Local Spectral Analysis of Turbulent Flows

- 2d case, we consider the vorticity field $\omega = \nabla \times \vec{u}$

(Farge, Meneveau, Perrier, Philipowich, Basdevant 1990-3)

1. Recall Fourier spectra

$$\hat{\omega}(\mathbb{k}) = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \omega(\mathbb{x}) e^{-i \mathbb{k} \cdot \mathbb{x}} \, d\mathbb{x} \qquad \mathbb{k} \in \mathbb{Z}^2$$

$$Z(k) = \frac{1}{2} \int_{|\mathbb{k}| = k} |\hat{\omega}(\mathbb{k})|^2 \, |\mathbb{k}| \, d\theta$$

$$Z = \int_0^{+\infty} Z(k) \, dk$$

• all spatial information is lost!

3. Local spectra using the DWT

with

$$\omega(\mathbf{x}) = \underset{\substack{\uparrow \\ \text{mean}}}{c_{000}} + \sum_{j\geq 0} \sum_{k_x=0}^{2^j-1} \sum_{k_y=0}^{2^j-1} \sum_{\lambda=h,v,d} d^{\lambda}_{j,k_x,k_y} \psi^{\lambda}_{j,k_x,k_y}(\mathbf{x})$$

we have

$$Z = \frac{1}{2} |c_{000}|^2 + \frac{1}{2} \sum_{j\geq 0} \sum_{k_x=0}^{2^j-1} \sum_{k_y=0}^{2^j-1} \sum_{\lambda=h,v,d} | d^{\lambda}_{j,k_x,k_y} |^2$$

To be consistent with the CWT (we used a isotropic WL) we define a mean coefficient at scale 2^{-j} and at position $2^{-j}(k_x + \frac{1}{2}, k_y + \frac{1}{2})$

$$D(2^{-j}, 2^{-j}(k_x,k_y)) = \left(\frac{(d^h_{j,k_x,k_y})^2 + (d^h_{j,k_x+1,k_y})^2}{2} + \frac{(d^v_{j,k_x,k_y})^2 + (d^v_{j,k_x,k_y+1})^2}{2} + (d^d_{j,k_x,k_y})^2 \right)$$

$D(\cdot,\cdot)$: enstrophy of ω, integrated in physical space on a square of side 2^{-j}, integrated in Fourier space on a wavenumber intervall of length $\sim 2^j$

With $D(\cdot,\cdot)$ the enstrophy conservation becomes

$$Z = \frac{1}{2} |c_{000}|^2 + \frac{1}{2} \sum_{j\geq 0} \sum_{k_x,k_y=0}^{2^j-1} D(2^{-j}, 2^{-j}(k_x,k_y))$$

Remark:

The global wavelet spectra are smoothed/averaged Fourier spectra

$$\tilde{E}_{WL}(k) = \frac{1}{C_\psi k_0} \int_0^\infty \tilde{E}_F(\xi) \left| \hat{\psi}\left(\frac{k_0 \xi}{k}\right)\right|^2 d\xi$$

- the larger k, the wider the averaging intervall
- used in statistics, consistent estimator for Periodogram
- to detect power law behaviour $k^{-\alpha}$ at least $m > (\alpha - 1)/2$ vanishing moments of ψ necessary.

3. Perrier - Wavelet

$$\hat{\psi}_n(k) = \alpha_n \exp\left(-\frac{1}{2}\left(k^2 + \frac{1}{k^{2n}}\right)\right), \quad n \geq 1$$

- α_n for normalization
- infinite number of vanishing moments
  ⇝ could detect any any power law behaviour $k^{-\alpha}$.

4. Discrete Wavelets

- quintic splines
- 5 vanishing moments
- estimates power-law exponents up to $\alpha < 11$

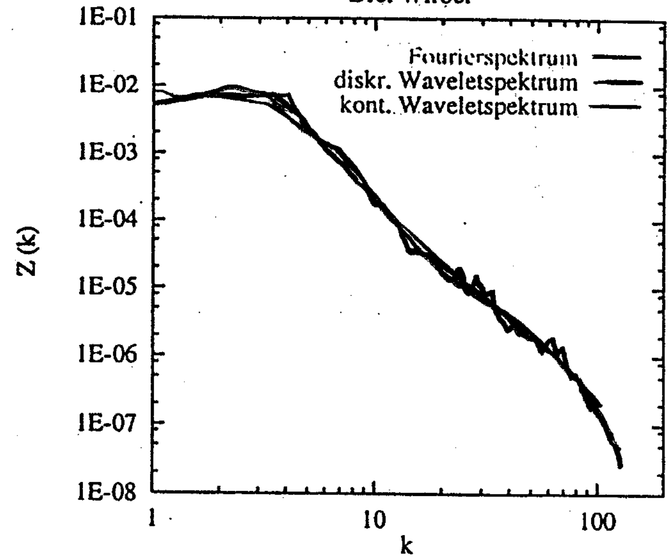
vorticity at t=13

$$\omega = \nabla \times \vec{u}$$

Global spectra

J. Ziuber, 1997

3 Vortices



Fully developed turbulence

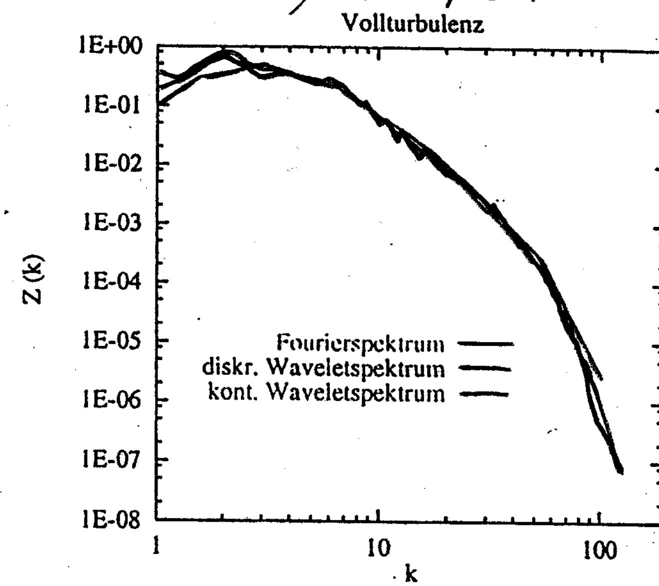


Abbildung 7.1: Globale Enstrophiespektren der Wirbelstärkenfelder der Fälle *I* und *II*. Für die Berechnung des kontinuierlichen Waveletspektrums wurde in der Transformation das Perrier-Wavelet verwendet (s. Anhang A).

— Fourier Spectrum
— discrete Wavelet Spectrum
— continuous Wavelet Spectrum

Intermittency in Wavelet Space

$$\omega(x,y) = \sum_{\lambda \in \Lambda} d_\lambda \psi_\lambda(x,y)$$

Scalewise energy/enstrophy distribution

$$Z_j = \sum_{i_x=0}^{2^j-1} \sum_{i_y=0}^{2^j-1} \sum_{\varepsilon=1}^{3} (d^{\varepsilon}_{j,i_x,i_y})^2$$

Scale dependent flatness factor (Meneveau 91)

$$F_j = M_{4,j}/(M_{2,j})^2 \quad // \quad Q_{p,j} = \frac{M_{p,j}}{(M_{2,j})^{p/2}}$$

Moments of the wavelet coefficients at scale j

$$M_{n,j} = \frac{1}{3 \cdot 2^{2j}} \sum_{i_x=0}^{2^j-1} \sum_{i_y=0}^{2^j-1} \sum_{\varepsilon=1}^{3} |d^{\varepsilon}_{j,i_x,i_y}|^n$$

Wavelets and structure functions

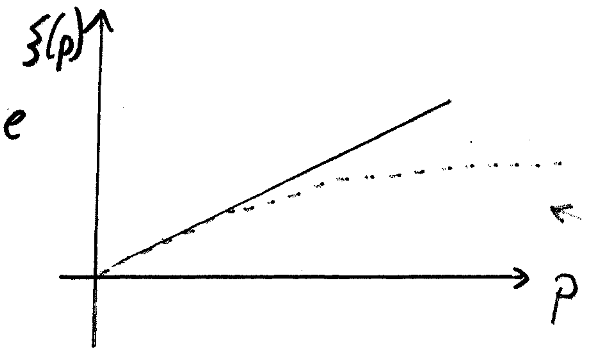
- classical structure functions

$$S_p(\ell) = \int |u(x+\ell) - u(x)|^p \, dx$$

- in turbulence

$$S_p(\ell) \propto \ell^{\xi(p)} \quad \text{with} \quad \xi(p) = p/3$$

but in practice



intermittency

Case $p=2$:

- autocorrelation function

$$R(\ell) = \int u(x+\ell)\, u(x)\, dx$$

$$S_2(\ell) = 2R(0) - 2R(\ell)$$

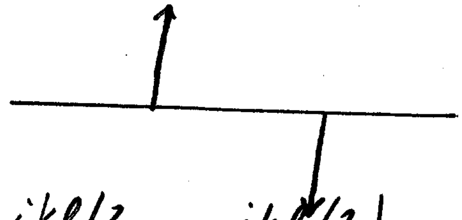
$$= 2\int (1 - \cos k\ell)\, E(k)\, dk$$

with

$$E(k) = \int R(\ell)\, e^{-ik\ell}\, d\ell$$

Interpretation with wavelets

$$\psi(x) = \delta(x+\ell) - \delta(x)$$



$$\hat{\psi}(k) = e^{ik\ell} - 1 = e^{ik\ell/2}\left(e^{ik\ell/2} - e^{-ik\ell/2}\right)$$

Wavelet coefficients of u:

$$\tilde{u}_{\ell,x} = \langle u, \psi_{\ell,x} \rangle = \int u(y) \frac{1}{\ell}\left[\delta\left(\frac{y-x}{\ell}+1\right) - \delta\left(\frac{y-x}{\ell}\right)\right] dy$$

$$= u(x-\ell) - u(x)$$

$$S_p(\ell) = \int |\tilde{u}_{\ell,x}|^p \, dx$$

$=$ p-th order moments of the wavelet coeff. on each scale ℓ

For $\psi = \delta(x+1) - \delta(x)$ [illegible] that if $S_p(\ell) \propto \ell^{\zeta(p)}$

$$\zeta(p) \leq 1$$

$\Rightarrow$ use wavelets with more vanishing moments

Besov norms, structure functions and wavelets

$$B^s_{p,q} = \left\{ f \in L^p(\mathbb{R});\ \ell^{-s}\left(\int |f(x+\ell)-f(x)|^p dx\right)^{1/p} \in L^q\left(\mathbb{R}^*_+, \frac{d\ell}{\ell}\right) \right\}$$

with $0 < s < 1$, $p,q \geq 1$, $q < \infty$

This means that $f \in B^s_{p,q}$ iff $f \in L^p$ and

$$\left(\int_0^\infty \ell^{-sq} \underbrace{\left(\int |f(x+\ell)-f(x)|^p dx \right)}_{S_p(\ell)}{}^{q/p} \frac{d\ell}{\ell} \right)^{\frac{1}{q}} < \infty$$

$$\Leftrightarrow \left(\int_0^\infty \ell^{-sq}\, S_p(\ell)^{q/p} \frac{d\ell}{\ell} \right)^{1/q} < \infty$$

$$\simeq \left(\int_0^\infty \left[a^{-s} \|\tilde{f}(a,\cdot)\|_{L^p} \right]^q \frac{da}{a} \right)^{1/q}$$

Self-similar behaviour:

$$S_p(\ell) \propto \ell^{\zeta(p)}$$

$$\Leftrightarrow f \in B^{\zeta(p)/p}_{p,\infty}$$

with $\zeta(p) < p$ as $s < 1$.

$\Rightarrow$ generalization for $s > 1$

- use wavelets with at least $[s]+1$ vanishing moments
- use higher order stencils, e.g. $f(x+\ell) - 2f(x) - f(x-\ell)$.

Scale dependent moments of the wavelet coefficients and structure functions

$$M_{p,j} = \frac{1}{2^j} \sum_{i=0}^{2^j-1} |\tilde{f}_{j,i}|^p$$

Using the L^1 normalization of the wavelets we have:

$$S_p^{wL}(2^{-j}) = 2^{jp/2} M_{p,j}$$

$\Rightarrow$ generalised extended self similarity in wavelet space

$$Q_{p,q,j} = \frac{M_{p,j}}{(M_{q,j})^{p/q}}$$

- scale dependent flatness

$$F_j = Q_{4,2,j}$$

.... scale dependent skewness

Case $p=2$:

$$\widetilde{E}_{wl}(k) = \frac{1}{C_\psi k} \int_0^\infty \underline{E(k')} \left|\hat{\psi}\left(\frac{k_0 k'}{k}\right)\right|^2 dk'$$

wavelet spectrum

Fourier spectrum

For $\psi(x) = \delta(x+1) - \delta(x)$ we get

$$\widetilde{E}_{wl}(k) = \frac{1}{C_\psi k} \int_0^\infty E(k') \left[2 - 2\cos\left(\frac{k_0 k'}{k}\right)\right] dk' \qquad \ell = \frac{k_0}{k}$$

$$= \frac{1}{C_\psi k} \underbrace{\int_0^\infty E(k') \left[2 - 2\cos(\ell k')\right] dk'}_{S_2(\ell)}$$

$$\boxed{\widetilde{E}_{wl}(k) = \frac{1}{C_\psi k} \, S_2(\ell)}$$

If $E(k) \propto k^{-\alpha}$ (for $k \to \infty$)

we have $\widetilde{E}_{wl}(k) \propto k^{-\alpha}$

if $\alpha < 2M+1$ (here $M=1$)

$\Rightarrow S_2(\ell) \propto \ell$

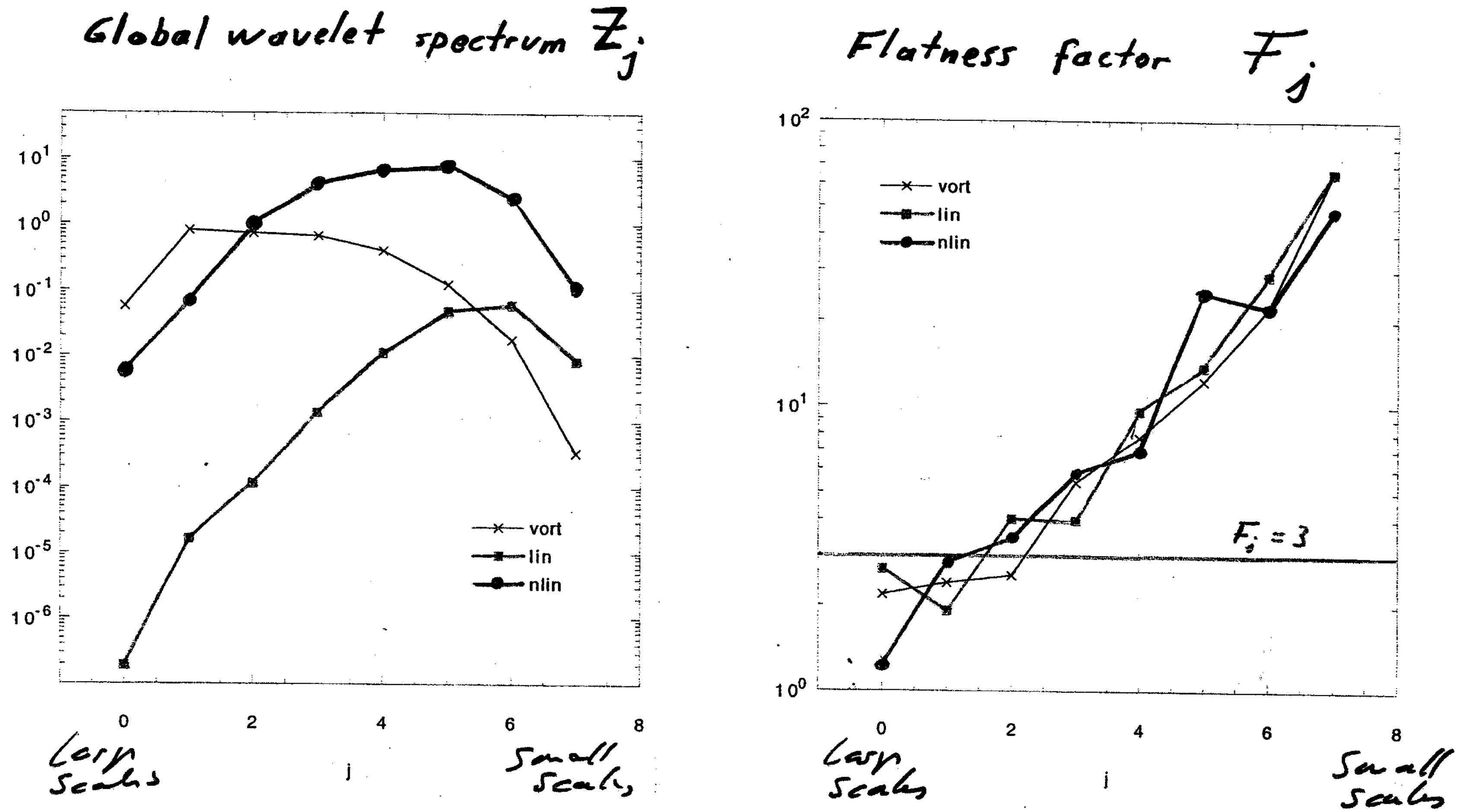


FIG. 13. Properties of the wavelet coefficients of vorticity ($\times$) as well as linear (■) and non–linear term (●) on different scales. LEFT: Scale–wise contributions, i.e. global discrete wavelet spectrum, RIGHT: Flatness of the wavelet coefficients on each scale.

Decaying turbulence 256, t=3

J. Zieber, 1997
Karlsruhe

Local Wavelet spectra

Z(k,x)

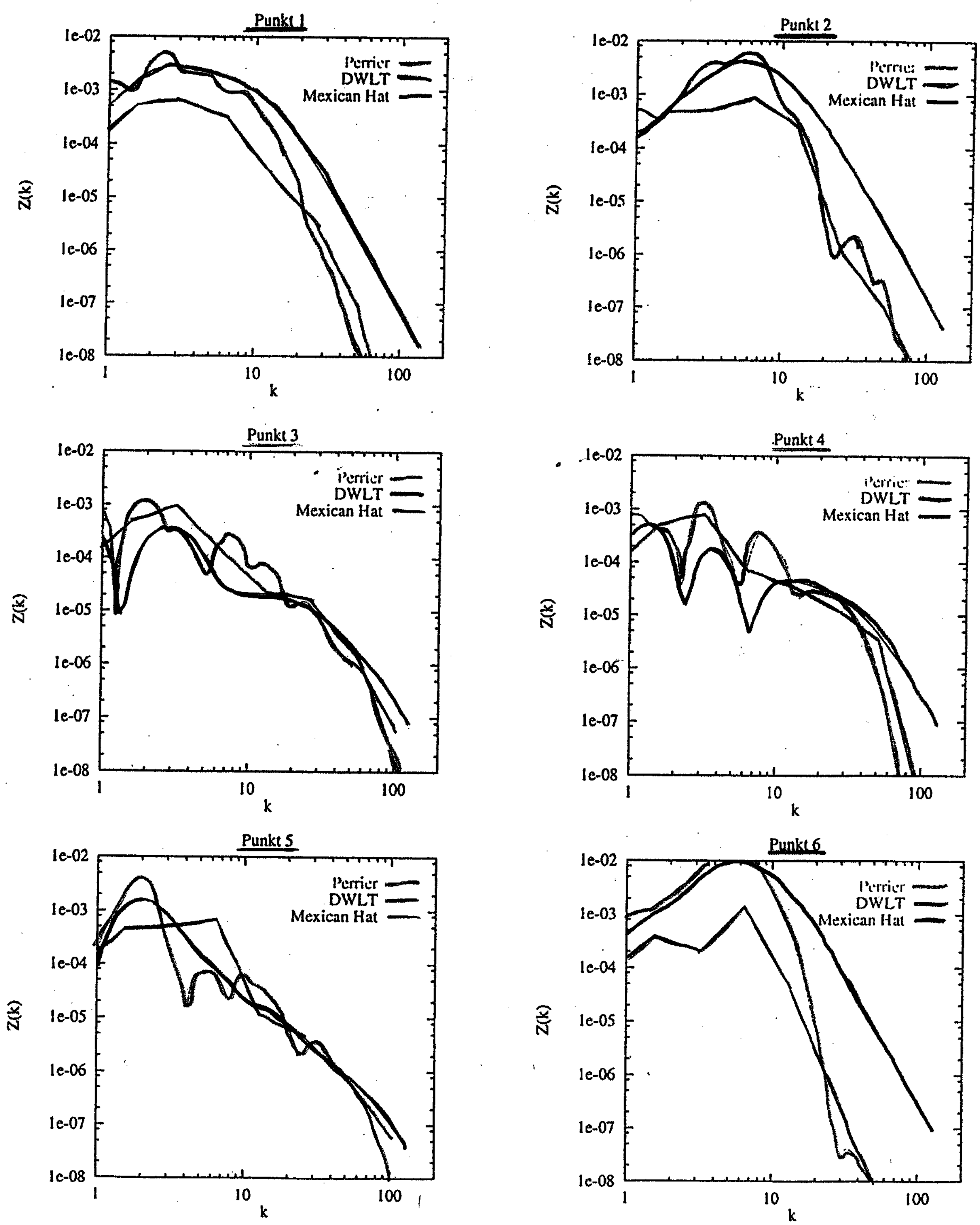
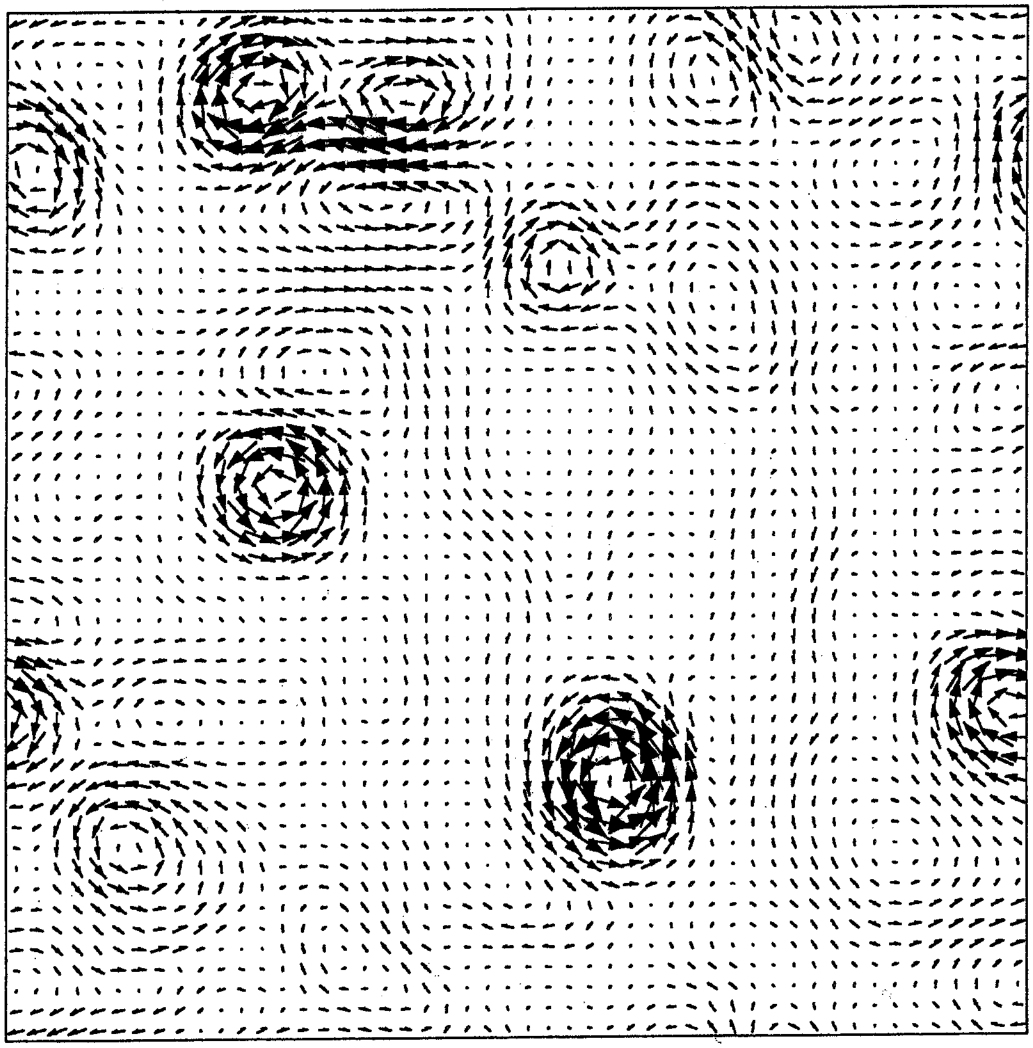


Abbildung 7.15: Lokale Enstrophiespektren des Wirbelstärkenfeldes in Abb. 7.14

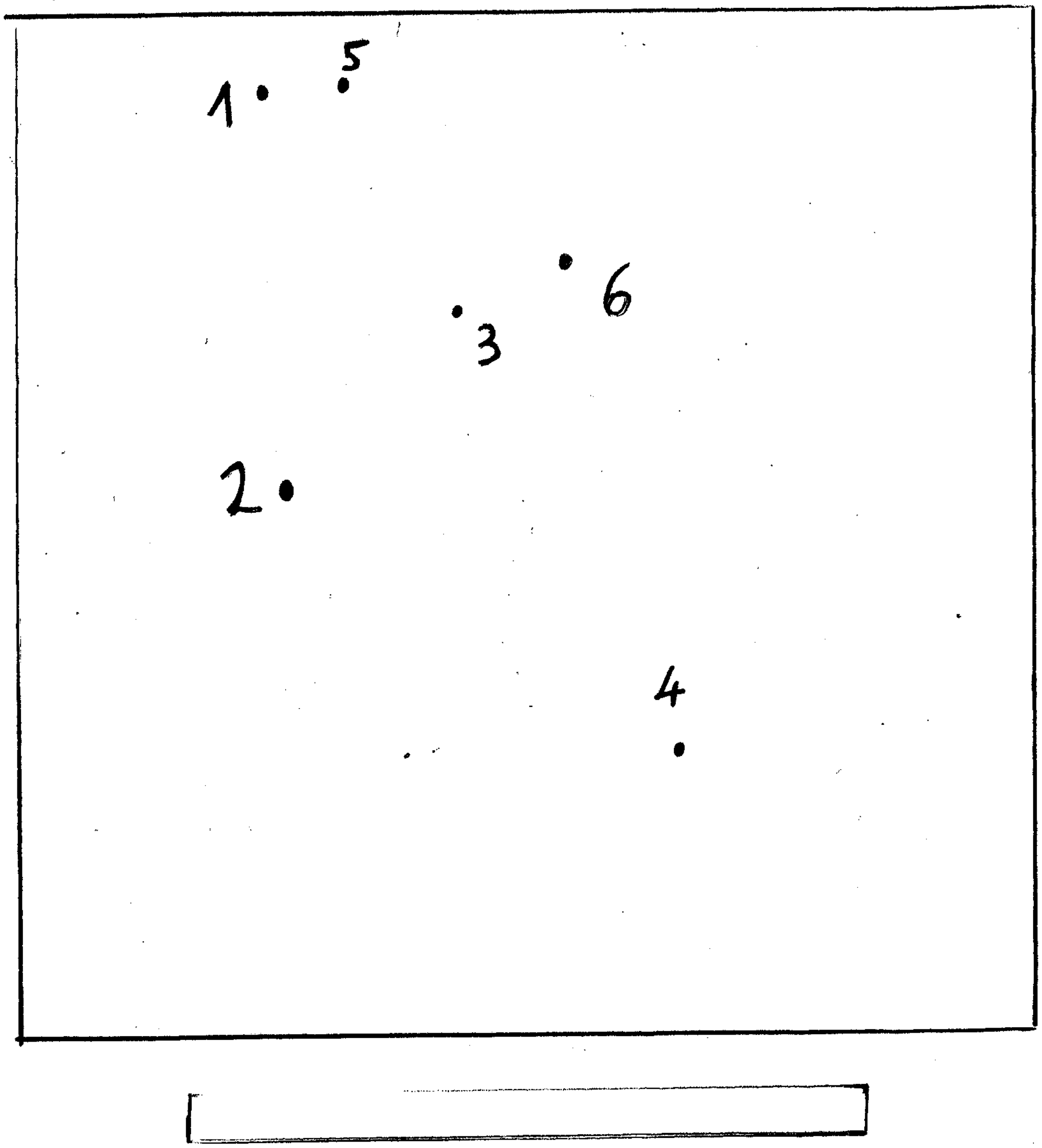
— Mexican Hat

— Perrier Wavelet

$\vec{u}$



Local Wavelet spectra : Points



1, 2, 6 : center of vortices

3, 4, 5 : shear region

Examples for wavelets:

1. Mexican hat

$$\psi(x) = -\frac{d^2}{dx^2} \exp(-x^2)$$

$$\hat{\psi}(k) = k^2 \exp(-k^2/2)$$

- two vanishing moments
- scaling exponents up to $\alpha < 5$ (k^α)

2. Morlet wavelet

- complex valued

$$\psi(x) = e^{-(|x|^2/2)} \; e^{i k_\psi \cdot x}$$

mostly $|k_\psi| = 6$

$$\hat{\psi}(k) = \begin{cases} \frac{1}{2\pi} \exp(-(k-k_\psi)^2/2) & \text{for } k > 0 \\ 0 & \text{for } k \le 0 \end{cases}$$

- only zeroth order moment is zero
but higher n^{th} order moments are very small
$(\propto k_\psi^n \exp(-k_\psi^2/2)$ provided that k_ψ is sufficiently large

- estimates power-low exponents up to

$\alpha < 7$ (if $k_\psi = 6$)

The local enstrophy spectrum is then defined as

$$\tilde{Z}_{DWT}(k_j, x_{k_x,k_y}) = \frac{1}{2} |D(2^{-j}, 2^{-j}(k_x, k_y)|^2 \frac{2^{2j}}{\Delta k_j}$$

with

$$\Delta k_j = \sqrt{k_j k_{j+1}} - \sqrt{k_j k_{j-1}},$$

$$x_{k_x,k_y} = 2^{-j}(k_x + \tfrac{1}{2}, k_y + \tfrac{1}{2})$$

and $k_j = \frac{k_0}{a_j}$, $a_j = 2^{-j}$, k_0 mean wavenumber of ψ.

The global discrete wavelet spectrum can be reconstructed by:

$$\tilde{Z}_{DWT}(k_j) = \sum_{k_x=0}^{2^j-1} \sum_{k_y=0}^{2^j-1} \tilde{Z}_{DWT}(k_j, x_{k_x,k_y}) 2^{-2j}$$

and for the total enstrophy it follows

$$Z = \sum_{j \geq 0} \tilde{Z}_{DWT}(k_j) \Delta k_j$$

2. Local spectra using the CWT

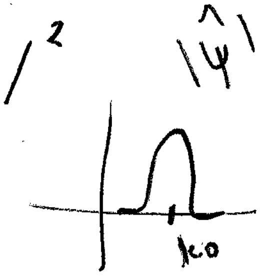
$$W(a, \mathbb{b}) = \iint_{\mathbb{R}^2} f(\mathbb{x}) \frac{1}{a} \psi\left(\frac{\mathbb{x} - \mathbb{b}}{a}\right) d\mathbb{x}$$

$$\mathbb{b} \in \mathbb{R}^2, \quad a > 0$$

isotropic ψ

We have

$$Z = \frac{1}{2 C_\psi} \int_0^\infty \frac{da}{a^3} \iint_{\mathbb{R}^2} d\mathbb{b} \, |W(a, \mathbb{b})|^2$$



With the mean wavenumber

$$k_0 = \frac{\int_0^\infty k |\hat{\psi}(k)| dk}{\int_0^\infty |\hat{\psi}(k)| dk} \qquad \text{(center of gravity)}$$

and the scale-wavenumber relation

$$a = \frac{k_0}{k}$$

we can define a <u>local enstrophy spectrum</u>

$$\tilde{Z}_{CWT}(k, \mathbb{b}) = \frac{1}{2 C_\psi} \frac{k}{k_0^2} \left| W\left(\frac{k_0}{k}, \mathbb{b}\right)\right|^2$$

at $\mathbb{b} \in \mathbb{R}^2$

The <u>global wavelet spectrum</u> can be calculated by integration over all positions

$$\tilde{Z}_{CWT}(k) = \iint_{\mathbb{R}^2} \tilde{Z}_{CWT}(k, \mathbb{x}) \, d\mathbb{x}$$

and for the total enstrophy follows

$$Z = \int_0^\infty \tilde{Z}_{CWT}(k) \, dk$$