Linear Approximation

\[ \langle f, \psi_{j,k} \rangle = 0 \text{ for } j > J_{MAX} \]
Nonlinear Approximation

\[ \langle f, \psi_{j,k} \rangle = 0 \text{ for } |\langle f, \psi_{j,k} \rangle| \leq \epsilon \]
2D Multi Resolution Analysis

\[ V_J \]
\[ \downarrow \]
\[ V_{J-1} \oplus W_{J-1}^h \oplus W_{J-1}^v \oplus W_{J-1}^d \]
\[ \downarrow \]
\[ W_{J-1} \]
\[ \vdots \]
\[ V_{J-2} \oplus W_{J-2}^h \oplus W_{J-2}^v \oplus W_{J-2}^d \]
\[ \downarrow \]
\[ W_{J-2} \]
\[ \vdots \]
\[ V_0 \oplus W_0^h \oplus W_0^v \oplus W_0^d \]
\[ \downarrow \]
\[ W_0 \]
$L^2(\mathbb{R}^2)$ \quad f(x,y) \quad 2D Multi Resolution Analysis (Fast WLT)

\begin{align*}
V_j & \quad f_j \left( \frac{k_x}{2^j}, \frac{k_y}{2^j} \right) \quad k_x, k_y = 0, ..., 2^j - 1 \\
L_j & \\
V_J & \quad c_{J,k_x,k_y} \quad k_x, k_y = 0, ..., 2^j - 1 \\
H_J & \quad H_JG_J \quad G_JH_J \quad G_JG_J \\
V_{J-1} & \quad c_{J-1,k_x,k_y} \quad d_{J-1,k_x,k_y} \quad d_{J-1,k_x,k_y} \quad d_{J-1,k_x,k_y} \quad W_{J-1} \quad k_x, k_y = 0, ..., 2^{j-1} - 1 \\
H_{J-1} & \quad H_{J-1}G_{J-1} \quad G_{J-1}H_{J-1} \quad G_{J-1}G_{J-1} \\
V_{J-2} & \quad c_{J-2,k_x,k_y} \quad d_{J-2,k_x,k_y} \quad d_{J-2,k_x,k_y} \quad d_{J-2,k_x,k_y} \quad W_{J-2} \quad k_x, k_y = 0, ..., 2^{j-2} - 1 \\
& \quad \vdots \\
V_0 & \quad c_{0,0,0} \quad d_{0,0,0} \quad d_{0,0,0} \quad d_{0,0,0} \quad W_0
\end{align*}
Scheme of the wavelet coefficients for a two-dimensional MRA.
Schematic localization in Fourier space of orth. Wavelets, $\psi^d$, $\psi^v$, $\psi^j$,
Conclusions DWT

- Fast Wavelet Transform $O(N)$ algorithm
- Compression property of signals/images/vorticity fields
- Transforms available for:
  - the real line
  - periodic signals
  - intervals (with boundary conditions)
- Orthogonal, biorthogonal constructions
- Higher dimensions $\rightarrow$ tensor product construction
- Divergence free wavelets ($\nabla \cdot \psi = 0$)
  not yet used in CFD (Lemarié, Federbush, Urban)
- No constructions for complex geometries (domain dec.)
- No translation invariance
b) two dim. case

Tensor product of two one-dimensional MRA

\[ V_j = V_j \otimes V_j = \text{span}\{ \phi_{j,k_1,k_2}(x,y) \} = \phi_{j,k}(x) \phi_{j,k}(y) \]

Orthogonal complement space

\[ W_j \text{ in } V_{j+1} \text{ of } V_j \]

\[ V_{j+1} = V_{j+1} \otimes V_{j+1} = (V_j \otimes W_j) \otimes (V_j \otimes W_j) \]

\[ = \left( V_j \otimes V_j \right) \oplus \left[ (W_j \otimes V_j) \oplus (V_j \otimes W_j) \right. \]

\[ \left. \oplus (W_j \otimes W_j) \right] \]

\[ = V_j \oplus W_j \]

It follows that \( W_j \) consists of three pieces

\[ \psi^H(x,y) = \varphi(x) \varphi(y) \]

\[ \psi^V(x,y) = \psi(x) \varphi(y) \]

\[ \psi^D(x,y) = \varphi(x) \psi(y) \]

with horizontal, vertical and diagonal wavelets.

\[ \varrho(x,v) = \sum C_{0,k} \varphi_0(x,y) + \sum \sum \sum d_{ik} \psi_{ik}(x,y) \]
Compression of functions

\[ f(x) = \sum_{j,k} \langle f, \psi_{j,k} \rangle \psi_{j,k}(x) \]

with

\[ \langle f, \psi_{j,k} \rangle = \int f(x) \psi_{j,k}(x) \, dx \]