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WAVELETS AND WAVELET PACKETS TO ANALYZE, FILTER, AND COMPRESS TWO-DIMENSIONAL TURBULENT FLOWS

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1. INTRODUCTION

The useful information in a signal is usually carried by both its frequency content and its time evolution. If we consider only the time representation, we do not know the spectrum, whereas the Fourier spectral representation does not give information on the time of occurrence of each frequency. A more appropriate representation should combine these two complementary descriptions. This is true in particular for turbulent signals, especially those presenting bursts or some intermittent, quasi-singular behaviours. The uncertainty principle precludes analysis of the signal from both sides of the Fourier transform at the same time because of the condition $\Delta t \cdot \Delta v \ge 1$ (normalized information cell). Therefore it is always a compromise: either good time resolution Δt but loss of spectral resolution Δv , which is the case when we sample a signal by convolving it with a Dirac comb (Fig. 1a), or good spectral resolution Δv but loss of time resolution Δt , which is the case with the Fourier transform (Fig. 1b). These two transforms are the most commonly used in practice because they allow construction of orthogonal bases onto which the signal can be projected for analysis and eventual computation.

In order to improve time resolution while using the Fourier transform, Gabor (1946) has proposed the windowed Fourier transform, which consists of convolving the signal with a set of Fourier modes localized in a Gaussian envelope of constant width a_0 (Fig. 1c). This transform allows then a time-frequency decomposition of the signal at a given scale a_0 , which is kept fixed. But unfortunately, as shown by Balian (1981), the bases constructed with such windowed Fourier modes cannot be orthogonal. More recently, Grossmann and Morlet (1984, 1985) have devised a new transform, the so-called wavelet transform, which consists of convolving the signal with a set of affine functions all presenting the same frequency v_0 ; the family of analysing wavelets $\psi_{a,b}$ is obtained by dilation and translation of a given function ψ presenting at least one oscillation. The wavelet transform allows therefore a time-scale decomposition of the signal at a given frequency v_0 , which is kept fixed. Actually the wavelet transform realizes the best compromise of the uncertainty principle, because it adapts the time-frequency resolution $\Delta t \cdot \Delta v$ to each scale a. In fact it gives a good spectral resolution Δv with a limited time resolution Δt in the large scales, but also gives a good time localization Δt with a limited spectral resolution Δv in the small scales (Fig. 1d). The continuous wavelet transform has been extended to n dimensions by Murenzi (1989).



c. Windowed Fourier transform

d. Wavelet transform

Figure 1. Comparison between different types of transforms.

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In 1985 Meyer, while trying to prove the impossibility of constructing orthogonal bases, as Balian had earlier done for the case of the windowed Fourier transform, was surprised to discover an orthogonal wavelet basis built with spline functions, now called the Meyer-Lemarié wavelets (Lemarié and Meyer, 1986). In fact the Haar orthogonal basis, which had been proposed in 1909, is now recognized as the first orthogonal wavelet basis known, but the functions it uses are not regular, which drastically limits its application. In practice one likes to build orthogonal wavelet bases using functions having a prescribed regularity to provide enough spectral decay depending on the application. In particular, following Meyer's work, Daubechies (1988) has proposed new orthogonal wavelet bases built with compactly supported functions of prescribed regularity defined by discrete quadratic mirror filters (QMF) of different lengths, the longer the filter, the more regular the associated functions. Mallat (1989) has devised a fast algorithm to compute the orthogonal wavelet transform using wavelets defined by QMF; it has been used in particular to devise more efficient techniques for numerical analysis (Beylkin, Coifman, and Rokhlin, 1992). Then, more recently, Malvar (1990), Coifman and Meyer (1991) found a new kind of window of variable width, which allows the construction of orthogonal adaptative local cosine bases. The elementary functions of such bases are then parametrized by their position b, their scale a (width of the window), and their wavenumber k (proportional to the number of oscillations inside each window). In the same spirit, Coifman et al. (1990), Wickerhauser (1990), and Coifman, Meyer, and Wickerhauser (1992) have proposed the so called wavelet packets which, similarly to compactly supported wavelets, are wavepackets of prescribed regularity defined by discrete QMF, from which one can construct orthogonal bases. A review of the different types of wavelet transforms and their applications to analysis and computation of turbulent flows in 2D and 3D is given in Farge (1992a.b).

2. THE CONTINUOUS WAVELET TRANSFORM

The only condition a function $\psi(x) \in L^2(\Re)$, real or complex-valued, should satisfy to be called a wavelet is the admissibility condition:

$$C(\hat{\psi}) = 2\pi \int_{-\infty}^{\infty} |\psi(k)|^2 \frac{dk}{|\vec{k}|} < \infty$$
⁽¹⁾

with

$$\hat{\psi}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx.$$
⁽²⁾

If ψ is integrable, this condition implies that the wavelet has a zero mean :

$$\int_{-\infty}^{\infty} \psi(x) dx = 0 \text{ or } \hat{\psi} = 0.$$
(3)

In practice one also wishes the wavelet to be as localized as possible on both sides in Fourier transform, namely that

$$\left|\psi(x)\right| < \frac{1}{1+\left|x\right|^{n}},\tag{4}$$

and

$$|\hat{\psi}(k)| < \frac{1}{1+|k-k_0|^n},$$
(5)

with k_0 being the frequency of the wavelet and n as large as possible.

Figure 2 shows examples of the most commonly used wavelets: the Marr wavelet (Fig. 2a), also called the Mexican hat, a real-valued function used for the isotropic continuous wavelet transform, the Morlet wavelet (Fig. 2b), a complex-valued function used for the non-isotropic continuous wavelet transform, the Meyer-Lemarié wavelet (Fig. 2c), and the Daubechies wavelet (Figs. 2d,2e), real-valued functions used to build orthogonal bases.

For several applications, in particular to study fractals, one also wishes the wavelet to have a good regularity, namely that $\hat{\psi}(k)$ decays rapidly near zero or, equivalently, that the wavelet has enough cancellations such as

$$\int_{-\infty}^{\infty} \psi(x) x^n dx = 0 \tag{6}$$

with n as large as possible.

Then, after having chosen the so-called 'mother wavelet' ψ , one generates the family of wavelets $\Psi_{b,a}(x)$, by continuously translating the 'mother wavelet' ψ along the signal b and continuously dilating it to all accessible scales a, which gives

$$\Psi_{b,a}(x) = \frac{1}{N(a)} \psi\left(\frac{x-b}{a}\right) \tag{7}$$

(8)

with N(a) a normalization coefficient equal, either to $a^{1/2}$ if one wishes the squared modulus of the wavelet coefficients to correspond to an energy density (L² norm), or to a if one uses the wavelet coefficients to analyze the local regularity of the signal (L¹ norm).

The continuous wavelet analysis of the function $f(x) \in L^2(\Re)$ is then the inner product between f(x) and the set of all translated and dilated wavelets $\Psi_{b,a}(x)$, such as

$$\tilde{f}(b,a) = \int_{-\infty}^{\infty} f(x) \Psi_{b,a}^* dx,$$

where * indicates the complex conjugate.





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The wavelet transform therefore projects the $L^2(\Re)$ space of finite energy functions into the $L^2(\Re \times \Re^+)$ space of wavelet coefficients having a measure $da db/a^2$, which is the Haar measure associated to the affine group. Figure 3 shows five examples of wavelet analysis of academic signals: a Dirac spike (Fig. 3a), the superposition of two cosine functions having different frequencies (Fig. 3b), the superposition of two cosine functions of very different amplitudes (Fig. 3c), a tchirp (Fig. 3d), a Gaussian white noise (Fig. 3e), and finally a tchirp in the presence of a strong noise (Fig. 3f).

From the wavelet coefficients $\overline{f}(b,a)$, one is able to reconstruct the function f(x) using the inverse wavelet transform, defined as

$$f(x) = \frac{1}{C(\hat{\psi})} \int_{-\infty}^{\infty} \int_{0}^{\infty} \tilde{f}(b,a) \Psi_{b,a}(x) \frac{dadb}{a^{2}}$$
(9)

with

$$C(\hat{\psi}) = 2\pi \int_{-\infty}^{\infty} \left| \hat{\psi}(k) \right|^2 \frac{dk}{\left| \vec{k} \right|},$$

a finite valued coefficient given by the admissibility condition (1).

One verifies that the wavelet transform conserves energy (as the Plancherel identity for the Fourier transform), namely that

$$\int_{-\infty}^{\infty} \left| f(x) \right|^2 dx = \frac{1}{C(\hat{\psi})} \int_{-\infty}^{\infty} \int_{0}^{\infty} \left| \tilde{f}(b,a) \right|^2 \frac{dadb}{a^2}.$$
 (10)

If the function f(x) belongs to the functional space $L^2(\Re)$, and if the wavelet is regular enough and therefore well localized in Fourier space (5), the wavelet analysis may be interpreted as a pass-band filter with dk/k being constant (Fig. 1d)

$$\bar{f}(b,a) = \frac{1}{2\pi N(a)} \int_{-\infty}^{\infty} \hat{f}(k) \hat{\psi}^{*}(ak) e^{ibk} dk.$$
(11)

The extension of the continuous wavelet transform to analyze signals in *n* dimensions has been done by Murenzi (1989), considering in this case the Euclidean group with dilations. The generation of the wavelet family $\Psi_{ar,b}(x)$ is obtained by translation (vector **b**), dilation (parameter *a*) and rotation (corresponding to the operator *r* defined in \Re^n), such as

$$\Psi_{ar,\tilde{b}}(\vec{x}) = N(a)^{-n} \, \psi(a^{-1}r^{-1}(\vec{x} - \vec{b})). \tag{12}$$



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the signal

the wavelet coefficient modulus

the wavelet coefficient phase



f. A tchirp $sin(t^2)$ in presence of a strong noise

Figure 3. (continued) Wavelet transforms of several academic signals using a Morlet wavelet. [We have used the code TecLet 1D (copyright Science & Tcc.)]

For \Re^2 , r is the rotation matrix:

$$r = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$
(13)

with θ the rotation angle.

In n dimensions the admissibility condition becomes

$$C(\hat{\psi}) = (2\pi)^n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \psi(\vec{k}) \right|^2 \frac{d^n \vec{k}}{\left| \vec{k} \right|^n} < \infty.$$
(14)

The analysis and synthesis are then

$$\tilde{f}(a,r,\bar{b}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\bar{x}) \Psi_{ar,\bar{b}}^{*}(\bar{x}) d^{n}\bar{x}$$
(15)

$$f(\bar{x}) = \frac{1}{C(\hat{\psi})} \int_0^\infty \int_0^{2\pi} \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \tilde{f}(a, r, \bar{b}) \Psi_{ar, \bar{b}}(\bar{x}) \frac{da \, dr \, d^n b}{a^{n+1}}.$$
 (16)

The energy conservation still holds:

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| f(\bar{x}) \right|^2 d^n \bar{x} = \frac{1}{C(\hat{\psi})} \int_0^{\infty} \int_0^{2\pi} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| \bar{f}(a, r, \bar{b}) \right|^2 \frac{dadr \, d^n b}{a^{n+1}} \tag{17}$$

Holschneider (1988) has shown that one can reconstruct the function f(x) from its wavelet coefficients $\tilde{f}(b,a)$ by using any other function $\phi(x)$, which verifies a modified admissibility condition such as

$$\int_{-\infty}^{\infty} \hat{\psi}(k) \,\hat{\phi}^*(k) \frac{dk}{|k|} < \infty. \tag{18}$$

This, for instance, allows us to reconstruct f(x) by a simple summation of all wavelet coefficients along the verticals b = constant. This in fact corresponds to using a Dirac function as the function $\phi(x)$ to reconstruct the signal, which gives

$$f(x) = \frac{1}{C(\hat{\psi})} \int_{-\infty}^{\infty} \tilde{f}(x,a) \frac{da}{a}$$
(19)

with

$$C(\hat{\psi}) = \sqrt{2\pi} \int_{-\infty}^{\infty} \hat{\psi}(k) \frac{dk}{\left|\vec{k}\right|} < \infty.$$

3. PROPERTIES OF THE CONTINUOUS WAVELET TRANSFORM

3.1 Covariance by Translation and Dilation

One property of the continuous wavelet transform, which is lost in the case of the orthogonal wavelet transform, is its covariance, by both translation, i.e., shift by x_0

$$W[f(x-x_0)] = \tilde{f}(b-x_0,a)$$
⁽²⁰⁾

with W the continuous wavelet transform operator, and dilation, i.e., under scale changes by a factor λ

$$W\left[f(\frac{x}{\lambda})\right] = \frac{1}{\lambda} \tilde{f}\left(\frac{b-b_0}{\lambda}, \frac{a}{\lambda}\right).$$
(21)

3.2 Linearity

The continuous wavelet transform is a linear transform; therefore we have the following superposition principle:

$$W[\alpha f_1(x) + \beta f_2(x)] = \alpha \tilde{f_1}(b, a) + \beta \tilde{f_2}(b, a)$$
(22)

with a and b two arbitrary constants.

3.3 Locality in Both Space and Scale

The localization of wavelets by both position b and scale a yields both values from the wavelet coefficients. This is not the case with the Fourier coefficients because the basis functions are nonlocal: a given Fourier coefficient therefore depends on the behaviour of the whole signal. On the contrary a given wavelet coefficient $\tilde{f}(b_0, a_0)$ does not depend on the value of the signal outside the so called 'influence cone' localized in $b_0 + \Delta b / a$, with Δb depending on the support of the wavelet (Fig. 4a). Likewise the wavelet coefficients at a given scale a_0 depend only on the spectral behaviour of the signal in the bandwidth $[k_{\min}/a_0, k_{\max}/a_0]$ with k_{\min} and k_{\max} given by the support of $\hat{\psi}$ (Fig. 4b). The support of $\hat{\psi}$ is defined as the region where ψ is larger than a given value, because wavelet ψ has at least an exponential decay.

3.4 Characterization of the Local Regularity of a Function

One of the most useful properties of the wavelet transform in analyzing turbulent flows is the fact that the local scaling of the wavelet coefficients computed in L^1 norm, i.e., with the normalization N(a)=a in (7), allows us to characterize the regularity of the signal



b. the spectral band attached to wavenumber k_0

Figure 4. Locality in wavelet coefficient space.

(Holschneider 1988) and (Jaffard 1989). Thus, if $d^m f / dx^m$ exists, i.e., if f is m times continuously differentiable in x_0 , then

 $\left\|\tilde{f}(x_0,a)\right\|_{L^1} \sim a^m \tag{23}$

when a tends to 0.

If $f \in \Lambda^{\alpha}(x_0)$, the space of Lipschitz functions having exponent $-1 < \alpha < 1$, which are continuous functions non differentiable in x_0 , such that

$$f(x) - f(x_0) \le C |x - x_0|^{\alpha}$$
(24)

with constant C>0. Then

$$\tilde{f}(x_0, a) - a^{\alpha} \tag{25}$$

when a tends to 0.

Thus the behaviour of the wavelet coefficients $\tilde{f}(x_0, a)$ at x_0 in the limit $a \to 0$ measures the local regularity of the function f in x_0 , which is given by the slope of the modulus of (x_0, a) represented in log-log coordinates. For instance, the wavelet coefficients computed in norm L¹ of a function presenting a Lipschitz singularity a in x_0 will diverge in the very small scale limit (Fig. 5a), while those of a function which is regular in x_0 will tend to zero in the same limit (Fig. 5b).

4. ANALYSIS OF TWO-DIMENSIONAL TURBULENT FLOWS

"In the last decade we have experienced a conceptual shift in our view of turbulence. For flows with strong velocity shear... or other organizing characteristics, many now feel that the spectral description has inhibited fundamental progress. The next "El Dorado" lies in the mathematical understanding of coherent structures in weakly dissipative fluids: the formation, evolution and interaction of metastable vortex-like solutions of nonlinear partial differential equations..." Norman Zabusky (1984).

As Norman Zabusky stated, it is essential before modelling turbulent flows to understand the dynamical role of coherent structures and analyze their contribution to the different nonlinear interactions. Because the Fourier modes contain nonlocal information, we are unable to discriminate the role of coherent structures and we cannot separate the coherent structures from the rest of the flow. However, this local spectral analysis becomes possible



a. f is a function presenting a Lipschitz singularity α in x_0



Figure 5. Analysis of the local regularity of a function f in x_0 (given by the slope of the modulus of $\tilde{f}(x_0, a)$ represented in log-log coordinates).

when using the wavelet transform and with it we can devise new types of diagnostics. After defining them, we will apply them to analyze some vorticity fields corresponding to long-time evolution of a forced two-dimensional flow, computed with a resolution 128².

4.1 The Wavelet Coefficients

If we denote the position as b, the scale as a, and the angle as θ , the wavelet coefficients computed in LP norm are

$$\tilde{f}(a,\theta,\vec{b}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\vec{x}) \Psi_{ar,\vec{b}}^{*}(\vec{x}) d^{2}\vec{x}$$
(26)

with

$$\Psi_{ar,\bar{b}}(\bar{x}) = N(a)^{-n} \,\psi(a^{-1}r^{-1}(\bar{x}-\bar{b})), \text{ and } r = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$
(27)

If $N(a) = a^{1/2}$, the wavelet coefficients are in L² norm and the squared wavelet coefficients correspond to the local energy density of the signal at location b, scale a and direction θ . If N(a) = a, the wavelet coefficients are in L¹ norm and in this case the local scaling of the wavelet coefficients gives information on the local regularity, or the Lipschitz exponent in the case of discontinuities, of the signal at location b, scale a and direction θ .

In Figure 6 we show the 1D continuous wavelet analysis along a cut done in a twodimensional turbulent vorticity field. The wavelet coefficients are computed, either in L^2 norm (Fig. 6a), or in L¹ norm (Fig. 6b), using the Morlet wavelet with $k_0=5$.

In Figure 7 we show the 2D continuous wavelet analysis of a two-dimensional turbulent vorticity field. The wavelet coefficients are computed in L^2 norm at three different scales, namely 32 pixels (Fig. 7b), 16 pixels (Fig. 7c), and 2 pixels (Fig. 7d), using the isotropic Marr wavelet (in this case, there is no angular dependence of the wavelet coefficients resulting from to the wavelet isotropy).

4.2 The Intermittency Factor

The intermittency factor is given by the wavelet coefficients renormalized by the space averaged energy at each scale, such that

$$I(a, \vec{b}) = \frac{\left| \tilde{f}(a, \theta, \vec{b}) \right|^2}{\int_0^{2\pi} \int_{-\infty}^{\infty} \left| \tilde{f}(a, \theta, \vec{b}) \psi_{ar, \vec{b}}(\vec{x}) \right|^2 d\theta d^2 b / a^3}.$$
 (28)



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d'



Figure 7. The wavelet coefficients in L^2 norm computed using the Marr wavelet.

It gives information on the space variance of the energy spectrum, namely if $I(a, \vec{b})=1$ the field is homogeneous and there is no space variance of the energy at scale a. If $I(a, \vec{b})$ is large, the field is intermittent, namely all the energy contribution at scale a comes from a few very excited regions, while the rest of the field has little energy at this scale.

Figure 8 shows the intermittency factor computed at three different scales, namely 32 pixels (Fig. 8b), 8 pixels (Fig. 8c), and 2 pixels (Fig. 8d) using the isotropic Marr wavelet (there is no angular dependence of the wavelet coefficients resulting from the wavelet isotropy in this case).

4.3 The Local Energy Spectrum

The local energy spectrum is defined from the wavelet coefficients, such that

$$E(a,\bar{b}_{0}) = \frac{\int_{0}^{2\pi} \left| \tilde{f}(a,\theta,\bar{b}_{0}) d\theta \right|^{2}}{\mu^{2}}.$$
(29)

Figure 9 shows the local energy spectra (Fig. 9d) computed by integrating in space the Marr wavelet coefficients after segmenting the vorticity field (Fig. 9a) into three different regions using the Weiss criterium (Weiss 1981): the elliptical region corresponding to the cores of the coherent structures (Fig. 9b), the parabolic region corresponding to the shear layers at the periphery of the coherent structures (Fig. 9c), and the hyperbolic region corresponding to the vorticity filaments of the incoherent background flow. We observe that the elliptic region scales as k^{-6} , the parabolic region as k^{-4} , while the hyperbolic region scales as k^{-3} . Therefore the more coherent the region is, the steeper its spectrum, whereas an incoherent region, such as the background flow, is much more homogeneous and has a flatter spectrum—similar to noise.

5. FILTERING OF TWO-DIMENSIONAL TURBULENT FLOWS USING CONTINUOUS WAVELETS

Because the wavelet transform is invertible it is always possible to select a subset of the coefficients and reconstruct a filtered version of the field from them. We propose several filtering techniques to extract coherent structures from the background vorticity in two-dimensional turbulent flows. The first one consists of discarding all wavelet coefficients outside the influence cones (Fig. 4a) attached to the local maxima of the vorticity field that corresponds to the coherent structures' cores. The second method consists of discarding all wavelet coefficients smaller than a given threshold that depends on the quantity of enstrophy we want to retain in the filtered vorticity field.

Figure 10 shows the extraction of one coherent structure, done by filtering all wavelet coefficients outside the influence cone attached to the center of this coherent structure,



d. small scale, 2 pixels (min 0, max 44)

Figure 8. The intermittency factor computed using the Marr wavelet.



a. The complete vorticity field





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b. The elliptical region corresponding to the coherent structures c. The parabolic region corresponding to the shear layers at the periphery of the coherent structures d. The hyperbolic region corresponding to the vorticity filaments of the incoherent background flow



Figure 9. Local energy spectra computed from the wavelet coefficients after segmenting the vorticity field into three different regions.

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Figure 10. Extraction of one coherent structure, done by filtering all wavelet coefficients outside the influence cone attached to the center of this coherent structure, before computing the inverse wavelet transform.

before computing the inverse wavelet transform. We display the complete vorticity field (Fig. 10a), the coherent structure alone (Fig. 10b), the vorticity field without the coherent structure (Fig. 10c), and the energy spectra of the three previous fields (Fig. 10d).

Figure 11 shows the extraction of the 40 most excited coherent structures, done by filtering all wavelet coefficients outside the influence cones attached to the centers of these coherent structures, before computing the inverse wavelet transform. We display the complete vorticity field (Fig. 11a), the 40 coherent structures alone (Fig. 11b), the vorticity field without the coherent structures (Fig. 11c), and the energy spectra of the three previous fields (Fig. 11d).

Figure 12 shows the extraction of all excited coherent structures, done by filtering all wavelet coefficients smaller than a given threshold and then computing the inverse wavelet transform. We display the complete vorticity field (Fig. 12a), the coherent structures alone (Fig. 12b), the vorticity field without the coherent structures (Fig. 12c), and the energy spectra of the three previous fields (Fig. 12d).

As seen with the local energy spectra, these filtering techniques show again that the spectral behaviour depends on the region of the flow, with a tendency to scale around k^{-6} near the cores of the coherent structures, between k^{-4} and k^{-5} at their periphery, and around k^{-3} in the background.

6. COMPRESSION OF TWO-DIMENSIONAL TURBULENT FLOWS USING WAVELET PACKETS

Wavelet packets represent a family of orthogonal bases that unifies wavelets with Dirac. Fourier and wavepacket functions, affording increased flexibility in tiling the information plane, because now each element of the basis is parametrized independently in position b. scale a and wavenumber k (cf. Coifman et al., 1992). For a given signal sampled on Npoints the wavelet packet algorithm generates 2^N possible orthogonal bases and then selects the one that minimizes the number of coefficients having significant contributions to the total signal. In this sense, the wavelet packet algorithm defines the most efficient basis, so called the Best Basis, upon which to expand a given signal. If the flow is dominated by point vortices, then it is optimally represented using the Dirac grid point basis, and the output of the wavelet packet algorithm will reflect this. On the contrary, if the flow is dominated by waves, then it is optimally represented using the Fourier basis, and the output of the wavelet packet algorithm will again reflect this. If the flow behaviour is in between these two extreme situations, other bases will be more appropriate and the wavelet packet algorithm will give us the Best Basis in which the vorticity field can be represented with the smallest number of significant coefficients. The computation of the Best Basis for a signal sampled on N points is done in $N \log_2 N$ operations, while the reconstruction of the signal from its projection onto the Best Basis is done in Noperations.





Figure 11. Extraction of the 40 most excited coherent structures, done by filtering all wavelet coefficients outside the influence cone attached to the center of these coherent structures before computing the inverse wavelet transform.





Figure 13 shows the compression of a two-dimensional vorticity field using its wavelet packet coefficients with three different compression ratios. For a compression by 2 (Fig. 13a) we split the field into the 50% strongest wavelet packet coefficients and the 50% weakest wavelet packet coefficients. Then for a compression by 20 (Fig. 13b) we split the field into the 5% strongest wavelet packet coefficients and the 95% weakest wavelet packet coefficients, and for a compression by 200 (Fig. 13c) we split the field into the 0.5% strongest wavelet packet coefficients and the 99.5% weakest wavelet packet coefficients. For each of the three compression ratios we display the uncompressed field with its energy spectrum, the compressed field with its energy spectrum, and the discarded field with its energy spectrum. These results have been obtained in collaboration with Meyer, Pascal and Wickerhauser and are extensively discussed in Farge et al. (1992).

With these compression techniques we find as before that the spectral behaviour depends on the region of the flow we analyze, with a tendency to scale around k^{-6} near the cores of the coherent structures, around k^{-4} at their periphery, and around k^{-3} in the background.

7. CONCLUSION

Nowadays turbulence is commonly viewed from one of two alternative perspectives, depending upon which side of the Fourier transform one looks from. In physical space, we observe coherent vortices and wonder if there is universality in their structure and interactions. In Fourier space, we see transfers of energy and enstrophy between different scales of motion and ask, for example, if the slope of the energy spectrum is universal. The selection of bases in which turbulence may be examined must be extended if these perspectives are to be effectively reconciled. Through the use of wavelets and wavelet packets, we have constructed a class of bases, which includes grid point and Fourier representations as special cases, from which we select the basis which is optimal for a given flow, namely the one which compresses the most the information while keeping track of the behaviour of the flow in both space and scale.

With such a wavelet or wavelet packet representation we can compute a local energy spectrum. Using the continuous wavelet transform, we have shown that different regions of the flow present different slopes for the local energy spectrum. Clearly the Fourier transform is unable to detect these different spectral behaviours which vary in space, while the wavelet transform is here the appropriate tool. Typically we have observed that the cores of the coherent structures, which correspond to the elliptic regions, scale as k^{-6} , the shear layers around the coherent structures, which correspond to the parabolic regions, scale as k^{-4} , while the vorticity filaments in the background, which correspond to the hyperbolic regions, scale as k^{-3} . From this result we infer that the variation of the Fourier spectral slope we commonly observe for two-dimensional flows may be related to the density of coherent structures which varies depending on the initial conditions and



Figure 13a. Compression of a two-dimensional vorticity field using its wavelet packet coefficients, compression by a factor 2; (top) the uncompressed field and its energy spectrum, (center left and lower left) the compressed field and its energy spectrum, (center right and lower right) the discarded field and its energy spectrum. The visualisation was done in collaboration with Jean-Francois Colonna.



Figure 13b. Compression of a two-dimensional vorticity field using its wavelet packet coefficients, compression by a factor 20; (top) the uncompressed field and its energy spectrum, (center left and lower left) the compressed field and its energy spectrum, (center right and lower right) the discarded field and its energy spectrum. The visualisation was done in collaboration with Jean-Francois Colonna.

The uncompressed vorticity field and its Fourier spectrum

The vorticity field reconstructed from the 0.5 % strongest wavelet packet coefficients



The vorticity field reconstructed from the 99.5 % weakest wavelet packet coefficients



Figure 13c. Compression of a two-dimensional vorticity field using its wavelet packet coefficients, compression by a factor 200; (top) the uncompressed field and its energy spectrum, (center left and lower left) the compressed field and its energy spectrum, (center right and lower right) the discarded field and its energy spectrum. The visualisation was done in collaboration with Jean-Francois Colonna.

on the forcing. If this is true we may hope that the local scaling of the different regions may be universal enough in order to be able to model their behaviour, each region then having its own parametrization.

Using the orthogonal wavelet packet transform, we have shown that the significant coefficients correspond to the coherent structures, while the weak coefficients correspond to the vorticity filaments which are only passively advected by the coherent structures. One possible application of the wavelet packet algorithm is to apply it from time to time during a numerical simulation, in order to separate regions with highly active small scales, which need a better grid resolution, from regions with inactive small scales, which do not contribute much to the dynamics and can either be discarded or modelled. Indeed the wavelet packet Best Basis seems to distinguish the low-dimensional, dynamically active part of the flow from the high-dimensional, passive components. It gives us some hope of drastically reducing the number of degrees of freedom necessary to the computation of two-dimensional turbulent flows.

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