University of California, Santa Barbara Winter 2004
Department of Mechanical and Environmental Engineering
ME 252 B, Computational Fluid Dynamics:
Wavelet transforms and their applications to turbulence
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Exercise sheet 2:
Continuous wavelet transform
Due Friday January 30th 2004

Background material

Continuous wavelet transform:

- Let us consider a one-dimensional signal $f(x)$ in $L^2(I\!R)$ and a one-dimensional real or complex-valued wavelet $\psi(x)$ in $L^2(I\!R)$, the continuous wavelet coefficients are

$$\widetilde{f}(a,b) = \int_{-\infty}^{+\infty} f(x)\, \overline{\psi_{a,b}(x)}\, dx \tag{1}$$

  with $\psi_{a,b}(x) = a^{-1/2}\, \psi(\frac{x-b}{a})$.

- Applying Parseval's theorem, one gets

$$\widetilde{f}(a,b) = \int_{-\infty}^{+\infty} \hat{f}(k)\, \overline{\hat{\psi}_{a,b}(k)}\, dk \tag{2}$$

  with $\hat{\psi}_{a,b}(k) = a^{1/2}\, \hat{\psi}(ak)\, e^{-2\pi i b k}$.

Numerical implementation:

- The signal $f(x)$ is sampled on a grid x_n, $n \in [0, N-1]$ with $N = 2^J$.
  The sampled continuous wavelet coefficients computed with (1) are

$$\widetilde{f}(a_j, b_m) = \Delta x \sum_{n=0}^{N-1} f(x_n)\, \overline{\psi_{a_j,b_m}(x_n)} \tag{3}$$

  with $\Delta x = \frac{L}{N}$, $x_n = n\Delta x$, $b_m = m\Delta x$ and $a_j = a_0^{-j}$ for $j \in [0, J]$ and $a_0 = 2^{1/M}$, M being the number of voices per octave.

- If $f(x)$ is a real-valued periodic signal of period $[0, L]$ sampled on $N = 2^J$ grid points, the sampled continuous wavelet coefficients computed with (2) are

$$\widetilde{f}(a_j, b_m) = a_j^{1/2} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{f}(k)\, \overline{\hat{\psi}(a_j k)}\, e^{2\pi i b_m k} \tag{4}$$

  where $\hat{f}(k) = \Delta x \sum_{n=0}^{N-1} f(x_n) e^{-2\pi i x_n k}$.
  Note that in (4) $\hat{\psi}$ is sampled in Fourier space, therefore frequencies have to be centered around $k = 0$. Note also that $\hat{f}(-k) = \overline{\hat{f}(k)}$ since f is real-valued.

How to choose the wavelets:

We consider:

- either a real-valued wavelet, *e.g.* the mexican hat

$$\psi(x) = (1 - x^2)\, e^{-\frac{x^2}{2}} \tag{5}$$

$$\hat{\psi}(k) = k^2\, e^{-\frac{k^2}{2}} \tag{6}$$

- or a complex-valued wavelet,*e.g.* the Morlet wavelet

$$\psi(x) = e^{-ik_\psi x}\, e^{-\frac{x^2}{2}} \tag{7}$$

$$\hat{\psi}(k) = e^{-\frac{(k-k_\psi)^2}{2}} \quad \text{for} \quad k > 0 \tag{8}$$

$$0 \quad \text{for} \quad k \leq 0 \tag{9}$$

with k_ψ the mean frequency of the mother wavelet.

How to plot the wavelet coefficients:

The sampled wavelet coefficients $\tilde{f}(a_j, b_n)$ are plotted with a linear axis for the position b_n and a logarithmic (base 2) axis for the scale a_j, or equivalently with a linear axis for j. You can thus use the matlab commands *imagesc* with *colormap('jet')* to plot $\tilde{f}(n, j)$ with n on the horizontal axis and j and the vertical axis.

Superimpose on the plot:

- two curves to denote the influence cones at both ends of the signal. They are the envelops of the wavelet coefficients $\tilde{f}$ of two Diracs, located at positions x_0 and x_N). To draw them you can use matlab command *contour* and take for the isoligne a weak value of $\tilde{f}$, *e.g.* $\epsilon = 10^{-3}$.

- a line to denote the scale below which the wavelets are aliased (it corresponds to the Nyquist frequency given by the sampling theorem).

Exercices

Exercice 1:

Program the continuous wavelet transform of a periodic real-valued signal $f(x)$ of period $[0, L]$ with $L = 10$, sampled on $N = 1024$ grid points, using the Mexican hat wavelet $\psi(x)$. Choose $M = 4$ voices per octave to sample the scale $a_j = 2^{-j/M}$.

1. Use equation (3).

2. Use equation (4).

3. For both methods, plot $\tilde{f}(a_j, b_n)$ and $|\tilde{f}(a_j, b_n)|$ for several academic signals:

- a Dirac: $f(x) = \delta(x - x_1)$ for $x_1 = 5$,
- two sines: $f(x) = \sin 2\pi k_1 x + \sin 2\pi k_2 x$ for $k_1 = 5$ and $k_2 = 20$,
- a period doubling sine:
  $f(x) = \sin 2\pi k_1 x$ for $x \in [0, 5]$
  $f(x) = \sin 2\pi k_2 x$ for $x \in]5, 10]$,
- a chirp: $f(x) = \sin x^2$,
- other signals you like.

4. Compare the results obtained using equation (3) and equation (4).

5. Program equation (3) with a fast convolution (2).

Exercice 2:

1. Program the continuous wavelet transform of a periodic real-valued signal $f(x)$ of period $[0, L]$ with $L = 10$, sampled on $N = 1024$ grid points, for the Morlet wavelet $\psi(x)$. Take the best method you have found in Exercice 1. Choose $M = 4$ voices per octave to sample the scale.

2. Plot the real part, the modulus and the phase of the wavelet coefficients for the different signals you have analyzed in Exercice 1.

3. Compare the results obtained when you vary the mean frequency k_ψ of the mother wavelet.

4. Compare the results obtained using the Mexican hat wavelet with those using the Morlet wavelet.

5. By observing the wavelet transform of the two sine functions, give the relation between the scale a_j and the wavenumber k.