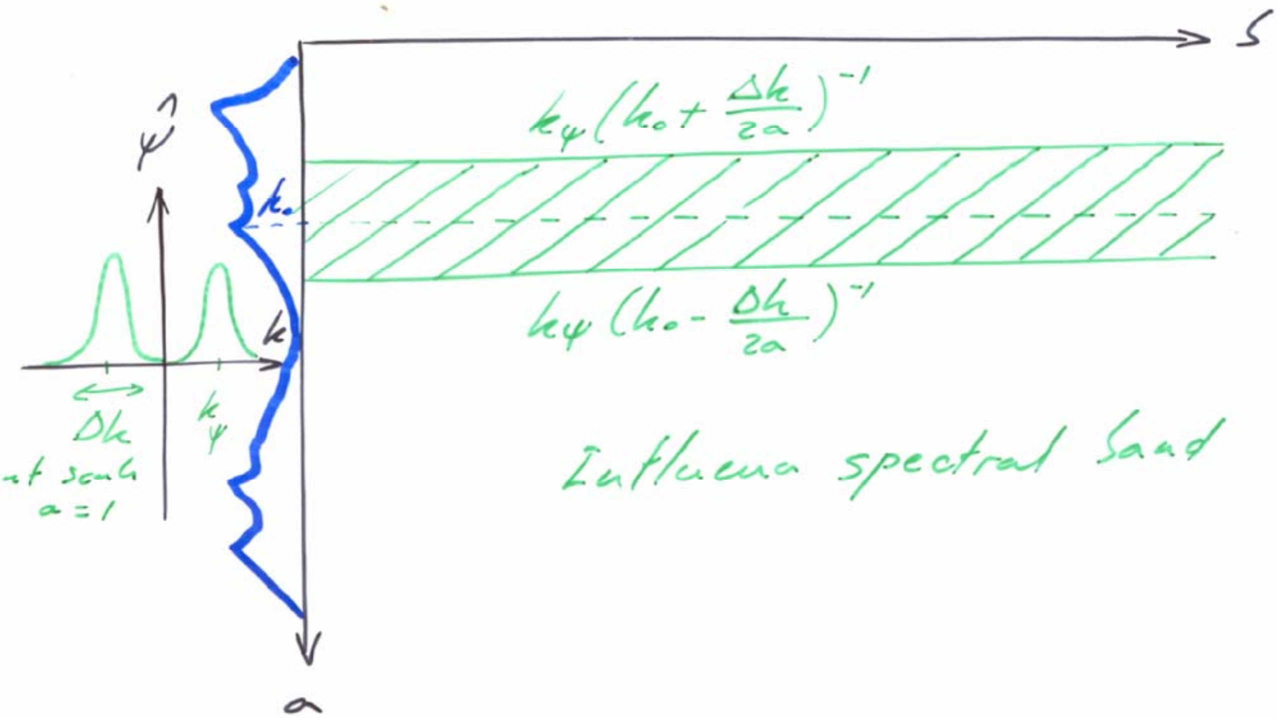
Differentiation

$$W\left[\frac{\partial^n f}{\partial x^n}\right] = (-1)^n \int_{-\infty}^{+\infty} f(x)\, \frac{\partial^n}{\partial x^n}\left[\bar{\Psi}_{\zeta,a}(x)\right] dx$$

CONSERVATION OF THE SPACE-SCALE LOCALITY



Influence cone



COVARIANCE BY TRANSLATION

$$W[f(x-x_0)] = \tilde{f}(a, b-x_0)$$

This implies that differentiation commutes with W:

$$\frac{\partial}{\partial x} W(f) = W\left(\frac{\partial f}{\partial x}\right)$$

$$\nabla[W(f)] = W(\nabla f)$$

$$\nabla\cdot[W(f)] = W(\nabla\cdot f)$$

A consequence of the covariance by translation is that the frequency of an harmonic signal can be read off from the phase of the wavelet coefficients. It corresponds to the number of zeros of the phase for a = cte.

COVARIANCE BY DILATION

$$W[f(\lambda x)] = \frac{1}{\lambda} \tilde{f}(\lambda a, \lambda b)$$

This is not the same as for the Fourier transform F:

$$F[f(\lambda x)] = \frac{1}{\lambda} \hat{f}\left(\frac{k}{\lambda}\right) = \frac{1}{\lambda} \hat{f}\left(\frac{2\pi}{\lambda \ell}\right)$$

ℓ wavelength

A consequence of the dilation covariance is that the wavelet transform of a power-law function is fully determined by its restriction to any line $a = \text{cste}$.

As a consequence, the lines of constant phase point out onto the singularities of the function f.

ENERGY CONSERVATION

The continuous wavelet transform is an isometry between

$L^2(\mathbb{R})$ and $H_\psi \subset L^2(\mathbb{R}^+ \times \mathbb{R})$

therefore it conserves energy:

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \frac{1}{C_\psi} \int_{0^+}^{+\infty} \int_{-\infty}^{+\infty} \tilde{f}(b,a)\, \bar{\tilde{f}}(b,a) \frac{da\, db}{a^2}$$

$$\text{with } C_\psi = 2\pi \int_{-\infty}^{+\infty} \frac{|\hat{\psi}(k)|^2}{|k|} dk$$

It conserves energy both globally and locally for all coefficients inside the influence cone.

The total energy can also be splitted among contributions at different scales:

$$E(a) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} |\tilde{f}(b,a)|^2 \frac{db}{a^2}$$

REPRODUCING KERNEL

The projection $L^2(\mathbb{R}^+ \times \mathbb{R}) \longrightarrow H_\psi$ is an integral operator with kernel:

$$k(s', a', s, a) = \langle \psi_{s'a'} \,|\, \psi_{sa} \rangle$$

autocorrelation function of ψ.

Therefore $\tilde{f}(s, a) \in L^2(\mathbb{R}^+ \times \mathbb{R})$ is the continuous wavelet transform of a function f iff it satisfies the reproducing kernel equation:

$$\tilde{f}(s', a') = \int_{0^+}^{+\infty} \int_{-\infty}^{+\infty} k(s', a', s, a)\, \tilde{f}(s, a)\, \frac{da\, ds}{a^2}$$

INSTANTANEOUS FREQUENCY

Frequency is a characteristic of a wave-like (harmonic) signal, which can be measured experimentally, to describe the rapidity of the wave oscillations.

If this wave has a variable amplitude we can still define a frequency iff the amplitude varies much slower than the oscillations, stationary phase hypothesis, namely:

$$f(t) = \underbrace{A(t)}_{\text{amplitude}} \, e^{\overbrace{i\varphi(t)}^{\text{phase}}} + \underbrace{n(t)}_{\text{noise}}$$

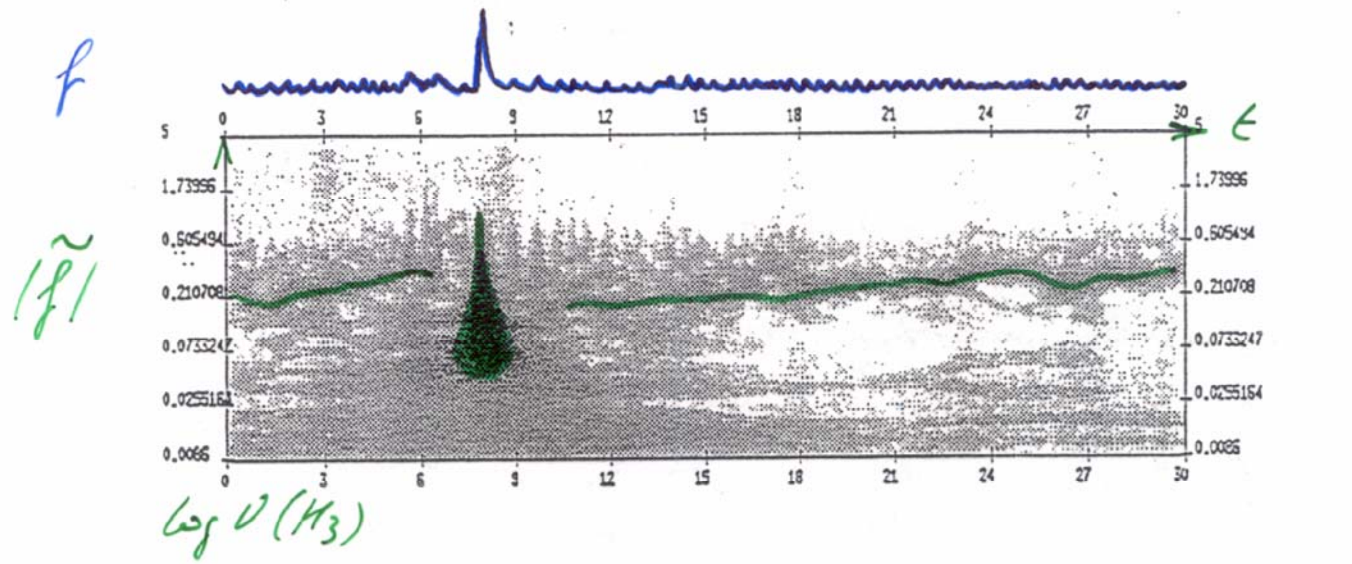
with $\frac{\partial A}{\partial t}$ and $\frac{\partial \varphi}{\partial t} << |e^{i\varphi}| = 1$

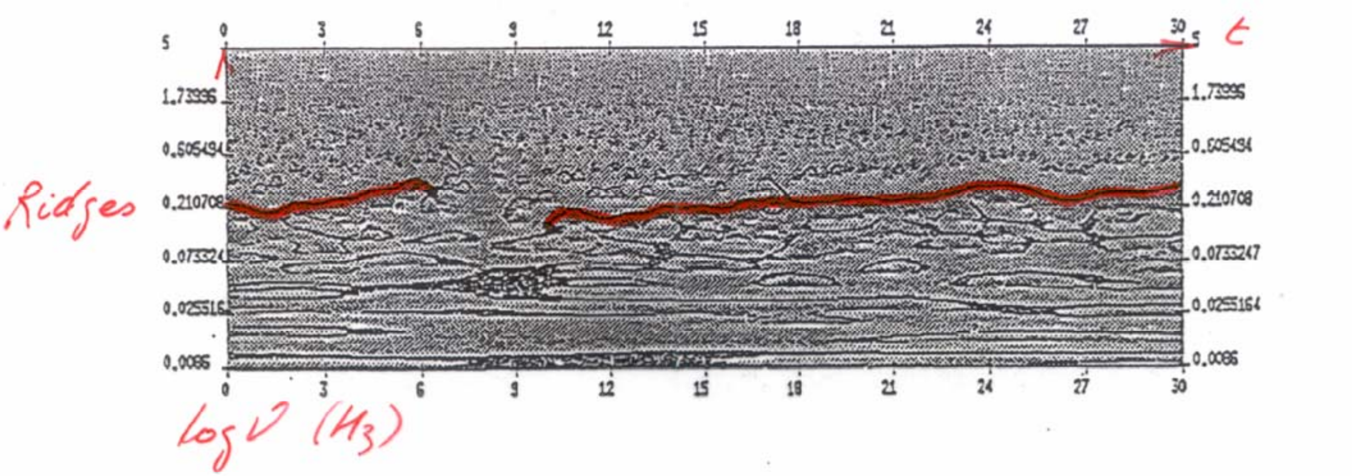
The instantaneous frequency is

$\nu(t) = \frac{\nu_\psi}{2\pi} \frac{\partial \varphi}{\partial t}$ which defines the ridge of the wavelet transform, curve in phase-space (b, a) where $\frac{\partial \varphi}{\partial t} = cst$

1) *Figure 1: Calcul de transformées en ondelettes (signal respiratoire)* MEDICAL SIGNAL

De haut en bas, le signal, le module, et la fréquence instantanée de la transformée en ondelettes (les lignes continues noires représentent les arètes de la transformée) (On s'intéresse à l'évolution temporelle des fréquences instantanées).





Ex. Virgo: $10 \log_{10} \frac{P_S}{P_B} = -40\,dB$ soit $\frac{P_S}{P_B} = 10^{-4}$

Reference:

Carmona, Hwang and Torresani
Characterization of signals by the ridges of their wavelet transforms, 1995
anonymous ftp from chelsea.math.uci.edu
http: cpt.sx1.univ-mrs.fr

CHARACTERIZATION OF THE LOCAL SCALING OF A SIGNAL

The signal f can be a function, a distribution or a measure, that we want to study in x_0.

$$\| \tilde{f}(b,a) \|_{L^1} \leq M_1 |b - x_0|^{\alpha} \|\psi\|_{L^1} + M_1 M_{\alpha} a^{\alpha}$$

$$\text{with } M_{\alpha} = \int_{-\infty}^{+\infty} x^{\alpha} \psi(x)\, dx$$

α is the degree of differentiability of f at x_0 if $\alpha \geq 1$, i.e. f regular in x_0

or α is the Lipschitz exponent of the singularity in x_0 if $-1 < \alpha < 1$.

If we want to eliminate the most regular (polynomial) contribution of f we have to choose ψ with cancellations up to order m (then $M_m = 0$).

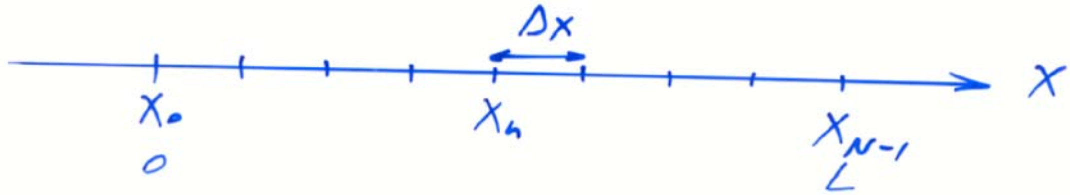
In this case $\| \tilde{f}(b,a) \|_{L^1}$ will only react to regions where f is less smooth as order m.

CWT ALGORITHM (1)

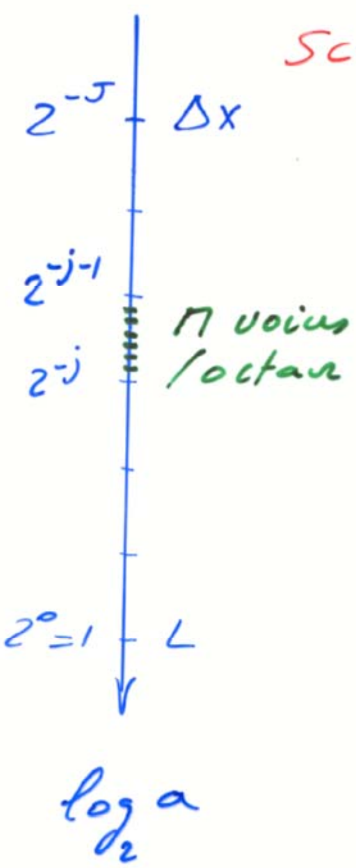
1D periodic signal $f(x)$
1D periodic wavelet $\psi(x)$
$x \in [0, L]$ with period L

Space discretization

Sampling on a regular grid
$\Delta x = \frac{L}{N}$ with $N = 2^J$ samples
$\Rightarrow$ $x_n = \frac{nL}{N}$ and $b_n = \frac{nL}{N}$



Scale discretization



Logarithmic sampling
$a_j = a_0^{-j}$ with $j \geq 0$
$a_0 = 2^{1/M}$ M number of voices/octave

$\Rightarrow$ MJ samples for scale

Largest scale $L \rightarrow a_0 = 1$
Smallest scale $\Delta x \rightarrow a_J = 2^{-J}$

CWT ALGORITHM (2)

L2-norm wavelet family

$$\psi_{a,s}(x) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{x-s}{a}\right)$$

$$\Longrightarrow \hat{\psi}_{a,s}(k) = \sqrt{a}\,\hat{\psi}(ak)\,e^{-2\pi i s k}$$

Wavelet coefficients

$$\tilde{f}(a,s) = \int_{-\infty}^{+\infty} f(x)\,\overline{\psi_{a,s}(x)}\,dx$$

$$\overset{\text{Parseval}}{=} \int_{-\infty}^{+\infty} \hat{f}(k)\,\overline{\hat{\psi}_{a,s}(k)}\,dk$$

$$= \sqrt{a}\int_{-\infty}^{+\infty} \hat{f}(k)\,\hat{\psi}(ak)\,e^{2\pi i k s}\,dk$$

Discrete wavelet coefficients

$$\tilde{f}(a_j, s_n) = a_j^{1/2} \sum_{k=0}^{N-1} \hat{f}_k\,\hat{\psi}(a_j k)\,e^{2\pi i s_n k}$$

where $\hat{f}_k = \frac{1}{N}\sum_{n=0}^{N-1} f(x_n)\,e^{-2\pi i x_n k}$

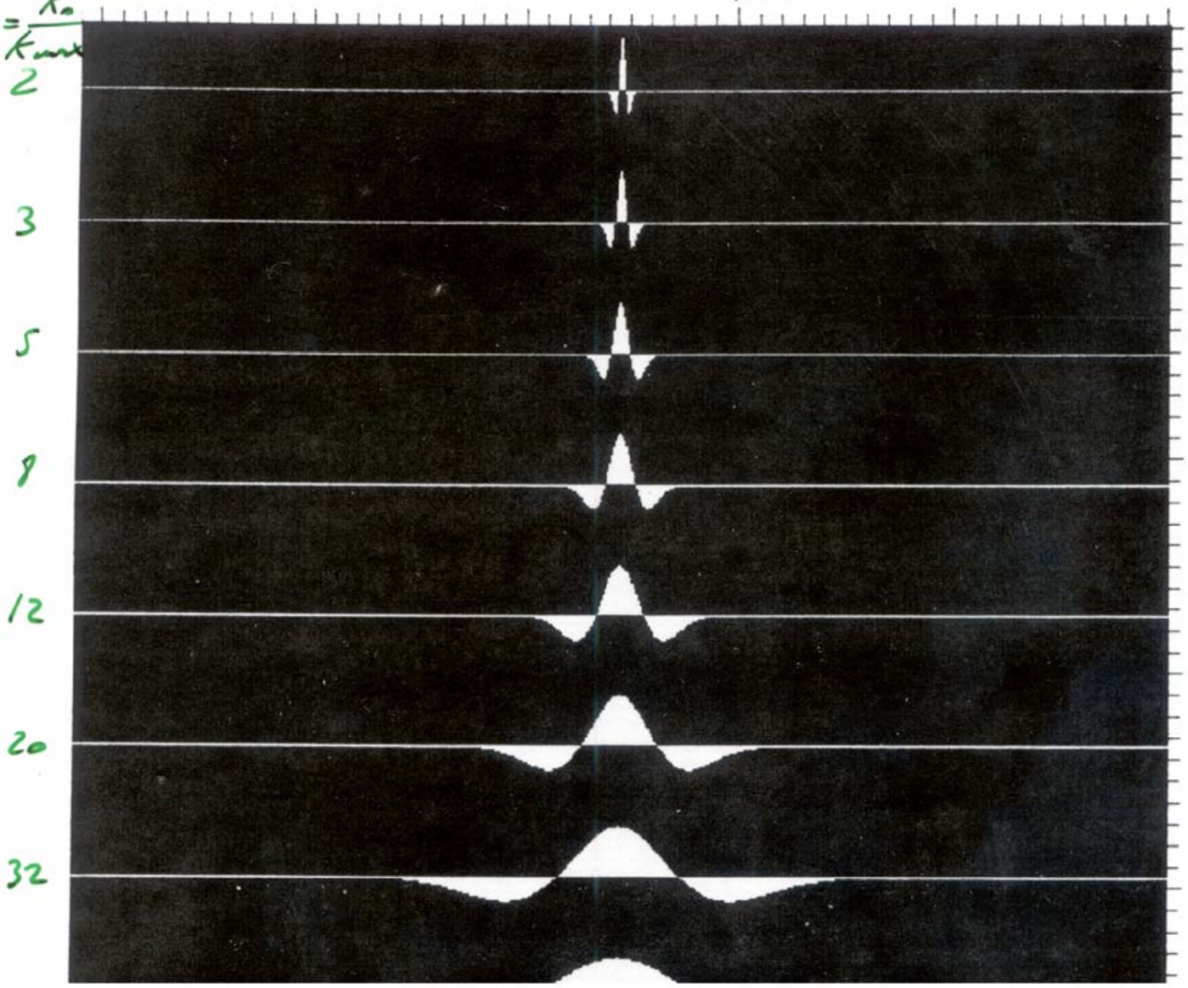
with $x_n = \frac{nL}{N}$, $s_n = \frac{nL}{N}$

WAVELET FAMILY IN L2-NORM
WITH REAL-VALUED WAVELETS.

Scale in pixel units

L signal duration or length
K_{max} Nyquist cut-off frequency

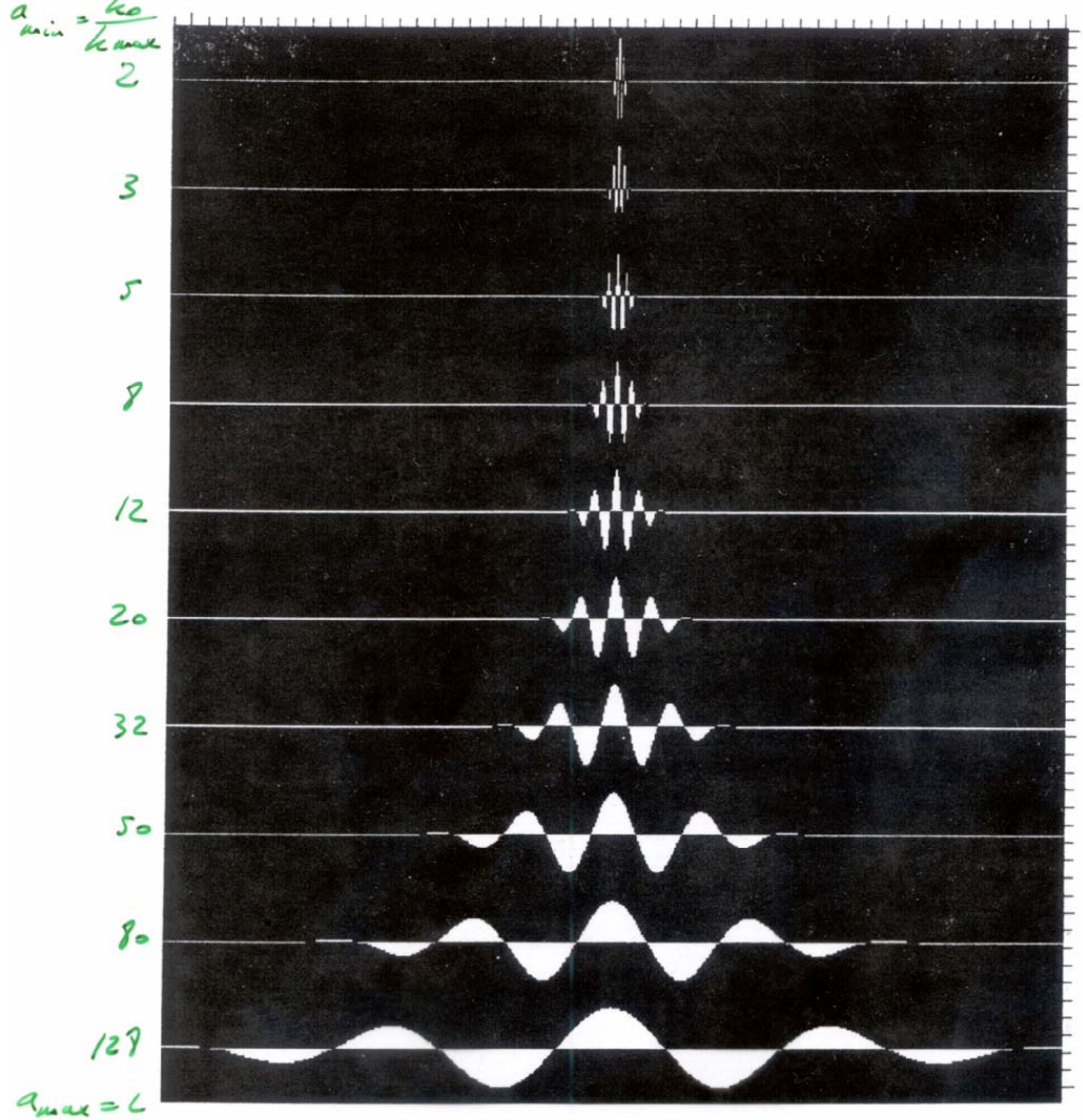
$a_{min} = \frac{K_0}{K_{max}}$



WAVELET FAMILY IN L2-NORM
WITH COMPLEX-VALUED WAVELETS

L signal duration or length
k_{max} Nyquist cut-off frequency

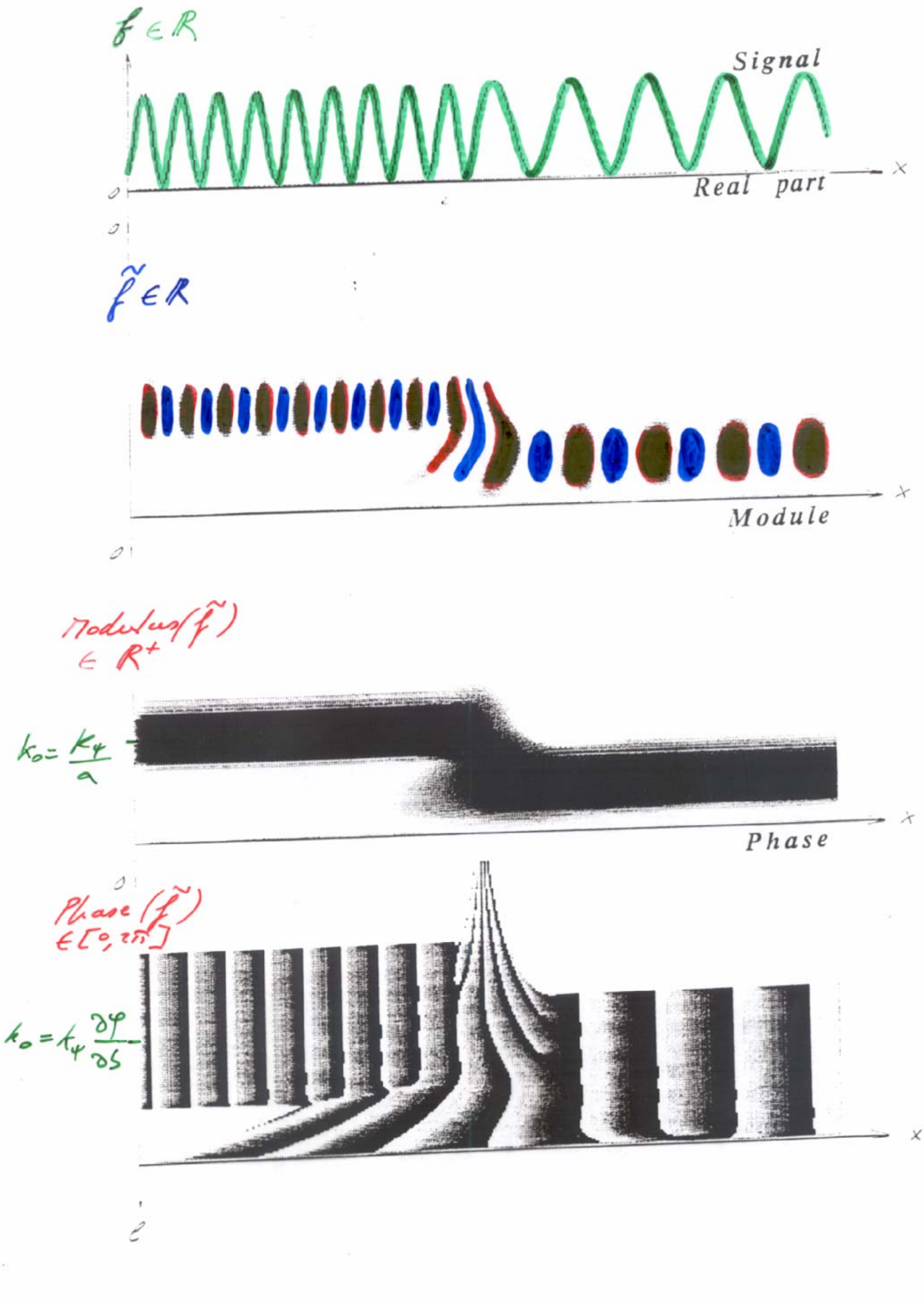
scale in pixel units

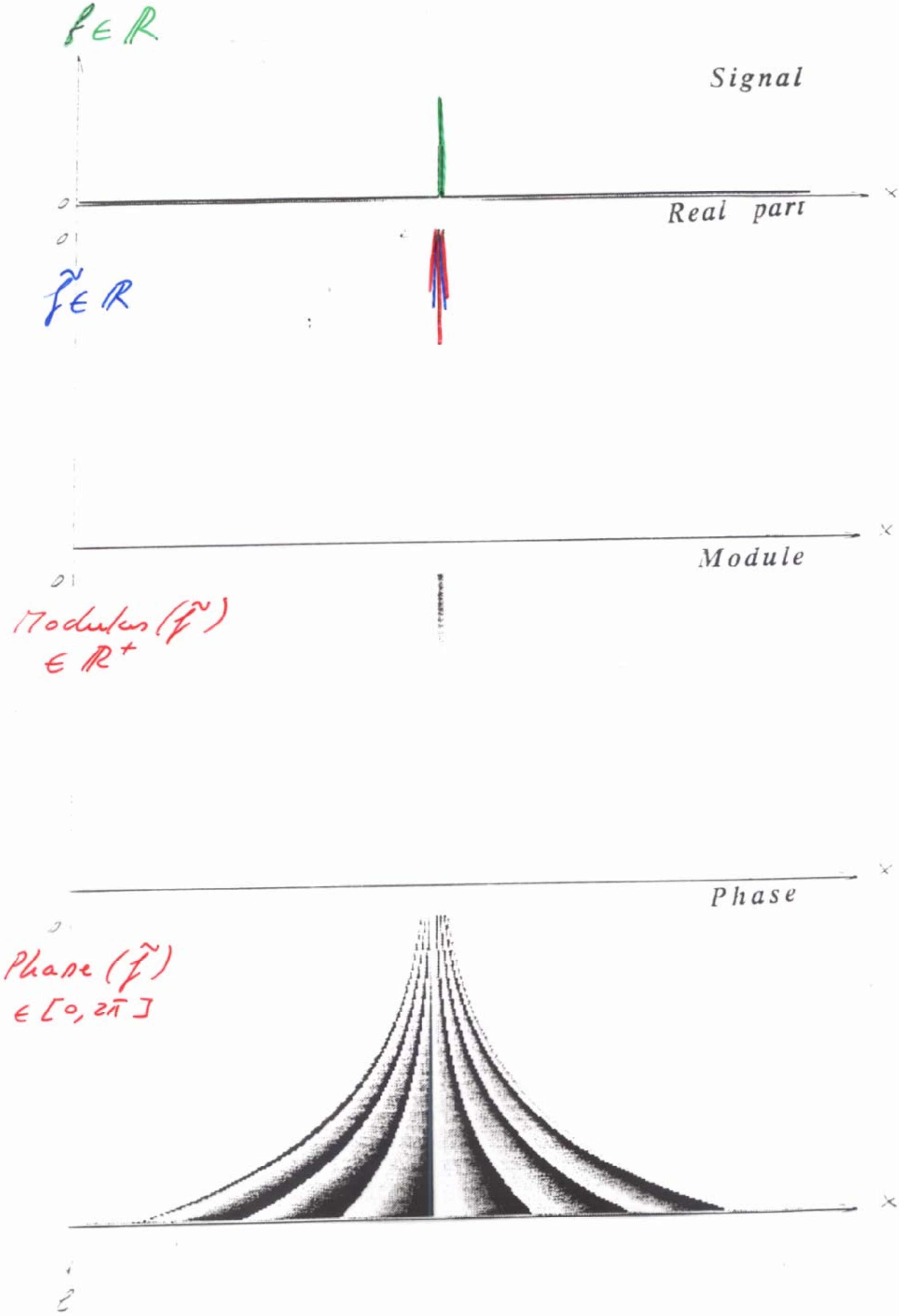


Morlet Wavelet

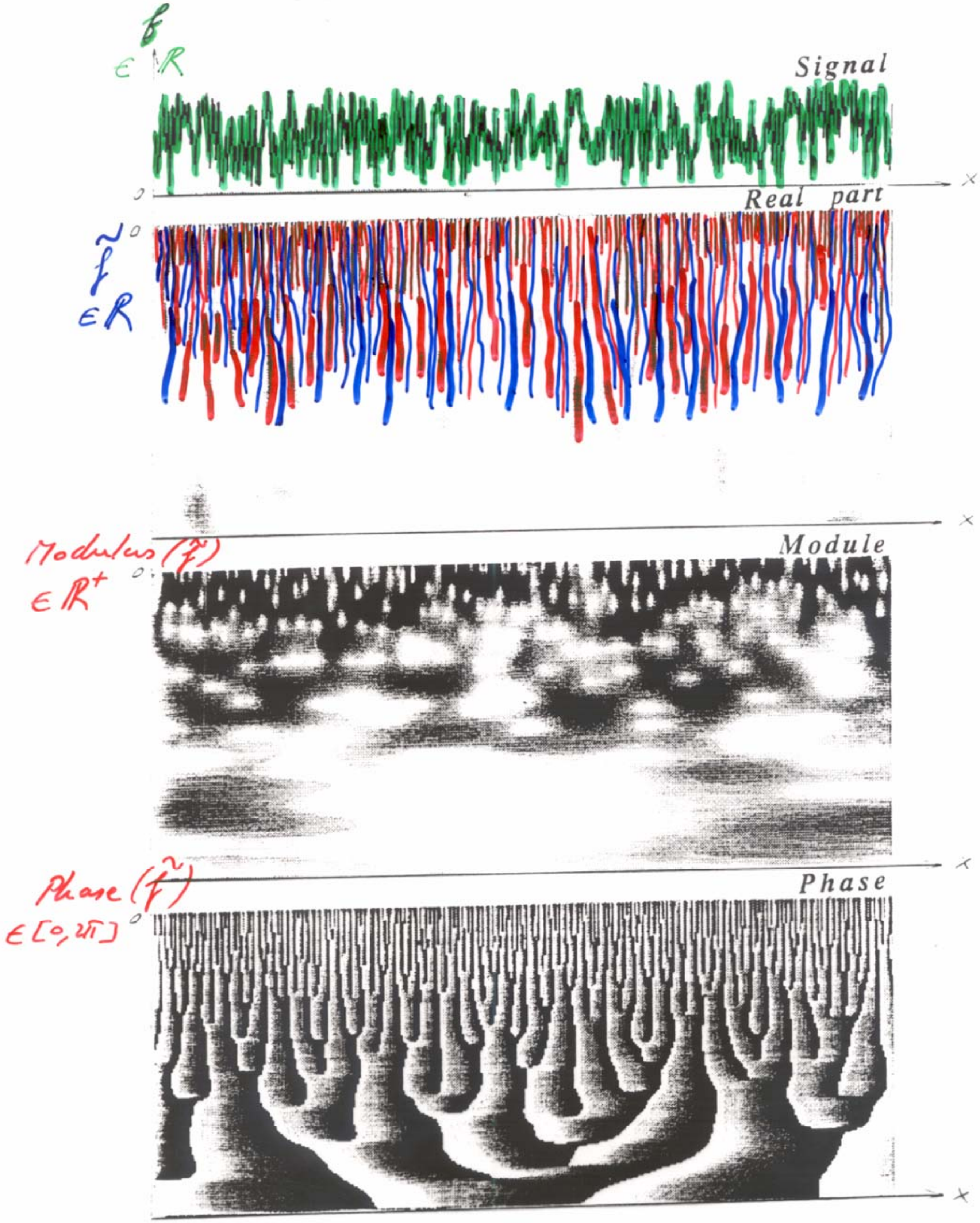
$$\psi(x) = e^{-ik_0 x} e^{-\frac{x^2}{2}}$$

$$\psi \in \mathbb{C}$$


$f \in \mathbb{R}$
Signal
Real part
x
$\tilde{f} \in \mathbb{R}$
Module
x
Modulus($\tilde{f}$) $\in \mathbb{R}^+$
$k_0 = \frac{k_\psi}{a}$
Phase
x
Phase($\tilde{f}$) $\in [0, 2\pi]$
$k_0 = k_\psi \frac{\partial \varphi}{\partial s}$
x



f ∈ ℝ
Signal
x
0
Real part
f̃ ∈ ℝ
x
Module
Modulus(f̃) ∈ ℝ+
x
Phase
Phase(f̃) ∈ [0, 2π]
x


Signal
Real part
Module
Phase
$f \in \mathbb{R}$
$\tilde{f} \in \mathbb{R}$
Modulus($\tilde{f}$) $\in \mathbb{R}^+$
Phase($\tilde{f}$) $\in [0, 2\pi]$