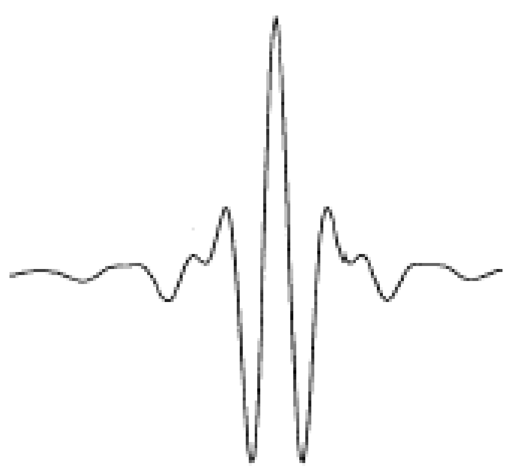
ME 252 B

Computational Fluid Dynamics: Wavelet transforms and their applications to turbulence

Marie Farge[1] & Kai Schneider[2]



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University of California, Santa Barbara

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[1] LMD-CNRS, Ecole Normale Supérieure
24 rue Lhomond
75231 Paris Cedex 05, France
Email : farge@lmd.ens.fr

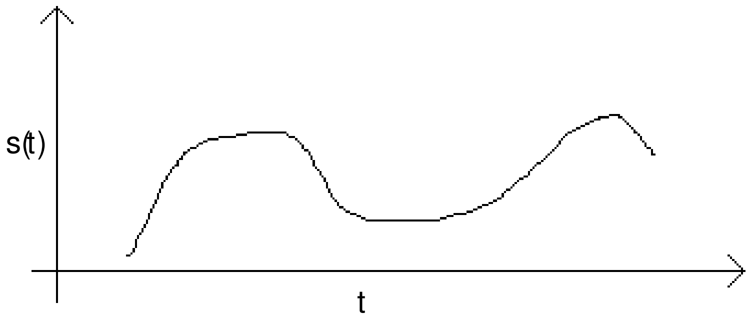
[2] CMI, Université de Provence
39 rue Joliot-Curie
13453 Marseille Cedex 13, France
Email : kschneid@cmi.univ-mrs.fr

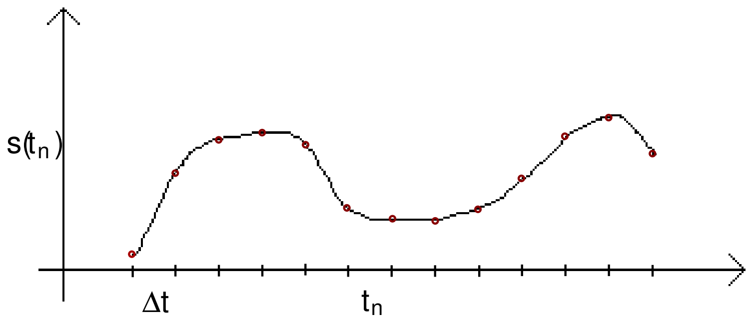
Classification of signals (1d)

$$s : t \rightarrow s(t) \qquad \text{with} \quad t \in \mathbb{R},\ s(t) \in \mathbb{R} \quad \text{or} \quad \mathbb{C}$$

1) **Continuous** $\longleftrightarrow$ **discrete**

$$s(t), t \in \mathbb{R} \qquad\qquad s(t_n), n \in \mathbb{Z}$$





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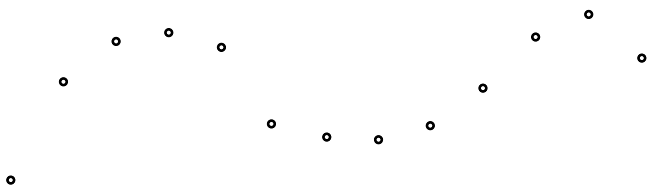
2) Nonperiodic $\longleftrightarrow$ periodic

- continuous

Period T, $s(t) = s(t + nT), n \in \mathbb{Z}$

- discrete



Period T, $s(t_n) = s(t_n + nT)$,

$n \in \mathbb{Z}$ with $T = N \Delta T$

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3) Compact support

$$s(t) \neq 0 \quad \text{for} \quad t \in [A, B] \quad \text{and} \quad s(t) = 0 \quad \text{else}$$

4) Signals with finite energy

- continuous, nonperiodic

$$E = \int_{-\infty}^{\infty} |s(t)|^2 dt < \infty$$

- continuous, periodic

$$E = \int_{0}^{T} |s(t)|^2 dt < \infty$$

- discrete, nonperiodic

$$E = \sum_{n=-\infty}^{\infty} |s(t_n)|^2 < \infty$$

- discrete, periodic

$$E = \sum_{n=0}^{N-1} |s(t_n)|^2 < \infty$$

For mathematicians: spaces of square-integrable functions (norm + scalar product)

$$s(t) \in L^2(\mathbb{R}), L^2(\mathbb{T}), l^2(\mathbb{R}), l^2(\mathbb{T}) \quad \text{where} \quad \mathbb{T} = \mathbb{R}/\mathbb{Z}$$

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5) Absolutely integrable signals

$$S = \int_{-\infty}^{\infty} |s(t)| dt < \infty \quad s(t) \in L^1(I\!R)$$

Classification of signals (higher dimensions)

2d $\longrightarrow$ images $s(\vec{x}) = s(x,y), x, y \in I\!R$ or $s(m,n), n, m \in Z\!\!Z$

3d, n-d

scalar-valued $\longleftrightarrow$ vector-valued signals

- temperature
- pressure

- velocity
- RGB signal

$\Longrightarrow$ similar classification possible.

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The Fourier transform

Motivation

- representation of signals with sine and cosine functions
- transformation of signals into the frequency plane
- fast algorithms (FFT), $N \log_2 N$ complexity
- correlation and convolution can be efficiently computed in the frequency domain
- system theory:
  sine and cosine are eigenfunctions of linear time-shift invariant systems

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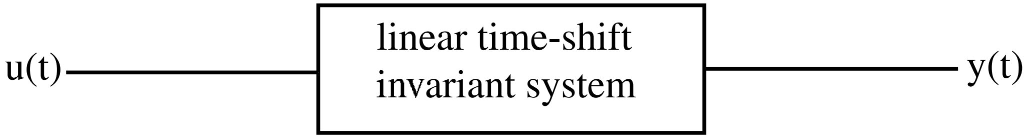

$$u(t) = \sin 2\pi f t \qquad\qquad y(t) = a \sin(2\pi f t + \phi)$$

$$\cos 2\pi f t \qquad\qquad a \cos(2\pi f t + \psi)$$

For simplification one uses complex exponentials:

$$e^{it} = \cos t + i \sin t$$

Recall complex numbers: $z \in \mathbb{C}, z = x + iy = re^{i\theta}$

$$x = \Re z, y = \Im z$$

$$r^2 = x^2 + y^2, \theta = \arctan y/x$$

Recall trigonometric polynomials:

$$s(t) = \sum_{k \geq 0} a_k \cos 2\pi k t + b_k \sin 2\pi k t$$

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Fourier transforms

1) Continuous signals

We consider an absolutely integrable signal $s(t) \in L^1(\mathbb{R})(\cap L^2(\mathbb{R}))$, $t, s \in \mathbb{R}$

The Fourier transform is defined as:

$$\widehat{S}(f) = \int_{-\infty}^{\infty} s(t) e^{-i2\pi f t} dt$$

$$= \int_{-\infty}^{\infty} s(t) \cos 2\pi f t dt + i \int_{-\infty}^{\infty} s(t) \sin 2\pi f t dt$$

Note that in general $\widehat{S}(f) \in \mathbb{C}$.

Define modulus $|\widehat{S}(f)|$ and phase $\phi = \arctan \Im \widehat{S}(f) / \Re \widehat{S}(f)$

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The inverse Fourier transform is defined as:

$$s(t) = \int_{-\infty}^{\infty} \widehat{S}(f) e^{i2\pi ft} df$$

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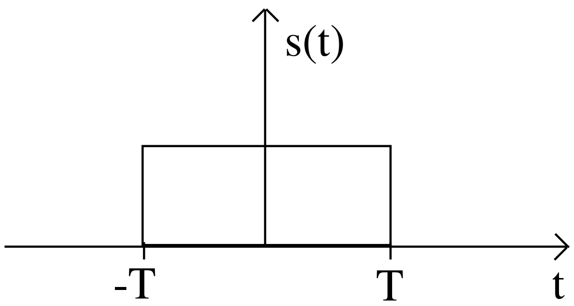
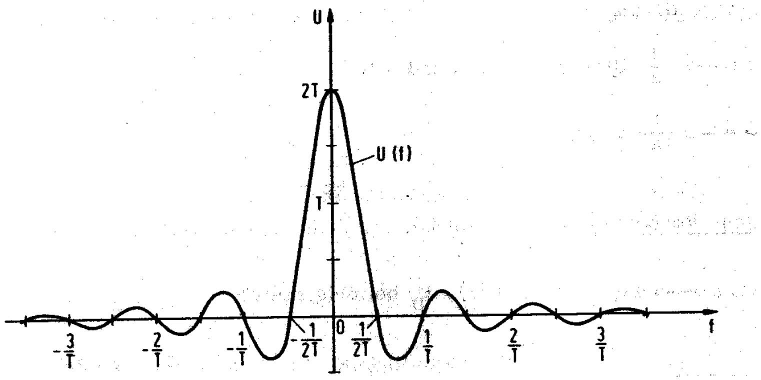
Example:

$$s(t) = \begin{cases} 1 & \text{for} \quad -T \leq t \leq T, \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

$$\widehat{S}(f) = \frac{\sin 2\pi f T}{\pi f}$$





2) Properties

a) scaling

$$s(at) \quad \Longleftrightarrow \quad \frac{1}{|a|}\widehat{S}(\frac{f}{a}) \quad a \in I\!R, a \neq 0$$

$$\widehat{S}(af) \quad \Longleftrightarrow \quad \frac{1}{|a|}s(\frac{t}{a})$$

b) time-shift

$$s(t-t_0) \quad \Longleftrightarrow \quad \exp(-i2\pi f t_0)\widehat{S}(f) \quad t_0 \in I\!R$$

c) frequency-shift

$$\widehat{S}(f-f_0) \quad \Longleftrightarrow \quad \exp(i2\pi f_0 t)s(t) \quad f_0 \in I\!R$$

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d) differentiation (with respect to time)
If $s(t)$ is n-times continuously differentiable and $s^{(n)}(t) \in L^1(\mathbb{R})$, then

$$s^{(n)}(t) \quad \Longleftrightarrow \quad (i2\pi f)^n \widehat{S}(f)$$

e) differentiation (with respect to frequency)
If $t^m s(t) \in L^1(\mathbb{R})$ for $m = 0, 1, ..., M$, then $\widehat{S}^{(m)}(f)$ exists and

$$(-i2\pi t)^m s(t) \quad \Longleftrightarrow \quad \widehat{S}^{(m)}(f)$$

f) multiple application of the Fourier transform

$$\mathcal{F}\{s(t)\}(f) = \int_{-\infty}^{\infty} s(t) e^{-i2\pi f t} dt = \widehat{S}(f)$$

$$\mathcal{F}^2\{\{s(t)\}(f)\}(t) = \mathcal{F}\{\widehat{S}(f)\}(t) = \int_{-\infty}^{\infty} \widehat{S}(f) e^{-i2\pi f t} df = s(-t)$$

$\longrightarrow \mathcal{F}^2$ corresponds to time inversion
and hence $\mathcal{F}^4 =$ Identity

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$\mathcal{F}^3 = \mathcal{F}^{-1} = \mathcal{F}^\star$ (inverse Fourier transform)
Remark: The Fourier transform is a cyclic operator of 4th degree.

g) convolution
given $s_1(t)$ and $s_2(t)$ with $s_1(t) \in L^2(\mathbb{R})$ and $s_2(t) \in L^\infty(\mathbb{R})$.

$$s_1(t) \star s_2(t) = \int_{-\infty}^{\infty} s_1(\tau) s_2(t-\tau) d\tau$$

$\star$ commutes, i.e. $s_1 \star s_2 = s_2 \star s_1$
$\star$ is associative, i.e. $s_1 \star s_2 \star s_3 = s_1 \star (s_2 \star s_3) = (s_1 \star s_2) \star s_3$

$$s_1(t) \quad \Longleftrightarrow \quad \widehat{S}_1(f) \quad \text{and} \quad s_2(t) \quad \Longleftrightarrow \quad \widehat{S}_2(f)$$

$$s_1(t) \star s_2(t) \quad \Longleftrightarrow \quad \widehat{S}_1(f)\widehat{S}_2(f)$$

h) correlation
- cross-correlation: $s_1(t), s_2(t) \in L^2(\mathbb{R})$

$$\phi_{12}(t) = \int_{-\infty}^{\infty} s_1(\tau) s_2(t+\tau) d\tau = s_1(t) \star s_2(-t)$$

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$$\phi_{21}(t) = \int_{-\infty}^{\infty} s_1(t+\tau) s_2(\tau) d\tau = s_1(-t) \star s_2(t)$$

If $\widehat{S}_1(f)$ and $\widehat{S}_2(f)$ exist, then

$$\widehat{\Phi}_{12}(f) = \mathcal{F}\{\phi_{12}(t)\}(f) = \mathcal{F}\{s_1(t) \star s_2(-t)\}(f)$$

$$= \mathcal{F}\{s_1(t) \star \mathcal{F}^2\{s_2(t)\}\}(f) = \mathcal{F}\{s_1(t)\}(f)\mathcal{F}^3\{s_2(t)\}\}(f)$$

$$= \widehat{S}_1(f)\widehat{S}_2^\star(f)$$

and analogously

$$\widehat{\Phi}_{21}(f) = \widehat{S}_1^\star(f)\widehat{S}_2(f)$$

i) autocorrelation
$s_1(t) \in L^2(I\!R)$

$$\phi_{11}(t) = \int_{-\infty}^{\infty} s_1(\tau) s_1(t+\tau) d\tau = s_1(t) \star s_1(-t)$$

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and with $s_1(t) \quad \Longleftrightarrow \quad \widehat{S}_1(f)$
we obtain in frequency space

$$\widehat{\Phi}_{11}(f) = \mathcal{F}\{\phi_{11}(t)\}(f) = \widehat{S}_1(f)\widehat{S}_1^\star(f) = |\widehat{S}_1(f)|^2$$

j) multiplication

$$s_1(t)s_2(t) \quad \Longleftrightarrow \quad \widehat{S}_1(f) \star \widehat{S}_2(f) = \int_{-\infty}^{\infty} \widehat{S}_1(\xi)\widehat{S}_2(f-\xi)d\xi$$

k) Parseval's identity

$$\int_{-\infty}^{\infty} s_1(t)s_2(t)dt = \int_{-\infty}^{\infty} \widehat{S}_1(f)\widehat{S}_2(-f)df$$

$$\longrightarrow \int_{-\infty}^{\infty} s_1(t)s_2^\star(t)dt = \int_{-\infty}^{\infty} \widehat{S}_1(f)\widehat{S}_2^\star(f)df$$

and in particular for $s_1 = s_2 = s \quad \Longleftrightarrow \quad \widehat{S}(f)$ we have

$$\int_{-\infty}^{\infty} |s(t)|^2 dt = \int_{-\infty}^{\infty} |\widehat{S}(f)|^2 df$$

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l) energy spectrum
$E(f) = |\widehat{S}(f)|^2$ and $E = \int_0^\infty E(f) df$
$E(f)$ is called spectral energy density, or energy spectrum.

m) symmetries

$$s(t) = s_{even}(t) + s_{odd}(t)$$

with $s_{even}(t) = \frac{1}{2}(s(t) + s(-t))$ and $s_{odd}(t) = \frac{1}{2}(s(t) - s(-t))$
Decomposing the corresponding Fourier transform into real and imaginary part we obtain:

$$\widehat{S}(f) = \widehat{S}_r(f) + i\widehat{S}_i(f)$$

where $\widehat{S}_r(f) = \Re\widehat{S}(f)$ and $\widehat{S}_i(f) = \Im\widehat{S}(f)$

$$s_{even}(t) \quad \Longleftrightarrow \quad \widehat{S}_r(f)$$

$$s_{odd}(t) \quad \Longleftrightarrow \quad \widehat{S}_i(f)$$

and additionally, we have that $\widehat{S}_r(f)$ is even (cosine-transform) and $\widehat{S}_i(f)$ is odd (sine-transform).

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n) real valued signals

If $s(t)$ is real valued, then we have $\widehat{S}(-f) = \widehat{S}^\star(f)$

o) regularity

If $s^n(t) \in L^1(I\!R)$ then $\lim_{f \to \pm\infty} |(i2\pi f)^n \widehat{S}(f)| = 0$, i.e.

$$\widehat{S}(f) = O(|f|^{-n-\epsilon})$$

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Bandwidth of signals and Heisenberg's uncertainty principle

$$\theta^2 = \int_{-\infty}^{\infty} (t - t_0)^2 |s(t)|^2 dt$$

$$B^2 = \int_{-\infty}^{\infty} (f - f_0)^2 |\widehat{S}(f)|^2 df$$

where $\int_{-\infty}^{\infty} |s(t)|^2 dt = \int_{-\infty}^{\infty} |\widehat{S}(f)|^2 df = 1$
and t_0 and f_0 are the center of gravity in the t/f plane, respectively:

$$t_0 = \int_{-\infty}^{\infty} t|s(t)|^2 dt \qquad f_0 = \int_{-\infty}^{\infty} f|\widehat{S}(f)|^2 df$$

Heisenberg's uncertainty principle yields:

$$\theta B \geq \frac{1}{4\pi}$$

Proof:

w.l.o.g. let $t_0 = f_0 = 0$
Using Schwarz inequality

$$|\int_{-a}^{b} g_1(t) g_2(t) dt|^2 \leq \int_{-a}^{b} |g_1(t)|^2 dt \int_{-a}^{b} |g_2(t)|^2 dt$$

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for $a, b \to \infty$ and with
$g_1(t) = ts(t)$ and $g_2(t) = ds/dt$ we have

$$|\int_{-\infty}^{\infty} ts(t) ds/dt dt|^2 \leq \int_{-\infty}^{\infty} |ts(t)|^2 dt \quad \int_{-\infty}^{\infty} |ds/dt|^2 dt$$

As $s \in L^2(I\!R)$, $\lim_{t \to \pm\infty} |s(t)| \leq C \frac{1}{\sqrt{t}}$

$$\int_{-\infty}^{\infty} ts(t) ds/dt dt = -\frac{1}{2}$$

and

$$\int_{-\infty}^{\infty} |ds/dt|^2 \, dt = \int_{-\infty}^{\infty} |2\pi f \widehat{S}(f)|^2 df$$

$$\frac{1}{4} \leq 4\pi^2 \int_{-\infty}^{\infty} t^2 |s(t)|^2 dt \int_{-\infty}^{\infty} f^2 |\widehat{S}(f)|^2 df$$

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Distributions

$$\delta(t-t_0) = \begin{cases} \infty & \text{for} \quad t = t_0, \\ 0 & \text{elsewhere} \end{cases} \tag{2}$$

$$\int_{-\infty}^{\infty} \delta(t-t_0)dt = 1$$

Properties:

$$\int_{-\infty}^{\infty} s(t)\delta(t-t_0)dt = s(t_0)$$

$$\sqrt{\frac{n}{\pi}}\exp(-nt^2) \longrightarrow \delta(t) \quad \text{for} \quad n \longrightarrow \infty$$

$$\sqrt{\frac{n}{\pi}}\exp(-nt^2) \quad \Longleftrightarrow \quad \exp(\frac{-\pi^2 f^2}{n})$$

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$$\delta(t) \quad \Longleftrightarrow \quad 1$$

$$1 \quad \Longleftrightarrow \quad \delta(t)$$

$$\delta^{(k)}(t) \quad \Longleftrightarrow \quad (i2\pi f)^k$$

$$(-i2\pi t)^k \quad \Longleftrightarrow \quad \delta^{(k)}(f)$$

Scaling:

$$\delta^{(k)}(at) \quad \Longleftrightarrow \quad \frac{1}{|a|}(i2\pi fa)^k$$

$$\frac{1}{|a|}\delta^{(k)}(\frac{t}{a}) \quad \Longleftrightarrow \quad (i2\pi fa)^k$$

and for $k = 0$

$$\frac{1}{|a|}\delta(\frac{t}{a}) \quad \Longleftrightarrow \quad 1$$

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Shift:

$$\delta^{(k)}(t - t_0) \quad \Longleftrightarrow \quad \exp(-i2\pi f t_0)(i2\pi f)^k$$

$$\exp(i2\pi f_0 t)(-i2\pi t)^k \quad \Longleftrightarrow \quad \delta^{(k)}(f - f_0)$$

and for $k = 0$

$$\delta(t - t_0) \quad \Longleftrightarrow \quad \exp(-i2\pi f t_0)$$

$$\exp(i2\pi f_0 t) \quad \Longleftrightarrow \quad \delta(f - f_0)$$

$$\sin(2\pi f_0 t) \quad \Longleftrightarrow \quad \frac{1}{2i}(\delta(f - f_0) - \delta(f + f_0))$$

$$\cos(2\pi f_0 t) \quad \Longleftrightarrow \quad \frac{1}{2}(\delta(f - f_0) + \delta(f + f_0))$$

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Convolution:

$$s(t) \quad \Longleftrightarrow \quad \widehat{S}(f)$$

$$\delta^{(k)}(t) \star s(t) \quad \Longleftrightarrow \quad (i2\pi f a)^k \widehat{S}(f)$$

and for $k = 0$

$$\delta(t) \star s(t) = s(t)$$

$$\delta(t - t_0) \star s(t) = s(t - t_0)$$

Sampling theorem: Let $s(t) \in L^1(I\!R)$ with $\widehat{S}(f) = 0$ for $|f| > f_c$. Then we have

$$s(t) = \sum_{n=0}^{\infty} s(nT) \frac{\sin \pi(t - nT)/T}{\pi(t - nT)/T} \qquad \text{for} \quad T \leq \frac{1}{2f_c}$$

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2) Periodic signals

Periodic signal $s(t) = s(t + nT), t \in \mathbb{R}, n \in \mathbb{Z}$ with period T,

Discrete Fourier coefficients $\widehat{S}_k, k \in \mathbb{Z}$ with $\widehat{S}_k = \frac{1}{T}\int_0^T s(t)e^{-i2\pi kt}dt$

and $s(t) = \sum_{k\in\mathbb{Z}} \widehat{S}_k e^{i2\pi kt/T}$

3) Discrete signals

Discrete signal $s_n, n \in \mathbb{Z}$

Periodic Fourier transform

$$\widehat{S}(f) = \sum_{n\in\mathbb{Z}} s_n e^{-i2\pi nf}$$

4) Discrete periodic signals

Discrete periodic signal $s_n, 0 \leq n \leq N-1$ with $s_n = s_{n+mN}, m \in \mathbb{Z}$

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Periodic discrete Fourier transform

$$\widehat{S}_k = \frac{1}{N} \sum_{n=0}^{N-1} s_n e^{-i2\pi kn/N}, 0 \leq k \leq N-1$$

where $\widehat{S}_k = \widehat{S}_{k+mN}, m \in \mathbb{Z}$

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5) Summary

- Continuous signal $s(t), t \in \mathbb{R} \longleftrightarrow$ continuous spectrum, $\widehat{S}(f), f \in \mathbb{R}$

- Periodic signal $s(t), t \in \mathbb{T} \longleftrightarrow$ discrete spectrum, $\widehat{S}_k, k \in \mathbb{Z}$

- Discrete signal $s_n, n \in \mathbb{Z} \longleftrightarrow$ periodic spectrum, $\widehat{S}(f), f \in \mathbb{T}$

- Discrete periodic signal $s_n, 0 \leq n \leq N-1$ with $s_n = s_{n+mN}, m \in \mathbb{Z}$ $\longleftrightarrow$ periodic discrete spectrum $\widehat{S}_k, 0 \leq k \leq N-1$ and with $\widehat{S}_k = \widehat{S}_{k+mN}, m \in \mathbb{Z}$

Extention to higher dimensions: tensor product ansatz

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Ad Fourier transform

Continuous signals: $s(t), t \in I\!R$

$$\widehat{S}(f) = \int_{-\infty}^{\infty} s(t) e^{-i2\pi f t} dt, f \in I\!R \quad \text{and} \quad s(t) = \int_{-\infty}^{\infty} \widehat{S}(f) e^{i2\pi f t} df$$

Periodic signals (continuous): $\tilde{s}(t) = \tilde{s}(t + mT), m \in Z\!\!Z$

$$\widehat{S}_k = \frac{1}{T} \int_0^T s(t) e^{-i2\pi kt/T} \quad \text{and} \quad s(t) = \sum_{k \in Z\!\!Z} \widehat{S}_k e^{i2\pi kt/T}$$

with

$$e^{i2\pi kt/T} \quad \Longleftrightarrow \quad \delta(f - \frac{k}{T})$$

$$\widehat{S}(f) = \sum_{k \in Z\!\!Z} \widehat{S}_k \delta(f - \frac{k}{T})$$

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Dirac pulse:

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(f - k/T)$$

Periodisation

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} s(t - nT)$$

$$\tilde{s}(t) = \tilde{s}(t + mT), m \in \mathbb{Z}$$

$$\tilde{s}(t) \quad \Longleftrightarrow \quad \sum_{k \in \mathbb{Z}} \widehat{S}_k \delta(f - \frac{k}{T})$$

with $\widehat{S}_k = \frac{1}{T} \widehat{S}(k/T)$

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Discrete signals: $s_n, n \in \mathbb{Z}$

$$\widehat{S}(f) = \sum_{n \in \mathbb{Z}} s_n e^{-i2\pi nf}$$

Discrete periodic signals: $s_n,\ n = 0, ..., N-1$

$$\widehat{S}_k = \sum_{n \in \mathbb{Z}} s_n e^{-i2\pi kn/N}$$

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Sampled signals:

$$s_{samp}(t) = T \sum_{k \in \mathbb{Z}} s(kT)\delta(t - kT)$$

$$f_{samp} = 2 f_{limit} = \frac{1}{T}$$

$$s_{samp}(t) \quad \Longleftrightarrow \quad \widehat{S}_{samp}(f) = \sum_{k=-\infty}^{\infty} \widehat{S}(f - k f_{samp})$$

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