University of California, Santa Barbara | Winter 2004
Department of Mechanical and Environmental Engineering
ME 252 B, Computational Fluid Dynamics:
Wavelet transforms and their applications to turbulence
Prof. Marie Farge & Prof. Kai Schneider

Exercise sheet 3:
Fast orthogonal wavelet transform
Due Friday February 13, 2004

Background material

Multiresolution analysis

We consider a sequence $V_j, j \in \mathbb{Z}$ of closed subspaces of $L^2(\mathbb{R})$ which constitutes a one-dimensional orthogonal multiresolution analysis of $L^2(\mathbb{R})$. A scaling function $\phi(x)$ is required to exist. Its translates generate a basis in each V_j, *i.e.*

$$V_j = \overline{span}\{\phi_{ji}\}_{i\in\mathbb{Z}} \quad . \tag{1}$$

where

$$\phi_{ji}(x) = 2^{j/2}\phi(2^j x - i) \qquad\qquad j, i \in \mathbb{Z} \quad . \tag{2}$$

This basis is orthonormal, so that

$$\langle \phi_{ji}, \phi_{jk} \rangle = \delta_{ik} \tag{3}$$

with $\langle f, g \rangle = \int_{-\infty}^{+\infty} f(x)\, \overline{g}(x)\, dx$ being the inne-product in $L^2(\mathbb{R})$.
The scaling functions satisfy a refinment equation: $\phi_{j-1,n}(x) = \sum_{k\in\mathbb{Z}} h_{k-2n}\phi_{j,k}(x)$ with the filter coefficients $h_n = \langle \phi_{jn}, \phi_{j-1,0} \rangle$.
The orthogonal projection of a function $f \in L^2(\mathbb{R})$ on V_J is defined as

$$P_{V_J} : f \longrightarrow P_{V_J} f = f_J \tag{4}$$

with

$$f_J(x) = \sum_{k\in\mathbb{Z}} \langle f, \phi_{jk} \rangle \phi_{jk}(x) \quad . \tag{5}$$

As V_{J-1} is included in V_J we can define its orthogonal complement space in V_J:

$$V_J = V_{J-1} \oplus W_{J-1} \tag{6}$$

Correspondingly, the approximation of the function f at scale 2^{-J}, belonging to V_J, can be decomposed as a sum of orthogonal projections on V_{J-1} and W_{J-1}, *i.e.*

$$P_{V_J} f = P_{V_{J-1}} f + P_{W_{J-1}} f \quad . \tag{7}$$

Based on the scaling function ϕ, one can construct a function ψ, the so–called mother wavelet, such that

$$\psi_{jn}(x) = \sum_{k\in\mathbb{Z}} g_{k-2n}\phi_{j,k}(x) \tag{8}$$

with $g_n = \langle \phi_{jn}, \psi_{j-1,0} \rangle$, and where

$$\psi_{ji}(x) = 2^{j/2}\psi(2^j x - i) \qquad j, i \in \mathbb{Z} \quad . \tag{9}$$

The filter coefficients g_n can be computed from the filter coefficients h_n using the relation $g_n = (-1)^{1-n} h_{1-n}$.

The translates and dilates of the wavelet ψ constitute an orthonormal bases of the spaces W_j,

$$W_j = \overline{span}\{\psi_{ji}\}_{i\in\mathbb{Z}} \quad . \tag{10}$$

Any function $f \in L^2(\mathbb{R})$ can now be expressed as

$$f(x) = \sum_{i\in\mathbb{Z}} \overline{f}_{j_0 i}\, \phi_{j_0 i}(x) \quad + \quad \sum_{j=j_0}^{\infty} \sum_{i\in\mathbb{Z}} \widetilde{f}_{ji}\, \psi_{ji}(x) \tag{11}$$

where

$$\overline{f}_{ji} = \langle f, \phi_{ji} \rangle \qquad \widetilde{f}_{ji} = \langle f, \psi_{ji} \rangle \tag{12}$$

In numerical applications the sums in (11) have to be truncated, which corresponds to the projection of f onto a subspace of $V_J \subset L^2(\mathbb{R})$. The decomposition (11) is orthogonal, as, by construction,

$$\langle \psi_{ji}, \psi_{lk} \rangle = \delta_{jl}\ \delta_{ik} \tag{13}$$

$$\langle \psi_{ji}, \phi_{lk} \rangle = 0 \qquad j \geq l \tag{14}$$

in addition to (3).

Fast wavelet transform (FWT)

Starting with a function f given at a finite resolution 2^{-J}, *i.e.* we know $f_J \in V_J$ and hence the coefficients $\overline{f}_{Ji}$ for $i \in \mathbb{Z}$, the fast wavelet transform computes its wavelet coefficients $\widetilde{f}_{ji}$ by decomposing successively each approximation $P_{V_J} f = f_J$ into a coarser scale approximation $P_{V_{J-1}} f$ plus its differences $P_{W_{J-1}} f$. The algorithm uses a cascade of discrete convolutions with the filters h_n and g_n, plus downsampling.

Initialization: given $f \in L^2(\mathbb{R})$ and $\overline{f}_{J,n} = f\left(\frac{n}{2^J}\right)$ for $n \in \mathbb{Z}$.

Decomposition: for $j = J$ to 1, step -1, do:

$$\overline{f}_{j-1,n} = \sum_{k\in\mathbb{Z}} h_{k-2n} \overline{f}_{j,k} \tag{15}$$

$$\widetilde{f}_{j-1,n} = \sum_{k\in\mathbb{Z}} g_{k-2n} \overline{f}_{j,k} \qquad \text{for} \quad n \in \mathbb{Z}\ . \tag{16}$$

The inverse wavelet transform is based on successive reconstrunctions of a fine scale approximation $P_{V_J} f$ from a coarser scale approximation $P_{V_{J-1}} f$ plus its differences $P_{W_{J-1}} f$.

The algorithm uses a cascade of discrete convolutions with the filters h_n and g_n, plus upsampling.

Reconstruction: for $j = 1$ to J, step 1, do:

$$\overline{f}_{j,n} = \sum_{k \in \mathbb{Z}} h_{n-2k} \overline{f}_{j-1,k} + \sum_{k \in \mathbb{Z}} g_{n-2k} \tilde{f}_{j,k} \qquad \text{for} \quad n \in \mathbb{Z} \ . \tag{17}$$

Scalogram, intermittency measures

We define the scale distribution of energy, also called scalogram, as

$$\widetilde{E}_j = \sum_{i=0}^{2^j-1} |\tilde{f}_{j,i}|^2. \tag{18}$$

By summing the scalogram over all scales we get the total energy of the function, *i.e.*,

$$E = |\overline{f}_{00}|^2 + \sum_{j=0}^{J-1} \widetilde{E}_j. \tag{19}$$

Note that due to Parseval we also have $E = \sum_{n=0}^{2^J-1} |\overline{f}_{Jn}|^2$.
To measure intermittency we use the space-scale information contained in the wavelet coefficients to define scale-dependent moments and moment ratios. Useful diagnostics to quantify the intermittency of a field f are the moments of its wavelet coefficients at different scales j,

$$M_{p,j}(f) = 2^{-j} \sum_{i=0}^{2^j-1} |\tilde{f}_{j,i}|^p. \tag{20}$$

Note that $E_j = 2^j M_{2,j}$.

The sparsity of the wavelet coefficients at each scale is a measure of intermittency, and it can be quantified using ratios of moments at different scales,

$$Q_{p,q,j}(f) = \frac{M_{p,j}(f)}{(M_{q,j}(f))^{p/q}} \quad . \tag{21}$$

Classically, one chooses $q = 2$ to define typical statistical quantities as a function of scale. Recall that for $p = 4$ we obtain the scale dependent flatness $F_j = Q_{4,2,j}$. It is equal to 3 for a Gaussian white noise at all scales j, which proves that this signal is not intermittent. The scale dependent skewness, hyperflatness and hyperskewness are obtained for $p = 3, 5$ and 6, respectively. For intermittent signals $Q_{p,q,j}$ increases with j.

Examples of orthogonal wavelets

Filter coefficients of h_n.

- Haar D1 (1 vanishing moment):
  $h_0 = 1/\sqrt{2}, h_1 = 1/\sqrt{2}$.
- Daubechies D2 (2 vanishing moments):
  $h_0 = 0.482962913145, h_1 = 0.836516303736, h_2 = 0.224143868042, h_3 = -0.129409522551$.
- Daubechies D3 (3 vanishing moments):
  $h_0 = 0.332670552950, h_1 = 0.806891509311, h_2 = 0.459877502118, h_3 = -0.135011020010,$
  $h_4 = -0.085441273882, h_5 = 0.035226291882$.

Exercice 1:

- Verify that $\sum_k h_k = \sqrt{2}$ and that $\sum_k k^n g_k = 0$ for $n = 0, M-1$, where M corresponds to the number of vanishing moments of the wavelet ψ, for D1, D2 and D3.

The Fourier transform of the filters h and g is defined as $\widehat{H}(\omega) = \sum_n h_n e^{-2\pi i\omega n}$ and $\widehat{G}(\omega) = \sum_n g_n e^{-2\pi i\omega n}$ respectively.

- Compute $\widehat{H}(\omega)$ and $\widehat{G}(\omega)$ numerically for D1, D2, and D3, and plot $|\widehat{H}(\omega)|$ and $|\widehat{G}(\omega)|$. Discuss the properties of both filters.
- Show that $\widehat{G}(\omega) = e^{-i2\pi\omega}\widehat{H}^\star(\omega + 1)$.

Exercice 2:

Implement the fast orthogonal wavelet transform and its inverse using the formulas (15) and (17) using Haar wavelets (D1) and Daubechies wavelets (D2, D3).
Input: function values $(f_0, f_1, f_2, ..., f_{N-1})$, number of scales J with $N = 2^J$.
Output: wavelet coefficients $(\overline{f}_{0,0}|\widetilde{f}_{0,0}|\widetilde{f}_{1,0}, \widetilde{f}_{1,1}|...|\widetilde{f}_{J-1,0}, \widetilde{f}_{J-1,1}, ..., \widetilde{f}_{J-1,2^{J-1}-1}|)$.
In order to simplify the discrete convolutions, suppose the function is periodic, *i.e.* $f(n) = f(n + pN)$ for $p \in \mathbb{Z}$.

Given a discrete signal $f_n = f(t_n)$ with $t_n = n/N$ for $n = 0, ..., N-1$ and with $t_0 = 0.5$, $\sigma^2 = 1/500$ and $N = 2^{10} = 1024$.

- Plot the signal f_n.
- Apply the fast orthogonal wavelet transform to f_n and plot the magnitude of the wavelet coefficients $\widetilde{f}_{ji}$ in logarithmic scale using a scale-space representation (e.g. j vertical axis, i horizontal axis).
- Reconstruct the signal from its wavelet coefficients $\widetilde{f}_{ji}$ using the inverse FWT and compare the result with the signal f_n.
- Compute the energy of the signal, either in physical space ($E_p = \sum_{n=0}^{N-1} |f_n|^2$), or in wavelet space ($E_w = |\overline{f}_{00}|^2 + \sum_{j=0}^{J-1}\sum_{i=0}^{2^j-1} |\widetilde{f}ji|^2$). Discuss the result.

- Compute approximations of the signal at different scales J_f (linear filtering). Reconstruct different approximations f_{J_f} for $J_f = 4, ...9$. For this apply an inverse FWT to the wavelet coefficients $\widetilde{f}_{ji}$ and set the coefficients for $j = J_f - 1, ...J - 1$ and $i = 0, ..., 2^j - 1$ equal to zero.
  Plot the different approximations f_{J_f} for $Jf = 4, ...9$, assembled in one figure.

- Compute approximations of the signal using thresholding of the wavelet coefficients (nonlinear filtering). Reconstruct several nonlinear approximations f_T. For this apply an inverse FWT to the wavelet coefficients whose modulus is larger than the threshold T, which means that coefficients $\widetilde{f}_{ji}$ with $|\widetilde{f}_{ji}| \leq T$ are set to zero. Test different thresholds, $T = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$, count the number of retained coefficients and plot the results assembled in one figure.

Exercise 3 (optional):

Given a realization of a Gaussian white noise with variance $\sigma^2 = 0.1$ (using a random number generator), w_n for $n = 0, ..., N - 1$.

- Plot the noise w_n and construct a noisy signal $g_n = f_n + w_n$ using f_n from exercise 3.

- Apply the FWT to w_n and compute its scalogram E_j, its scale dependent flatness $F_j = Q_{4,2,j}$ and its scale dependent skewness $S_j = Q_{3,2,j}$. Discuss the results, knowing that the noise is uncorrelated, that the flatness of a Gaussian white noise is equal to three and its screwness is zero.

- Apply the wavelet transform to the noisy signal g_n.
  Threshold its wavelet coefficients $\widetilde{g}_{ji}$, by retaining only those values being larger than a threshold $T = \sigma(2 \log_2 N)^{1/2}$.
  Reconstruct the denoised signal using the inverse FWT, compute the difference between the denoised signal and the original f_n and plot the results.