The discrete Wavelet Transform

- continuous case

\[ f(x) \leftrightarrow W(a,b) \]

- discrete case

\[ f(x) \leftrightarrow \psi_{j,k}(x) \]

**Question:** Choice of \( a_j, b_k \)?

**Theory** → 

\[ a_j = a_0^j \text{ with } a_0 = 2 \]

\[ b_k = b_0 a_0^j \text{ with } b_0 = 1 \]

discrete wavelets

\[ \psi_{j,k}(x) = \frac{1}{\sqrt{a_j}} \psi \left( \frac{x - b_k}{a_j} \right) \]

\[ = 2^{-j/2} \psi \left( 2^{-j} x - k \right) \]
Kontinuierliche Wavelet Transformation

\[ f(x) \]

\[ d_{a,b} \]

\[ \tilde{f}(a, b) = \int \phi(x) \psi_{a,b}(x) \, dx \]

Diskrete Wavelet Transformation

\[ f(x) \]

\[ d_{j,k} \]

\[ \tilde{f}_j k = \int f(x) \psi_k(x) \, dx \]

Basiswavelet: Mexican Hat

Basiswavelet: quint. period. Spline
Orthogonal wavelets are the mathematical tool to describe the addition of details needed to go from a coarse approximation $\tilde{f}_j$ to a finer approximation $\tilde{f}_{j+1}$.

For this we need two functions $\Phi(x)$ such that $\int_{-\infty}^{\infty} \Phi(x) \, dx = 1$.

which will be needed to compute the approximations $\tilde{f}_j$ at different scales and the mother wavelet $\psi(x)$ such that $\int_{-\infty}^{\infty} \psi(x) \, dx = 0$.

which will be used to compute the details $\tilde{f}_j$ to go from scale $j$ to scale $j+1$.  

Haar scaling function

Haar wavelet

\[ f(x) \]
\[ \hat{f}_0(x) \in \mathcal{V}_0 \]
\[ \hat{f}_1(x) \in \mathcal{V}_1 \]

\[ w_0 \in \hat{f}_0 = \hat{f}_1 - \hat{f}_0 \]
2. Two-dimensional Multiresolution

a) One dim. case

Def: A multiresolution analysis (MRA) of $\mathcal{L}^2(\mathbb{R})$ is a set of closed subspaces $\{V_j\}_{j \in \mathbb{Z}}$ with

i) $\ldots V_{j-1} \subset V_j \subset V_{j+1} \ldots$

ii) $\bigcup_{j \in \mathbb{Z}} V_j = \mathcal{L}^2(\mathbb{R})$

iii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$

iv) $f(x) \in V_j \iff f(2x) \in V_{j+1}$

Scaling function $\varphi_{ji}(x)$ generates $V_j$

Orthogonality $\langle \varphi_{ji}, \varphi_{jk} \rangle = \delta_{ik}$

$f \in V_j$

$f_j(x) = \sum_k \langle f, \varphi_{jk} \rangle \varphi_{jk} (x)$
Main issue of the wavelet approach is to work with the orthogonal complement spaces $W_j$ defined by

$$W_j = V_{j+1} \oplus V_j$$

The space $W_j$ is generated by the wavelets

$$W_j = \text{span} \{ \psi_{j,k} \}_{k \in \mathbb{Z}}$$

Some properties:

- **Orthogonality** \(< \psi_{j,k}, \psi_{i,l} > = \delta_{ji} \delta_{kl}\)**

- **Vanishing moments** \(\int x^k \psi_{j,k}(x) \, dx = 0\)
  for $k = 0, \ldots, 2^j - 1$

- **Decay** \(|\psi(x)| \leq \frac{C}{(1 + |x|^n)}\)
  for $n \in \mathbb{N}$

\(L^2(\mathbb{R}) = \bigoplus_{j \geq 0} W_j\)

\(f \in L^2(\mathbb{R})\)

\(f(x) = \sum_k c_k \psi_{0,k}(x) + \sum_{j \geq 0} \sum_k d_{j,k} \psi_{j,k}(x)\)

with \(c_k = \langle f, \psi_{0,k} \rangle\)

\(d_{j,k} = \langle f, \psi_{j,k} \rangle\)
Figure 7
Fourier transforms of $\psi$ and $\bar{\psi}$
1. Projection onto $V_j$

given $f \in L^2(\mathbb{R})$ or samples $f(\frac{n}{2^j})$

Question: How to get an approximation $f_j$ of $f$ in $V_j$?

$$f_j(x) = \sum_k c_{jk} \phi_{jk}(x)$$

scaling coefficients

Theoretically $c_{jk} = \langle f, \phi_{jk} \rangle$

Practically different possibilities:

- $c_{jk} = f(\frac{k}{2^j}) \rightarrow$ accuracy $O(2^{-2j})$
  (or Coiflets to get $O(2^{-3j})$)

- quadrature rule $c_{jk} = \sum_n A_n f(\frac{n-k}{2^j})$
  (Sweldens)

- here collocation:

$$f_j(\frac{k}{2^j}) = f(\frac{k}{2^j})$$
Using the cardinal function $S_{\frac{1}{2}}$ in $V_2$ we can represent $f_2$ as

$$f_2(x) = \sum_{k} f\left(\frac{k}{2^{n}}\right) S_{\frac{1}{2}}\left(x - \frac{k}{2^{n}}\right)$$

where $S_{\frac{1}{2}}\left(\frac{k}{2^{n}}\right) = \delta_{k,0}$

and

$$V_2 = \text{span} \{ S_{\frac{1}{2}}(x) = S_{\frac{1}{2}}\left(x - \frac{k}{2^{n}}\right) \}$$

The coefficients $c_{2,k}$ of $f_2$ are then calculated by convolution of the samples

$$c_{2,n} = \sum_{k} f\left(\frac{k}{2^{n}}\right) I^{2}(n-k)$$

where

$$I^{2}(n) = \langle S_{2,n}, \phi_{2,0} \rangle$$
Projection onto $V_{j-1}$ and $W_{j-1}$

We have

$$f_j(x) = \sum_k c_{j,k} \varphi_{j,k}(x) \in V_j$$

$$= \sum_k g_{j-1,k} \varphi_{j-1,k}(x) + \sum_k d_{j-1,k} \psi_{j-1,k}(x)$$

$$\cap_{V_{j-1}} \oplus W_{j-1}$$

We can calculate the $c_{j,k}$ and $d_{j,k}$ from the $c_{j,k}$ using filters:

$$H_j(n) = \langle \varphi_{j,n}, \varphi_{j-1,0} \rangle$$

$$G_j(n) = \langle \varphi_{j,n}, \psi_{j-1,0} \rangle$$

$$c_{j+1,n} = \sum_k c_{j,k} H_j(k-2n)$$

$$d_{j+1,n} = \sum_k c_{j,k} G_j(k-2n)$$

Finally we represent $f_j$ as

$$f_j(x) = \sum_k c_{j-1,k} \varphi_{j-1,k}(x) + \sum_{j \geq 0} \sum_k d_{j,k} \psi_{j,k}(x)$$

Remark:

$H_j$: low pass filter

$G_j$: band pass filter

Reconstruction:

$$c_{j,n} = \sum c_{j-1,k} H_j(n-2k) + \sum_{j \geq 0} d_{j,k} G_j(n-2k)$$
1D Multi Resolution Analysis (Fast WLT)

\[ L^2(\mathbb{R}) \rightarrow f(x) \]

\[ V_J \rightarrow f(\frac{k}{2^J}) \quad k=0, \ldots, 2^J-1 \]

\[ L_J \rightarrow V_J \rightarrow c_{j,k} \quad k=0, \ldots, 2^J-1 \]

\[ H_J \rightarrow G_J \rightarrow V_{j-1} \rightarrow c_{j-1,k}, d_{j-1,k} \quad W_{j-1} \quad k=0, \ldots, 2^{J-1}-1 \]

\[ H_{j-1} \rightarrow G_{j-1} \rightarrow V_{j-2} \rightarrow c_{j-2,k}, d_{j-2,k} \quad W_{j-2} \quad k=0, \ldots, 2^{J-2}-1 \]

\[ H_1 \rightarrow G_1 \rightarrow V_0 \rightarrow c_{0,0}, d_{0,0} \quad W_0 \]

\[ f_j(x) = \sum_{k=0}^{2^j-1} c_{j,k} \phi_{j,k}(x) = \sum_{k=0}^{2^{j-1}-1} c_{j-1,k'} \phi_{j-1,k'}(x) + \sum_{k'=0}^{2^{j-1}-1} d_{j-1,k'} \psi_{j-1,k'}(x) \]
1D Multi Resolution Analysis

\[
\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k) \\
\phi_{j,k}(x) = 2^{j/2} \phi(2^j x - k)
\]

with

\[ k \in \mathbb{Z}, \quad j \in \mathbb{Z} \geq 0 \]

\[
c_{j,k} = \langle f | \phi_{j,k} \rangle \\
= \int f(x) \overline{\phi}_{j,k}(x) \, dx
\]

\[
d_{j,k} = \langle f | \psi_{j,k} \rangle \\
= \int f(x) \overline{\psi}_{j,k}(x) \, dx
\]

\[
f_j(x) = c_{0,0} \phi_{0,0}(x) + \sum_{j=0}^{J-1} \sum_{k \in \mathbb{Z}} d_{j,k} \psi_{j,k}(x)
\]

Complexity: \[ O(N) \quad N = 2^J \]
1. Interpolation

\[ (f_n)_{n=0}^{2^P-1} \]

\[ \text{FFT} (2^P) \]

\[ \hat{f}(k)_{k=0}^{2^P-1} \]

\[ \hat{c}_p(k) = 2^P \hat{f}(k) \hat{L}_p(k) \quad \text{for } k = 0, \ldots, 2^P-1 \]

2. Projection onto \( V_{p-1} \) and \( W_{p-1} \)

For \( j = p, \ldots, 1 \)

\[ \hat{c}_j(k) \]

\[ \hat{c}_j(k) = \hat{c}_j(k) \overline{H}_j(k) 2^i \quad \text{for } k = 0, \ldots, 2^{i-1} \]

\[ \hat{d}_j(k) = \hat{c}_j(k) \overline{G}_j(k) 2^i \quad \text{for } k = 0, \ldots, 2^{i-1} \]

\[ \hat{c}_{j-1}(k) = \hat{c}_j(k) + \hat{c}_j(k + 2^{i-1}) \quad \text{for } k = 0, \ldots, 2^{i-1} \]

\[ \hat{d}_{j-1}(k) = \hat{d}_j(k) + \hat{d}_j(k + 2^{i-1}) \quad \text{for } k = 0, \ldots, 2^{i-1} \]

\[ j = 1 \]

\[ c_{0,0} = \hat{c}_0(0) \]

\[ d_{j-1,n} \]

\[ \text{FFT}^{-1} (2^{j-1}) \]