Exercise sheet 3:  
Fast orthogonal wavelet transform  
Due Friday February 13, 2004  

Background material  

Multiresolution analysis  

We consider a sequence $V_j, j \in \mathbb{Z}$ of closed subspaces of $L^2(\mathbb{R})$ which constitutes a one-dimensional orthogonal multiresolution analysis of $L^2(\mathbb{R})$. A scaling function $\phi(x)$ is required to exist. Its translates generate a basis in each $V_j$, i.e.

$$V_j = \text{span}\{\phi_{ji}\}_{i \in \mathbb{Z}}.$$  

(1)

where

$$\phi_{ji}(x) = 2^{j/2}\phi(2^j x - i) \quad j, i \in \mathbb{Z}.$$  

(2)

This basis is orthonormal, so that

$$\langle \phi_{ji}, \phi_{jk} \rangle = \delta_{ik}$$  

(3)

with $\langle f, g \rangle = \int_{-\infty}^{+\infty} f(x) \overline{g}(x) \, dx$ being the inner-product in $L^2(\mathbb{R})$.

The scaling functions satisfy a refinement equation: $\phi_{j-1,n}(x) = \sum_{k \in \mathbb{Z}} h_{k-2n} \phi_{j,k}(x)$ with the filter coefficients $h_n = \langle \phi_{jn}, \phi_{j-1,0} \rangle$.

The orthogonal projection of a function $f \in L^2(\mathbb{R})$ on $V_j$ is defined as

$$P_{V_j} : f \longrightarrow P_{V_j} f = f_J$$  

(4)

with

$$f_J(x) = \sum_{k \in \mathbb{Z}} \langle f, \phi_{jk} \rangle \phi_{jk}(x).$$  

(5)

As $V_{J-1}$ is included in $V_J$ we can define its orthogonal complement space in $V_J$:

$$V_J = V_{J-1} \oplus W_{J-1}$$  

(6)

Correspondingly, the approximation of the function $f$ at scale $2^{-J}$, belonging to $V_J$, can be decomposed as a sum of orthogonal projections on $V_{J-1}$ and $W_{J-1}$, i.e.

$$P_{V_J} f = P_{V_{J-1}} f + P_{W_{J-1}} f.$$  

(7)

Based on the scaling function $\phi$, one can construct a function $\psi$, the so-called mother wavelet, such that

$$\psi_{jn}(x) = \sum_{k \in \mathbb{Z}} a_{k-2n} \phi_{j,k}(x)$$  

(8)
with $g_n = \langle \phi_{jn}, \psi_{j-1,0} \rangle$, and where

$$\psi_{ji}(x) = 2^{j/2} \psi(2^j x - i) \quad j, i \in \mathbb{Z} \ .$$ (9)

The filter coefficients $g_n$ can be computed from the filter coefficients $h_n$ using the relation $g_n = (-1)^{1-n} h_{1-n}$.

The translates and dilates of the wavelet $\psi$ constitute an orthonormal bases of the spaces $W_j$,

$$W_j = \text{span} \{ \psi_{ji} \}_{i \in \mathbb{Z}} \ .$$ (10)

Any function $f \in L^2(\mathbb{R})$ can now be expressed as

$$f(x) = \sum_{i \in \mathbb{Z}} \bar{f}_{ji} \phi_{ji}(x) + \sum_{j=j_0}^{\infty} \sum_{i \in \mathbb{Z}} \tilde{f}_{ji} \psi_{ji}(x)$$ (11)

where

$$\bar{f}_{ji} = \langle f, \phi_{ji} \rangle \quad \tilde{f}_{ji} = \langle f, \psi_{ji} \rangle$$ (12)

In numerical applications the sums in (11) have to be truncated, which corresponds to the projection of $f$ onto a subspace of $V_J \subset L^2(\mathbb{R})$. The decomposition (11) is orthogonal, as, by construction,

$$\langle \psi_{ji}, \psi_{lk} \rangle = \delta_{jl} \delta_{ik}$$ (13)

$$\langle \psi_{ji}, \phi_{lk} \rangle = 0 \quad j \geq l$$ (14)

in addition to (3).

**Fast wavelet transform (FWT)**

Starting with a function $f$ given at a finite resolution $2^{-J}$, i.e. we know $f_J \in V_J$ and hence the coefficients $\bar{f}_{ji}$ for $i \in \mathbb{Z}$, the fast wavelet transform computes its wavelet coefficients $\tilde{f}_{ji}$ by decomposing successively each approximation $P_{V_J} f = f_J$ into a coarser scale approximation $P_{V_{J-1}} f$ plus its differences $P_{W_{J-1}} f$. The algorithm uses a cascade of discrete convolutions with the filters $h_n$ and $g_n$, plus downsampling.

Initialization: given $f \in L^2(\mathbb{R})$ and $\bar{f}_{J,n} = f \left( \frac{n}{2^J} \right)$ for $n \in \mathbb{Z}$.

Decomposition: for $j = J$ to 1, step $-1$, do:

$$\bar{f}_{j-1,n} = \sum_{k \in \mathbb{Z}} h_{k-2n} \bar{f}_{j,k}$$ (15)

$$\tilde{f}_{j-1,n} = \sum_{k \in \mathbb{Z}} g_{k-2n} \tilde{f}_{j,k} \quad \text{for} \quad n \in \mathbb{Z} .$$ (16)

The inverse wavelet transform is based on successive reconstructions of a fine scale approximation $P_{V_J} f$ from a coarser scale approximation $P_{V_{J-1}} f$ plus its differences $P_{W_{J-1}} f$. 
The algorithm uses a cascade of discrete convolutions with the filters $h_n$ and $g_n$, plus upsampling.

Reconstruction: for $j = 1$ to $J$, step 1, do:

$$\tilde{f}_{j,n} = \sum_{k \in \mathbb{Z}} h_{n-2k} \tilde{f}_{j-1,k} + \sum_{k \in \mathbb{Z}} g_{n-2k} \tilde{f}_{j,k} \quad \text{for} \quad n \in \mathbb{Z}.$$ (17)

Scalogram, intermittency measures

We define the scale distribution of energy, also called scalogram, as

$$\tilde{E}_j = \sum_{i=0}^{2^j-1} |\tilde{f}_{j,i}|^2.$$ (18)

By summing the scalogram over all scales we get the total energy of the function, \textit{i.e.},

$$E = |\tilde{f}_{00}|^2 + \sum_{j=0}^{J-1} \tilde{E}_j.$$ (19)

Note that due to Parseval we also have $E = \sum_{n=0}^{2^j-1} |\tilde{f}_{j,n}|^2$.

To measure intermittency we use the space-scale information contained in the wavelet coefficients to define scale-dependent moments and moment ratios. Useful diagnostics to quantify the intermittency of a field $f$ are the moments of its wavelet coefficients at different scales $j$,

$$M_{p,j}(f) = 2^{-j} \sum_{i=0}^{2^j-1} |\tilde{f}_{j,i}|^p.$$ (20)

Note that $E_j = 2^j M_{2,j}$.

The sparsity of the wavelet coefficients at each scale is a measure of intermittency, and it can be quantified using ratios of moments at different scales,

$$Q_{p,q,j}(f) = \frac{M_{p,j}(f)}{(M_{q,j}(f))^{p/q}}.$$ (21)

Classically, one chooses $q = 2$ to define typical statistical quantities as a function of scale. Recall that for $p = 4$ we obtain the scale dependent flatness $F_j = Q_{4,2,j}$. It is equal to 3 for a Gaussian white noise at all scales $j$, which proves that this signal is not intermittent. The scale dependent skewness, hyperflatness and hyperskewness are obtained for $p = 3, 5$ and 6, respectively. For intermittent signals $Q_{p,q,j}$ increases with $j$. 

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Examples of orthogonal wavelets

Filter coefficients of $h_n$.

- Haar D1 (1 vanishing moment):
  \[ h_0 = \frac{1}{\sqrt{2}}, \quad h_1 = \frac{1}{\sqrt{2}}. \]

- Daubechies D2 (2 vanishing moments):
  \[ h_0 = 0.482962913145, \quad h_1 = 0.836516303736, \quad h_2 = 0.224143868042, \quad h_3 = -0.129409522551. \]

- Daubechies D3 (3 vanishing moments):
  \[ h_0 = 0.332670552950, \quad h_1 = 0.806891509311, \quad h_2 = 0.459877502118, \quad h_3 = -0.135011020010, \]
  \[ h_4 = -0.085441273882, \quad h_5 = 0.035226291882. \]

**Exercice 1:**

- Verify that $\sum_k h_k = \sqrt{2}$ and that $\sum_k k^n g_k = 0$ for $n = 0, \ldots, M - 1$, where $M$ corresponds to the number of vanishing moments of the wavelet $\psi$, for D1, D2 and D3.

The Fourier transform of the filters $h$ and $g$ is defined as $\hat{H}(\omega) = \sum_n h_n e^{-2\pi i n \omega}$ and $\hat{G}(\omega) = \sum_n g_n e^{-2\pi i n \omega}$ respectively.

- Compute $\hat{H}(\omega)$ and $\hat{G}(\omega)$ numerically for D1, D2, and D3, and plot $|\hat{H}(\omega)|$ and $|\hat{G}(\omega)|$. Discuss the properties of both filters.

- Show that $\hat{G}(\omega) = e^{-2\pi i \omega} \hat{H}^\star(\omega + 1)$.

**Exercice 2:**

Implement the fast orthogonal wavelet transform and its inverse using the formulas (15) and (17) using Haar wavelets (D1) and Daubechies wavelets (D2, D3).

Input: function values $(f_0, f_1, f_2, \ldots, f_{N-1})$, number of scales $J$ with $N = 2^J$.

Output: wavelet coefficients $(\tilde{f}_{0,0}, \tilde{f}_{0,1}, \tilde{f}_{1,1}, \ldots, \tilde{f}_{J-1,0}, \tilde{f}_{J-1,1}, \ldots, \tilde{f}_{J-1,2^J-1-1})$.

In order to simplify the discrete convolutions, suppose the function is periodic, i.e. $f(n) = f(n + pN)$ for $p \in \mathbb{Z}$.

Given a discrete signal $f_n = f(t_n)$ with $t_n = n/N$ for $n = 0, \ldots, N - 1$ and with $t_0 = 0.5$, $\sigma^2 = 1/500$ and $N = 2^{10} = 1024$.

- Plot the signal $f_n$.

- Apply the fast orthogonal wavelet transform to $f_n$ and plot the magnitude of the wavelet coefficients $\tilde{f}_{ji}$ in logarithmic scale using a scale-space representation (e.g. $j$ vertical axis, $i$ horizontal axis).

- Reconstruct the signal from its wavelet coefficients $\tilde{f}_{ji}$ using the inverse FWT and compare the result with the signal $f_n$.

- Compute the energy of the signal, either in physical space ($E_p = \sum_{n=0}^{N-1} |f_n|^2$), or in wavelet space ($E_w = |\tilde{f}_{00}|^2 + \sum_{j=0}^{J-1} \sum_{i=0}^{2^j-1} |\tilde{f}_{ji}|^2$). Discuss the result.
• Compute approximations of the signal at different scales $J_f$ (linear filtering). Reconstruct different approximations $f_{J_f}$ for $J_f = 4, ...9$. For this apply an inverse FWT to the wavelet coefficients $\tilde{f}_{ji}$ and set the coefficients for $j = J_f - 1, ... J - 1$ and $i = 0, ..., 2^j - 1$ equal to zero.

Plot the different approximations $f_{J_f}$ for $J_f = 4, ...9$, assembled in one figure.

• Compute approximations of the signal using thresholding of the wavelet coefficients (nonlinear filtering). Reconstruct several nonlinear approximations $f_T$. For this apply an inverse FWT to the wavelet coefficients whose modulus is larger than the threshold $T$, which means that coefficients $\tilde{f}_{ji}$ with $|\tilde{f}_{ji}| \leq T$ are set to zero. Test different thresholds, $T = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$, count the number of retained coefficients and plot the results assembled in one figure.

**Exercise 3 (optional):**

Given a realization of a Gaussian white noise with variance $\sigma^2 = 0.1$ (using a random number generator), $w_n$ for $n = 0, ..., N - 1$.

• Plot the noise $w_n$ and construct a noisy signal $g_n = f_n + w_n$ using $f_n$ from exercise 3.

• Apply the FWT to $w_n$ and compute its scalogram $E_j$, its scale dependent flatness $F_j = Q_{4,2,j}$ and its scale dependent skewness $S_j = Q_{3,2,j}$. Discuss the results, knowing that the noise is uncorrelated, that the flatness of a Gaussian white noise is equal to three and its skewness is zero.

• Apply the wavelet transform to the noisy signal $g_n$.
Threshold its wavelet coefficients $\tilde{g}_{ji}$, by retaining only those values being larger than a threshold $T = \sigma(2 \log_2 N)^{1/2}$.
Reconstruct the denoised signal using the inverse FWT, compute the difference between the denoised signal and the original $f_n$ and plot the results.