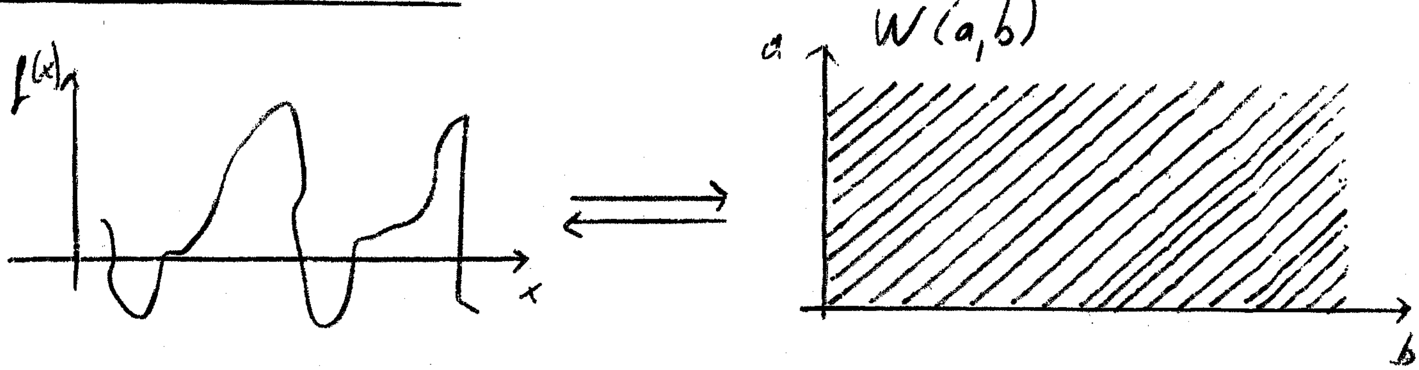
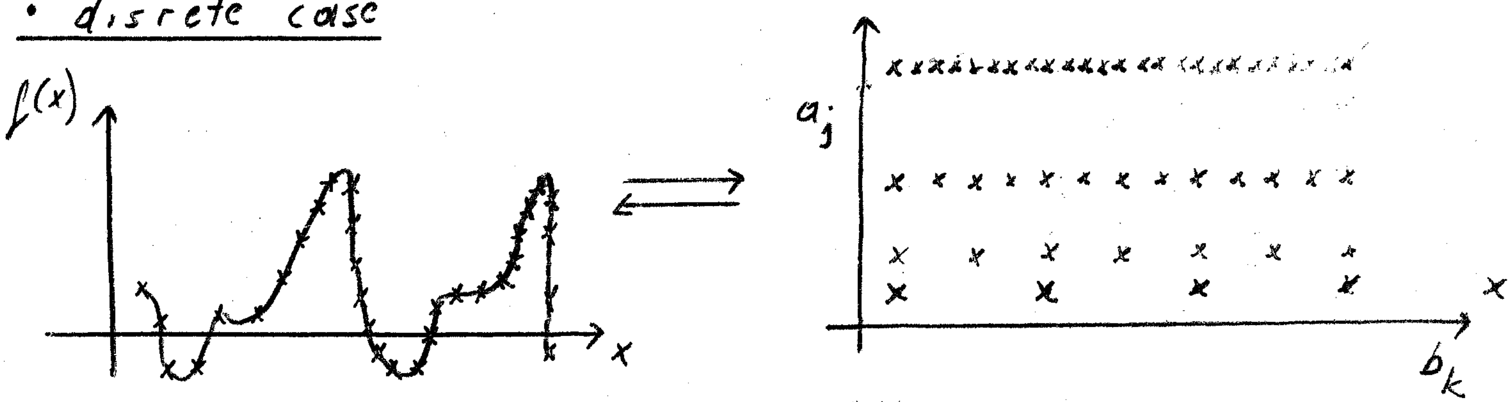
The discrete Wavelet Transform

• continuous case



• disrete case



Question: Choice of a_j, b_k ?

Theory $\rightarrow$ $a_j = a_0^{\,j}$ with $a_0 = 2$

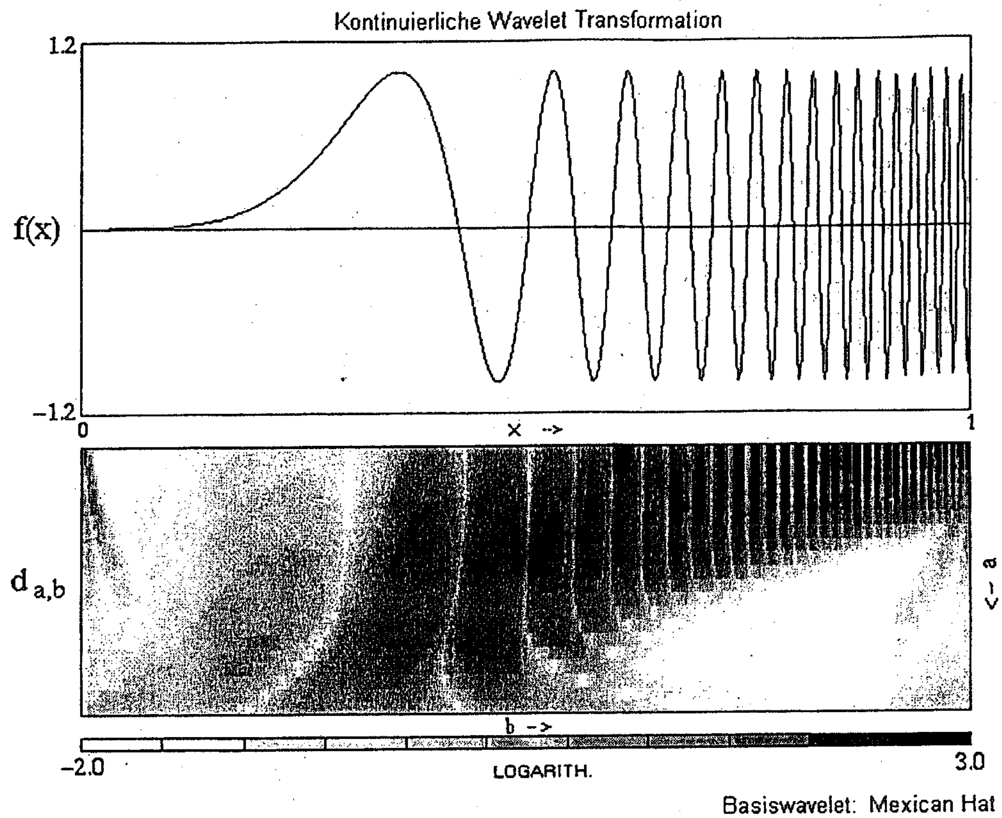
$b_k = k\, b_0\, a_0^{\,j}$ with $b_0 = 1$

discrete wavelets

$$\psi_{jk}(x) = \frac{1}{\sqrt{a_j}}\, \psi\left(\frac{x - b_k}{a_j} \right)$$

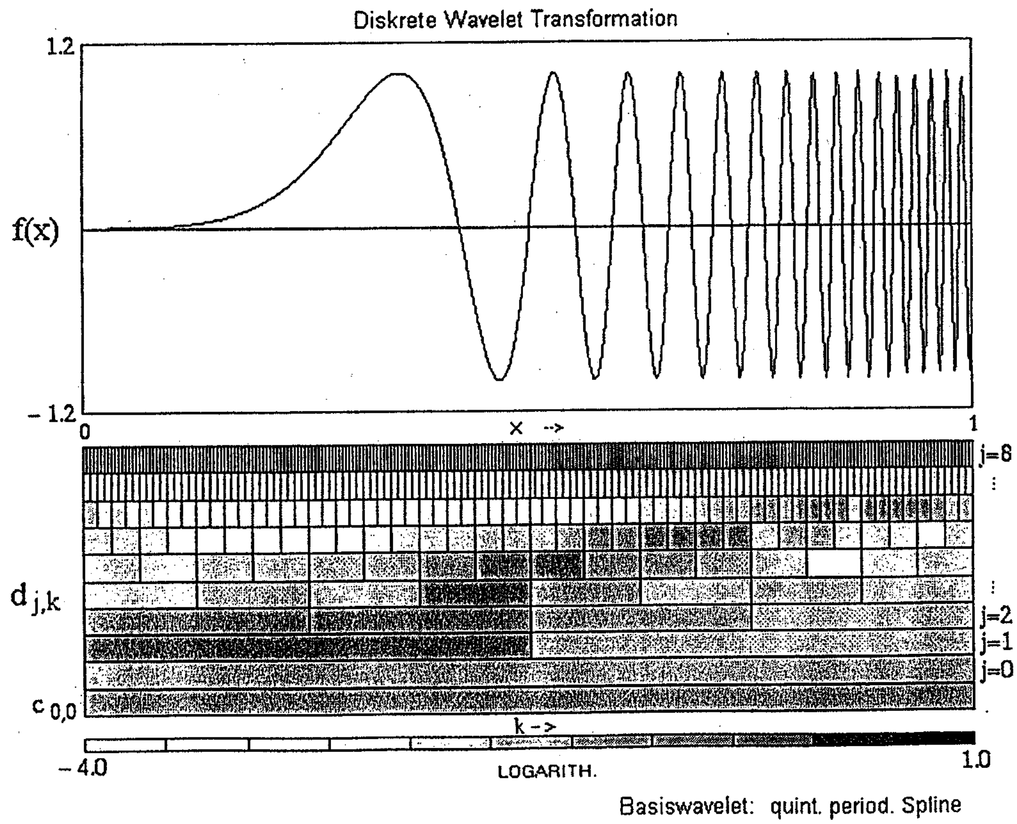
$$= 2^{-j/2}\, \psi\left(2^{-j} x - k\right)$$

Chirp



$\tilde{f}(a,b) =$

$= \int f(x)\,\psi_{a,b}(x)\,dx$



$\tilde{f}_{jk} =$

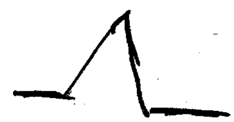
$\int f(x)\,\psi_{jk}(x)\,dx$

Multiresolution Analysis (MRA)

Orthogonal wavelets are the mathematical tool to describe the addition of details needed to go from a coarse approximation $\bar{f}_j$ to a finer approximation $\bar{f}_{j+1}$.

For this we need two functions

the scaling function $\varphi(x)$



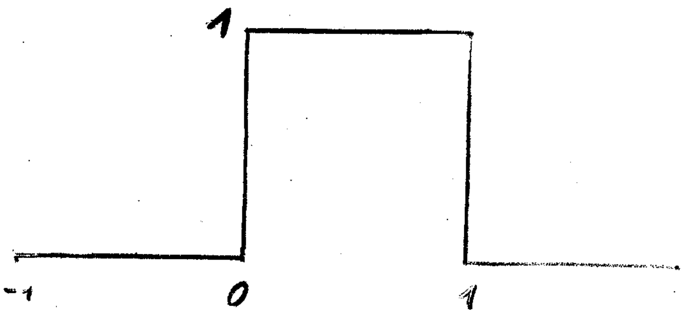
such that $\int_{-\infty}^{\infty} \varphi(x)\, dx = 1$

which will be needed to compute the approximations $\bar{f}_j$ at different scales and the mother wavelet $\psi(x)$

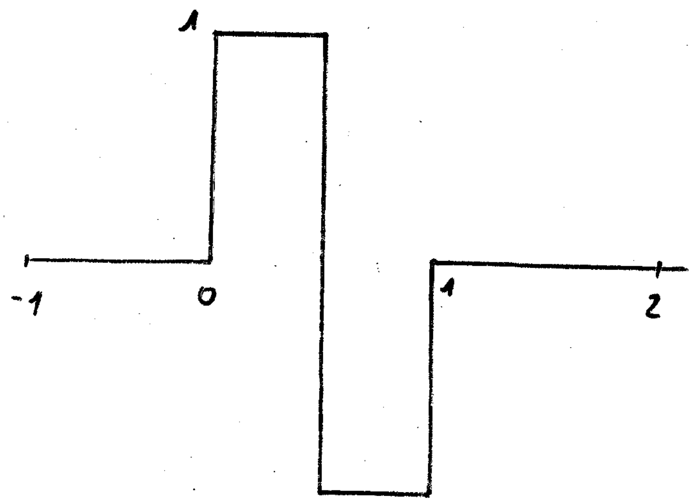
such that $\int_{-\infty}^{\infty} \psi(x)\, dx = 0$

which will be used to compute the details $\tilde{f}_j$ to go from scale j to scale $j+1$.

Exemple MRA: Haar Wavelets



Haar scaling function

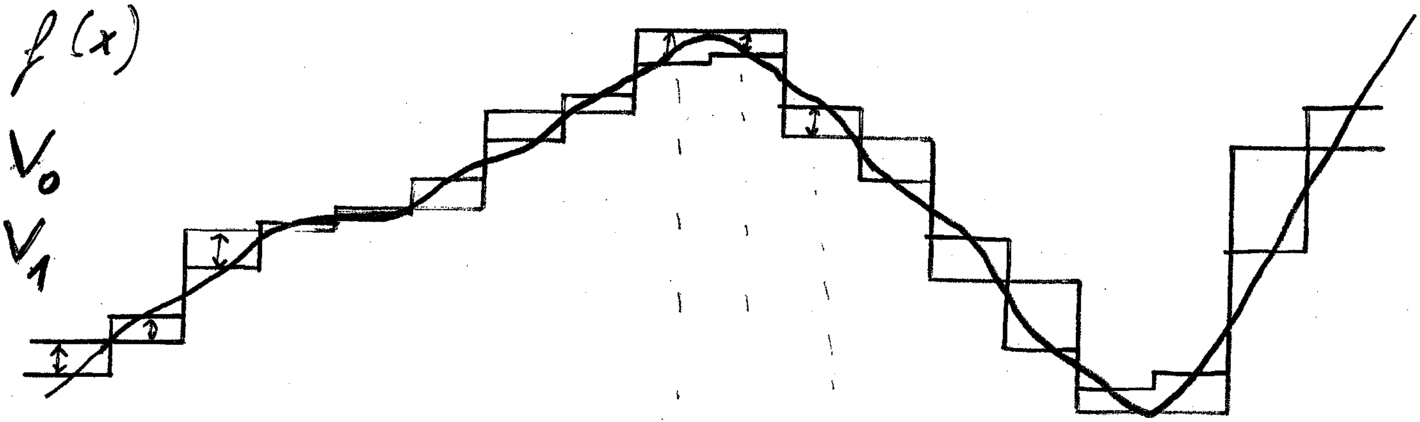


Haar wavelet

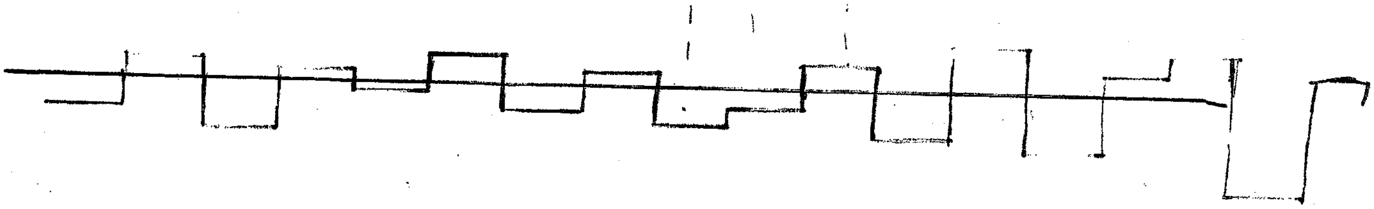
$f(x)$

$\bar{f}_0(x) \in V_0$

$\bar{f}_1(x) \in V_1$



$$W_0 \ni \tilde{f}_0 = \bar{f}_1 - \bar{f}_0$$



2. Twodimensional Multiresolution

a) One dim. case

Def.: A multiresolution analysis (MRA) of $L^2(\mathbb{R})$ is a set of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ with

i) $\dots V_{j-1} \subset V_j \subset V_{j+1} \dots$

ii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$

iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$

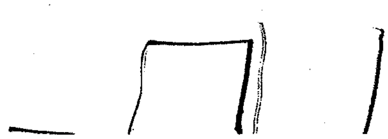
iv) $f(x) \in V_j \iff f(2x) \in V_{j+1}$

Scaling function $\varphi_{ji}(x)$ generates V_j.

Orthogonality $\langle \varphi_{ji}, \varphi_{jk} \rangle = \delta_{ik}$

$$f_j \in V_j$$

$$f_j(x) = \sum_k \langle f, \varphi_{jk} \rangle \varphi_{jk}$$



Main issue of the wavelet approach is to work with the orthogonal complement spaces W_j defined by

$$W_j = V_{j+1} \ominus V_j$$

The space W_j is generated by the <u>wavelets</u>

$$W_j = \overline{\text{span}} \{ \psi_{ji} \}_{i \in \mathbb{Z}}$$

<u>Some properties:</u>

- Orthogonality $\quad \langle \psi_{jk}, \psi_{il} \rangle = \delta_{ji} \delta_{kl}$
- vanishing moments $\quad \int x^k \psi_{jk}(x)\, dx = 0$ for $k = 0, .., M-1$
- decay $\quad |\psi(x)| \leq \dfrac{C}{(1 + |x|^n)} \quad n \in \mathbb{N}$

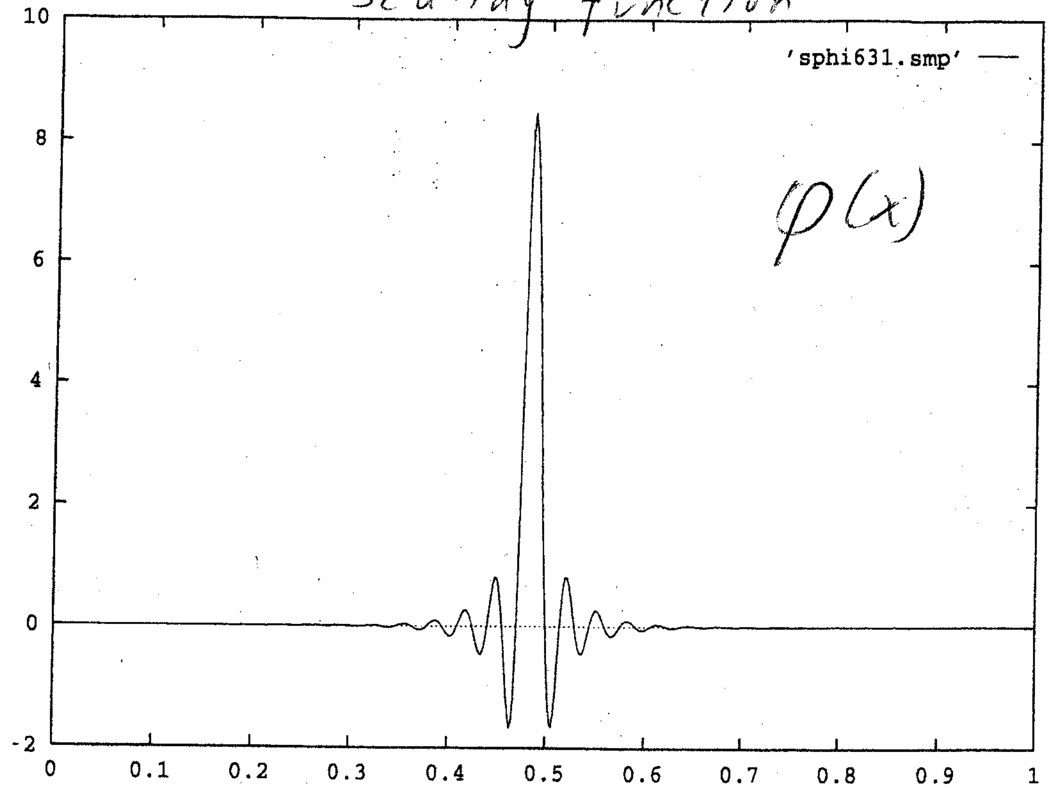
$$L^2(\mathbb{R}) = V_0 \bigoplus_{j \geq 0} W_j$$

$f \in L^2(\mathbb{R})$

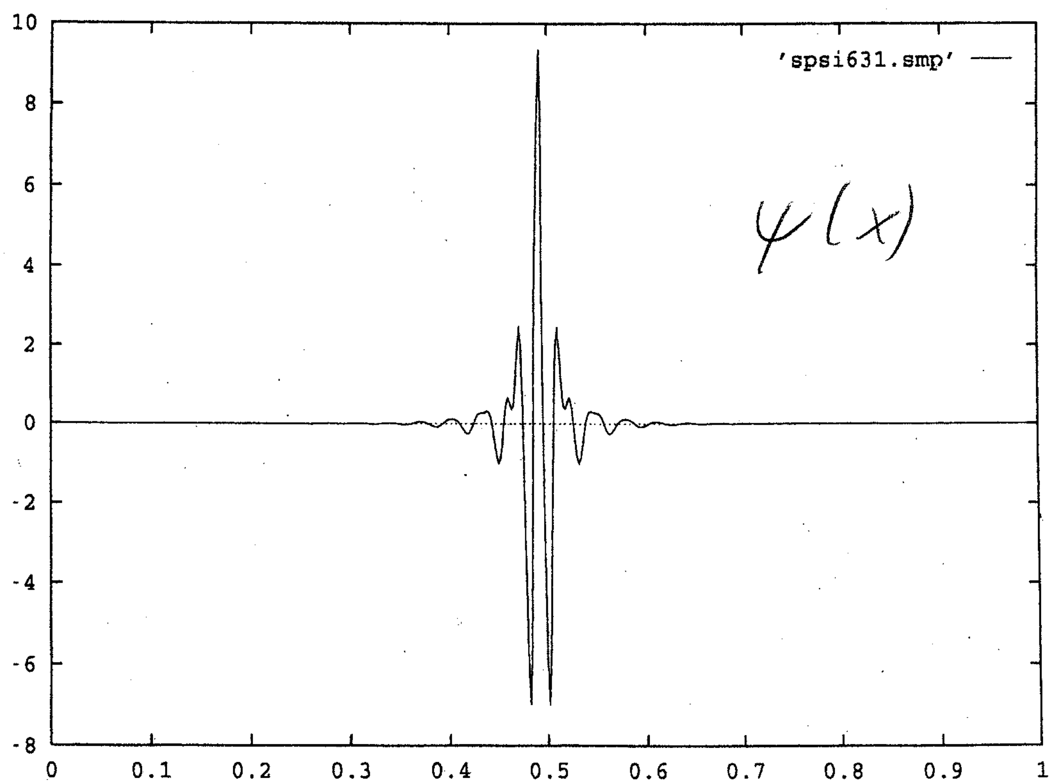
$$f(x) = \sum_k c_{0k} \varphi_{0k}(x) + \sum_{j \geq 0} \sum_k d_{jk} \psi_{jk}(x)$$

with $c_{0k} = \langle f, \varphi_{0k} \rangle \qquad d_{jk} = \langle f, \psi_{jk} \rangle$

Scaling function

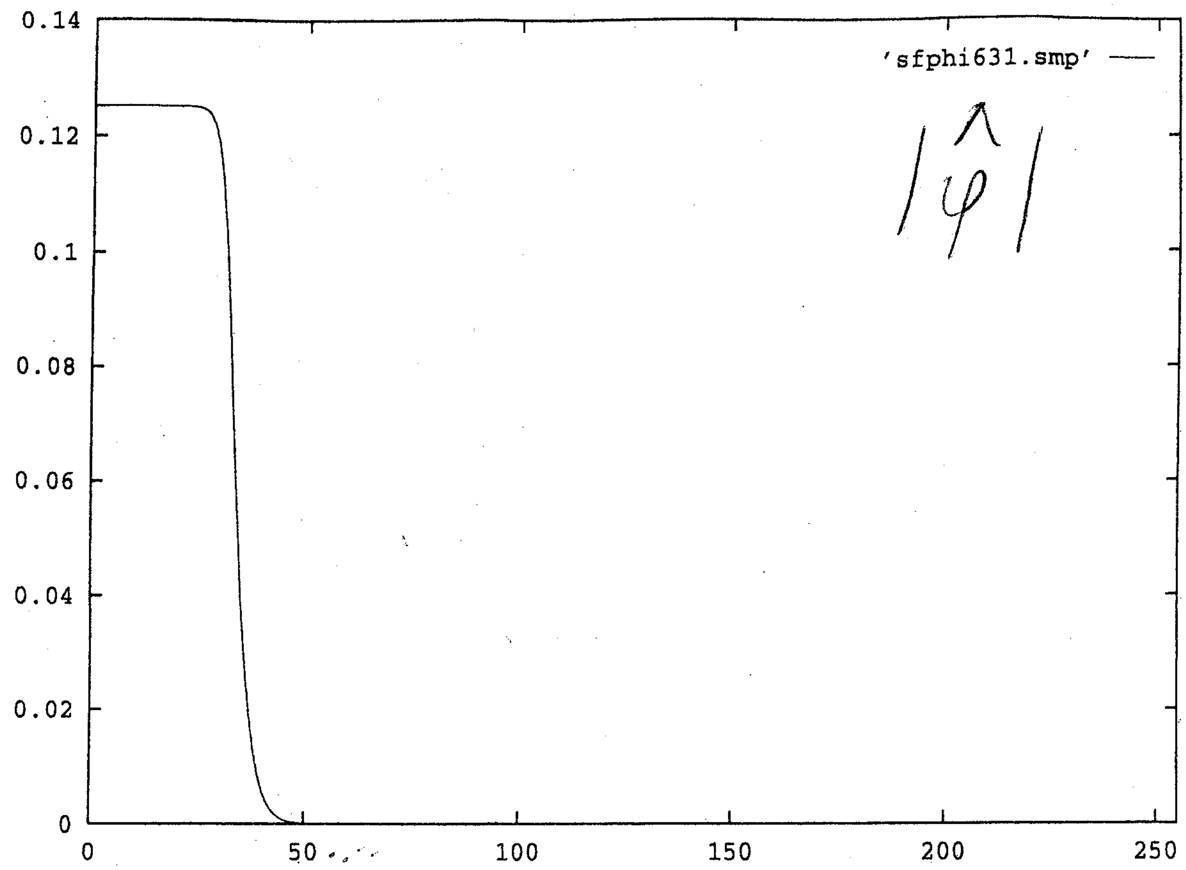


Wavelet

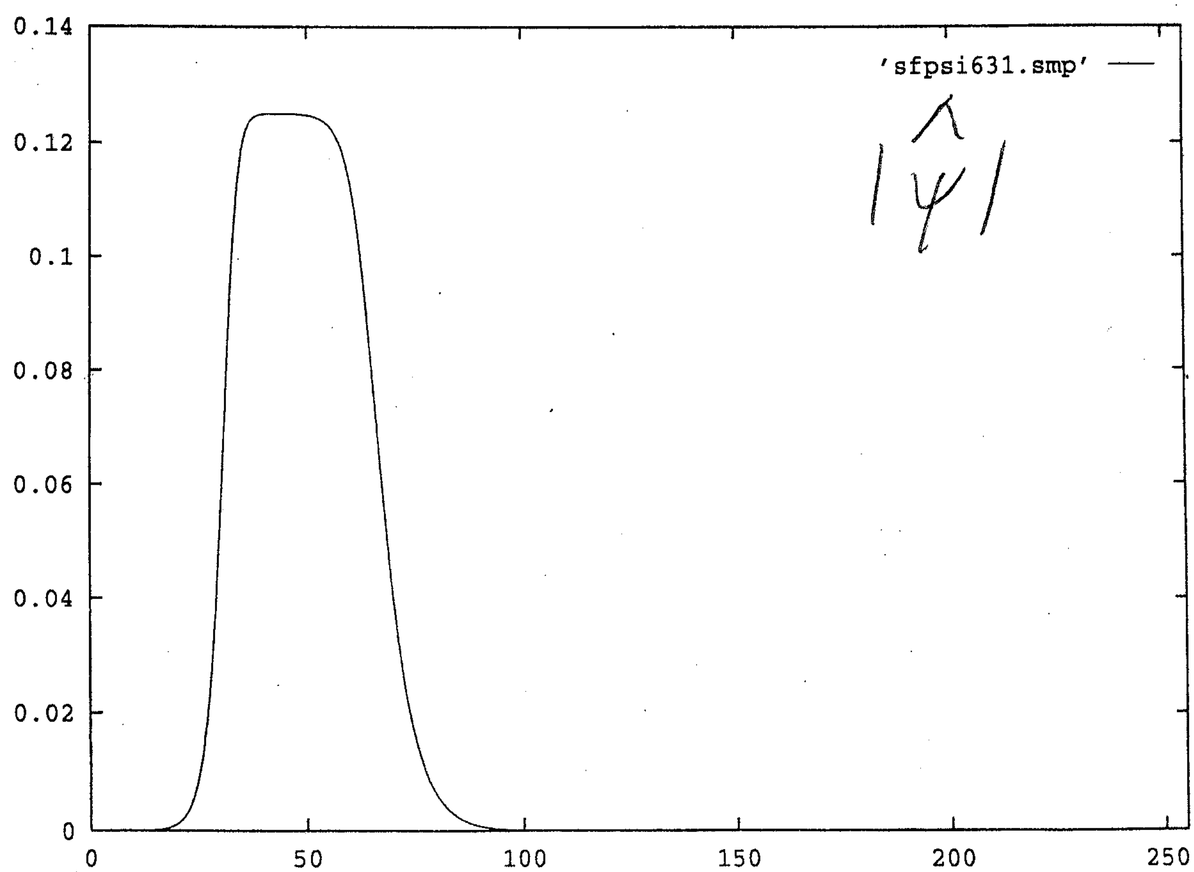


Quintic splines

Figure 2



Fourier transforms of φ and ψ



1. Projection onto V_j

given $f \in L^2(\mathbb{R})$ or samples $f\left(\frac{n}{2^j}\right)$

Question: How to get an approximation f_j of f in V_j?

$$f_j(x) = \sum_k c_{jk} \varphi_{jk}(x)$$

scaling coefficients

Theoretically $c_{jk} = \langle f, \varphi_{jk} \rangle$

Practically different possibilities:

- $c_{jk} = f\left(\frac{k}{2^j}\right) \longrightarrow$ accuracy $O(2^{-j})$
  (or Coiflets to get $O(2^{-jm})$
- quadrature rule $c_{jk} = \sum_n A_n f\left(\frac{n-k}{2^{j+\alpha}}\right)$
  (Sweldens)
- <u>here collocation:</u>

$$f_j\left(\frac{k}{2^j}\right) = f\left(\frac{k}{2^j}\right)$$

Using the cardinal function S_J in V_J we can represent f_J as

$$f_J(x) = \sum_k f\left(\frac{k}{2^J}\right) S_J\left(x - \frac{k}{2^J}\right)$$

where $S_J\left(\frac{k}{2^J}\right) = \delta_{k,0}$

and

$$V_J = \overline{\operatorname{span}} \left\{ S_{Jk}(x) = S_J\left(x - \frac{k}{2^J}\right) \right.$$

The coefficients $c_{J,k}$ of f_J are then calculated by convolution of the samples

$$c_{J,n} = \sum_k f\left(\frac{k}{2^J}\right) I^J(n-k)$$

where

$$I^J(n) = \langle S_{J,n}, \varphi_{J,0} \rangle$$

Projection onto V_{j-1} and W_{j-1}

We have

$$f_J(x) = \sum_k c_{J,k}\, \varphi_{J,k}(x) \qquad \in V_J$$

$$= \sum_k c_{J-1,k}\, \varphi_{J-1,k}(x) + \sum_k d_{J-1,k}\, \psi_{J-1,k}(x)$$

$$\underset{V_{J-1}}{\cap} \qquad \oplus \qquad \underset{W_{J-1}}{\cap}$$

We can calculate the $c_{J-1,k}$ and $d_{J-1,k}$ from the $c_{J,k}$ using filters:

$$H_J(n) = \langle \varphi_{J,n}, \varphi_{J-1,0} \rangle \qquad \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$G_J(n) = \langle \varphi_{J,n}, \psi_{J-1,0} \rangle \qquad \left(\tfrac{1}{2}, -\tfrac{1}{2}\right)$$

$$c_{J-1,n} = \sum_k c_{J,k}\, H_j(k-2n)$$

$$d_{J-1,n} = \sum_k c_{J,k}\, G_j(k-2n)$$

Finally we represent f_J as

$$f_J(x) = \sum_k c_{0,k}\, \varphi_{0,k}(x) + \sum_{j\geq 0}^{J-1} \sum_k d_{jk}\, \psi_{jk}(x)$$

Remark: H_j : low pass filter
G_j : band pass filter

Reconstruction:

$$c_{J,n} = \sum_k c_{J-1,k}\, H_j(n-2k) + \sum_k d_{J-1,k}\, G_j(n-2k)$$

1D Multi Resolution Analysis (Fast WLT)

$$\boxed{\begin{aligned} c_{j,k} &= \langle f | \phi_{j,k} \rangle \\ d_{j,k} &= \langle f | \psi_{j,k} \rangle \end{aligned}}$$

$$\begin{tikzcd}
L^2(\mathbf{R}) & f(x) \arrow[d] & & & \\
V_J & f\left(\frac{k}{2^J}\right) \arrow[d, "L_J"'] & & & k=0, \ldots, 2^J-1 \\
V_J & c_{J,k} \arrow[d, "H_J"'] \arrow[dr, "G_J"] & & & k=0, \ldots, 2^J-1 \\
V_{J-1} & c_{J-1,k} \arrow[d, "H_{J-1}"'] \arrow[dr, "G_{J-1}"] & d_{J-1,k} & W_{J-1} & k=0, \ldots, 2^{J-1}-1 \\
V_{J-2} & c_{J-2,k} \arrow[d, dotted, "H_1"'] \arrow[dr, dotted, "G_1"] & d_{J-2,k} & W_{J-2} & k=0, \ldots, 2^{J-2}-1 \\
V_0 & c_{0,0} & d_{0,0} & W_0 &
\end{tikzcd}$$

$$\boxed{f_j(x) = \underbrace{\sum_{k=0}^{2^j-1} c_{j,k}\phi_{j,k}(x)}_{V_j} = \underbrace{\sum_{k'=0}^{2^{j-1}-1} c_{j-1,k'}\phi_{j-1,k'}(x)}_{V_{j-1}} + \underbrace{\sum_{k'=0}^{2^{j-1}-1} d_{j-1,k'}\psi_{j-1,k'}(x)}_{W_{j-1}}}$$

1D Multi Resolution Analysis

$$
\begin{tikzcd}
\mathrm{V_J} \arrow[d] \arrow[drr] & & \\
\mathrm{V_{J-1}} \arrow[d] \arrow[drr] & \oplus & \mathrm{W_{J-1}} \\
\mathrm{V_{J-2}} \arrow[d, dotted] \arrow[drr, dotted] & \oplus & \mathrm{W_{J-2}} \\
\mathrm{V_0} & \oplus & \mathrm{W_0}
\end{tikzcd}
$$

$$
\begin{aligned}
\psi_{j,k}(x) &= 2^{j/2}\,\psi(2^j x - k) \\
\phi_{j,k}(x) &= 2^{j/2}\,\phi(2^j x - k)
\end{aligned}
$$

with

$$k \in \mathbf{Z} \quad , \; j \in \mathbf{Z} \geq 0$$

$$
\begin{aligned}
c_{j,k} &= \langle f | \phi_{j,k} \rangle \\
&= \int f(x)\,\bar{\phi}_{j,k}(x)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
d_{j,k} &= \langle f | \psi_{j,k} \rangle \\
&= \int f(x)\,\bar{\psi}_{j,k}(x)\,\mathrm{d}x
\end{aligned}
$$

$$\boxed{f_J(x) = c_{0,0}\,\phi_{0,0}(x) + \sum_{j=0}^{J-1} \sum_{k\in\mathbf{Z}} d_{j,k}\,\psi_{j,k}(x)}$$

Complexity: $O(N)$ $\quad N = 2^J$

Fast Periodic Wavelet Transform

1. Interpolation

$$(f_n)_{n=0,\dots,2^p-1}$$

$$\Big\downarrow \ \mathrm{FFT}\,(2^p)$$

$$\hat{f}(k)_{k=0,\dots,2^p-1}$$

$$\Big\downarrow$$

$$\hat{c}_p(k) = 2^p\, \hat{f}(k)\, \hat{L}_p(k) \qquad \text{for } k = 0,\dots,2^p-1$$

2. Projection onto V_{j-1} and W_{j-1}

For $j = p, \dots, 1$

$$\hat{c}_j(k)$$

$$\hat{\alpha}_j(k) = \hat{c}_j(k)\, \overline{\hat{H}_j(k)}\, 2^j \qquad \text{for } k = 0,\dots,2^j-1$$

$$\hat{\beta}_j(k) = \hat{c}_j(k)\, \overline{\hat{G}_j(k)}\, 2^j \qquad \text{for } k = 0,\dots,2^j-1$$

$$\hat{c}_{j-1}(k) = \hat{\alpha}_j(k) + \hat{\alpha}_j(k+2^{j-1}) \qquad \text{for } k = 0,\dots,2^{j-1}-1$$

$$\hat{d}_{j-1}(k) = \hat{\beta}_j(k) + \hat{\beta}_j(k+2^{j-1}) \qquad \text{for } k = 0,\dots,2^{j-1}-1$$

$$\Big\downarrow \ j=1$$

$$c_{0,0} = \hat{c}_0(0)$$

$$\Big\downarrow \ \mathrm{FFT}^{-1}\,(2^{j-1})$$

$$d_{j-1,n}$$