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Wavelet forcing for numerical simulation of two-dimensional turbulence

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Abstract. This note presents a new method to force turbulent flows computed by numerical simulations. The forcing term is defined in an inhomogeneous way, because it depends nonlinearly on the wavelet coefficients of the vorticity field. It injects energy and enstrophy as locally as possible in both physical and spectral spaces. This new forcing is defined in wavelet space in order to control the smoothness of the vortices thus excited and to model the local production of vortices by instabilities observed in turbulent flows.

Keywords: wavelet / two-dimensional turbulence / forcing

Forçage en ondelettes pour la simulation de la turbulence bidimensionnelle

Résumé. *Cette note présente une nouvelle méthode de forçage des écoulements turbulents calculés par simulation numérique. Le terme de forçage est défini de façon inhomogène car il dépend non linéairement de la valeur des coefficients d'ondelettes du champ de vorticité. Il injecte l'énergie et l'enstrophie aussi localement que possible, à la fois dans l'espace physique et dans l'espace spectral. Cette nouvelle façon d'effectuer le forçage est définie dans l'espace des coefficients d'ondelettes afin de pouvoir contrôler la régularité des tourbillons ainsi produits. Cela permet de modéliser la production locale de tourbillons par des instabilités que l'on observe dans les écoulements turbulents.*

Mots clés : *ondelette / turbulence bidimensionnelle / forçage*

Version française abrégée

Les forçages couramment utilisés pour la simulation numérique directe des écoulements turbulents bidimensionnels sont effectués à partir des coefficients de Fourier, c'est-à-dire de façon non locale dans l'espace physique (Basdevant *et al.*, 1981). Si l'on définit le forçage à partir des points de grille, on récupère la localisation mais on crée alors des discontinuités dans le champ ainsi forçé. Nous cherchons

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à mettre au point une méthode de forçage plus physique, permettant de simuler la formation des tourbillons produits localement par différentes instabilités (par exemple l'instabilité barocline ou l'instabilité de Kelvin–Helmholtz). Nous appliquons cette méthode au cas des écoulements bidimensionnels, mais elle peut également être utilisée pour les écoulements tridimensionnels.

Nous avons montré (Farge et Rabreau, 1988 ; Farge, 1992 ; Farge *et al.*, 1992, 1996) que les structures cohérentes correspondent aux coefficients d'ondelettes du champ de vorticité les plus forts, tandis que l'écoulement résiduel correspond aux coefficients faibles restants. Nous avons également observé (Farge *et al.*, 1992) que la dynamique non linéaire est préservée si l'on ne calcule que les coefficients d'ondelettes les plus forts, ce qui conduit à une réduction sensible du nombre de degrés de liberté nécessaires pour calculer l'évolution de l'écoulement. Cela nous a permis (Fröhlich et Schneider, 1996) de mettre au point un nouveau schéma d'intégration numérique des équations de Navier–Stokes en base d'ondelettes adaptives, pour lequel à chaque pas de temps on ne calcule que les coefficients d'ondelettes les plus forts. Le forçage que nous proposons ici consiste à amplifier les coefficients d'ondelettes les plus forts, afin de renforcer les tourbillons créés par la dynamique non linéaire de l'écoulement, cela de façon à la fois locale dans l'espace physique et aussi regulière que possible, c'est-à-dire sans créer de discontinuités dans le champ de vorticité, ce qui serait le cas si le forçage était effectué en points de grille.

Nous partons des équations de Navier–Stokes écrites en vitesse et vorticité, avec un terme de forçage, un terme de friction de Rayleigh et des conditions aux limites périodiques [éq. (1)]. Nous les intégrons à l'aide d'une méthode pseudo-spectrale de type Fourier et d'un schéma temporel semi-implicite du second ordre. A chaque pas de temps, on calcule la transformée en ondelettes du champ de vorticité, puis on annule les coefficients les plus faibles, ce qui simule la dissipation d'enstrophie dans l'écoulement résiduel, et l'on amplifie les coefficients les plus forts, ce qui simule l'excitation des tourbillons par instabilité non linéaire. Cette procédure force donc le champ de vorticité de façon inhomogène en ne renforçant que les structures cohérentes déjà formées. On ajuste la quantité d'enstrophie injectée par le forçage et la quantité d'énergie dissipée par la friction de Rayleigh afin d'obtenir un état de régime statistiquement stationnaire.

Nous constatons que l'énergie et l'enstrophie restent constantes pendant plus de 60 temps de retournement (*fig. 1*). Nous observons que les tourbillons gardent une forme quasi circulaire, car l'enstrophie injectée par le forçage leur permet de mieux résister à la déformation imposée par les tourbillons voisins (*fig. 2*). Pendant toute la durée de l'intégration les spectres d'énergie (*fig. 3*) ne changent pas et présentent une pente en k^{-6}, plus raide que celle en k^{-3} prédite par la théorie statistique de la turbulence homogène et isotrope (Kraichnan, 1967). Cet écart est également observé pour des forçages effectués en Fourier (Basdevant *et al.*, 1981), ce qui confirme le fait qu'il n'y a pas universalité de la pente spectrale en turbulence bidimensionnelle. On constate également qu'il n'y a pas de cascade inverse d'énergie, car le maximum d'énergie reste localisé autour du nombre d'onde $k_I = 4$. Les coefficients d'ondelettes de la vorticité mettent en évidence une forte intermittence de l'écoulement, car le support spatial des coefficients les plus forts se réduit très sensiblement avec l'échelle (*fig. 4*). Nous constatons également que la fonction de distribution de probabilité (PDF) de la vorticité (*fig. 5*) est gaussienne pour les valeurs faibles, correspondant à l'écoulement résiduel, et non gaussienne pour les valeurs fortes, correspondant aux tourbillons.

L'expérience numérique présentée ici a été faite en renforçant les tourbillons préalablement créés par la dynamique non linéaire de l'écoulement. Nous allons poursuivre cette étude en utilisant la même méthode de forçage en ondelettes pour créer cette fois-ci de nouveaux tourbillons, afin de simuler la production de tourbillons par instabilité, telle l'instabilité de Kelvin–Helmholtz, dans les régions de l'écoulement résiduel où la vorticité l'emporte sur la déformation.

1. Introduction

The direct numerical simulations of two-dimensional turbulent flows are used to study large-scale atmospheric and oceanic dynamics. In this context the forcing term is designed to model the baroclinic instability, which is supposed to maintain the motion of such large-scale geophysical flows. Although in this note we will consider only incompressible two-dimensional turbulent flows, the forcing method we present here could also be extended to three-dimensional turbulent flows.

Classically two forcing schemes are used (Basdevant *et al.*, 1981), which both operate in Fourier space: either a *negative dissipation* within a given wavenumber band, with a complex amplification coefficient that depends on the wavenumber, or a *white or coloured noise* in time with a prescribed isotropic spectral distribution, strongly peaked in the vicinity of a given wavenumber, with random phases. For both schemes the choice of the wavenumber band represents that part of the energy spectrum where the baroclinic instability has a significant growth rate. Neither of the two schemes is a satisfactory model of baroclinic instability (Basdevant *et al.*, 1981) because they inject energy and enstrophy locally in Fourier space and therefore non-locally in physical space. Another drawback of such a forcing is that the scale of the coherent vortices produced by the nonlinear dynamics of the flow is imposed by the scale at which the forcing is carried out. Our aim is to design a forcing scheme able to excite vortices locally in physical space and as smoothly as possible in order to avoid creating any unphysical discontinuities in the vorticity field.

We have shown (Farge and Rabreau, 1988; Farge, 1992; Farge *et al.*, 1992; Farge *et al.*, 1996) that vortices produced in two-dimensional turbulent flows correspond to the strongest wavelet coefficients of the vorticity field while the remaining weaker coefficients correspond to the residual background flow. We have also observed (Farge *et al.*, 1992) that the nonlinear dynamics is preserved if one only computes the strongest wavelet coefficients, which leads to an important reduction in the number of degrees of freedom necessary to compute the flow evolution. This has allowed us (Fröhlich and Schneider, 1996) to design a new numerical scheme to integrate Navier–Stokes equations in an adaptive wavelet basis, by only computing the strongest wavelet coefficients at each time step. Therefore the forcing scheme we propose injects enstrophy only into the strongest wavelet coefficients, hence in an inhomogeneous way, in order to excite the vortices without affecting the background flow. This procedure does not interfere with the emergence of vortices and does not impose on them a given scale, in contrast to the Fourier forcing; the distribution and size of the vortices depend only on the intrinsic nonlinear dynamics of the flow.

2. Governing equations and pseudo-spectral method

To numerically simulate non-decaying two-dimensional turbulence we consider the Navier–Stokes equations written in velocity–vorticity form with a forcing term F. Furthermore, we include an artificial dissipative term $\lambda\Psi$, a so-called Rayleigh friction (Basdevant *et al.*, 1981), to provide an energy sink at large scales. This is necessary because the energy injected by external forcing tends to accumulate in the large scales owing to the inverse energy cascade characteristic for two-dimensional turbulent flows (Kraichnan, 1967) and should therefore be dissipated there to reach a statistically stationary regime.

The governing equations are:

$$\partial_t \omega + \mathbf{v} \cdot \nabla\omega = \nu \nabla^2 \omega + \lambda\Psi + F \quad \text{and} \quad \nabla \cdot \mathbf{v} = 0 \tag{1}$$

with the velocity field $\mathbf{v} = (u,v)$, the vorticity $\omega = \nabla \times \mathbf{v}$ and the stream function $\Psi = \nabla^{-2} \omega$. Further parameters are the kinematic viscosity ν and the strength of the friction term λ. We assume periodic boundary conditions in both directions, *i.e.* our domain is the two-dimensional flat torus $\mathbb{T}^2$ with $\mathbb{T} = 2\pi\mathbb{R}/\mathbb{Z}$. This setting has been chosen in order to simulate turbulent flows far from the wall regions, which avoids the treatment of boundary layers. For the numerical solution of the system (1) we employ a classical Fourier pseudo-spectral method. Therefore the vorticity field and the other variables are represented as Fourier series: $\omega(\mathbf{x}) = \sum_{|\mathbf{k}| \leq N/2} \widehat{\omega}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}$, with $\mathbf{x} = (x,y)$ and $\mathbf{k} = (k_x, k_y)$. For the time discretization we use finite differences with a semi-implit scheme (Euler-backwards for the viscous term and Adams–Bashforth extrapolation for the nonlinear term, both of second order).

3. Wavelet compressed pseudo-spectral method

The wavelet compressed pseudo-spectral method is a classical spectral method supplemented by an additional wavelet transform performed at each time step. We employ a two-dimensional multiresolution analysis (MRA) (Farge, 1992) and develop ω^n at time step n as an orthonormal wavelet series from the largest scale $l_{\max} = 2^0$ to the smallest scale $l_{\min} = 2^J$:

$$\omega^n(x,y) = c^n_{0,0,0}\, \phi_{0,0,0}(x,y) + \sum_{j=0}^{J-1} \sum_{i_x=0}^{2^j-1} \sum_{i_y=0}^{2^j-1} \sum_{\mu=1}^{3} d^{\mu,n}_{j,i_x,i_y}\, \psi^{\mu}_{j,i_x,i_y}(x,y) \tag{2}$$

with $\quad \phi_{j,i_x,i_y}(x,y) = \phi_{j,i_x}(x)\, \phi_{j,i_y}(y)$, and

$$\psi^{\mu}_{j,i_x,i_y}(x,y) = \begin{cases} \psi_{j,i_x}(x)\, \phi_{j,i_y}(y) & \mu = 1 \\ \phi_{j,i_x}(x)\, \psi_{j,i_y}(y) & \mu = 2 \\ \psi_{j,i_x}(x)\, \psi_{j,i_y}(y) & \mu = 3 \end{cases} \tag{3}$$

where $\phi_{j,i}$ and $\psi_{j,i}$ are the 2π-periodic one-dimensional scaling function and the corresponding wavelet, respectively. Owing to the orthogonality the coefficients are given by $c^n_{0,0,0} = \langle \omega^n, \phi_{0,0,0} \rangle$ and $d^{\mu,n}_{j,i_x,i_y} = \langle \omega^n, \psi^{\mu}_{j,i_x,i_y} \rangle$ where $\langle \, . \, , . \, \rangle$ denotes the inner product.

At each time step a nonlinear wavelet thresholding technique is used, *i.e.* wavelet coefficients $d^{\mu,n}_{j,i_x,i_y}$ with absolute value below a given threshold ϵ are set to zero. In Farge *et al.* (1992) such a compression of the vorticity field (by means of wavelet packets) was applied to the initial state. This work has shown that wavelets and wavelet packets are appropriate tools to separate the coherent structures of the flow from the background components. It has also demonstrated that the coherent structures are the active components of the flow, responsible for its nonlinear dynamics, and that the vorticity filaments of the background flow are just passively advected by the velocity field resulting from the distribution of all coherent vortices.

4. Wavelet forcing

One classical way to force the flow is to inject energy and enstrophy at a fixed wave number k_I, therefore locally in Fourier space. For this, one or a couple of Fourier modes of the vorticity field are maintained constant, *i.e.* $\widehat{F}_k = constant$ in a narrow wavenumber band around $k = k_I$. The forcing term F has a narrow-band spectrum and is therefore non-local in physical space. This forcing mechanism is neither intrinsically related to the flow's chaotic dynamics, nor simulates the production of enstrophy in shear layers, which is local in physical space and therefore broad-band in Fourier space.

Here we apply a new nonlinear wavelet-based forcing approach, which is triggered directly by the intrinsic nonlinear dynamics of the flow. We use a Fourier pseudo-spectral method to solve equation (1) with an additional wavelet compression. An analysis is performed at each time step as decribed in the previous section. The forcing term F at time step $n+1$ is then defined as a function of ω at time step n:

$$F^{n+1}(x,y) = C \sum_{J_0<j<J_1} \sum_{k_x=0}^{2^j-1} \sum_{k_y=0}^{2^j-1} \sum_{\mu=1,2,3} \langle \omega^n(x,y), \psi^{\mu}_{j,k_x,k_y} \rangle \, \psi^{\mu}_{j,k_x,k_y}(x,y) \tag{4}$$

with $0 \le J_0 \le J_1 \le J$, where J denotes the finest scale in the simulation, $C>0$ and $|\langle \omega^n(x,y), \psi^{\mu}_{j,k_x,k_y} \rangle| > \epsilon$. The scale parameters J_0 and J_1 define the scale range of the forcing. The restriction to wavelet coefficients above a given threshold implies that only the dynamically active part of the flow, *i.e.* the coherent structures (Farge, 1992; Farge *et al.*, 1996), is forced. The constants C and λ, responsible for the strength of the forcing and the Rayleigh friction, respectively, are adjusted in such a way that we obtain a statistically stationary state.

5. Numerical results and discussion

The resolution is 128^2 corresponding to $J=7$, the viscosity $\nu = 2 \cdot 10^{-3}\,\mathrm{m}^2\,\mathrm{s}^{-1}$ and $\Delta t = 10^{-3}\,\mathrm{s}$. The forcing scale range lies between $J_0 = 3$ and $J_1 = 6$ with $C = 2.5 \cdot 10^{-1}\,\mathrm{s}^{-2}$ and $\lambda = 1.0\,\mathrm{m}^{-2}\,\mathrm{s}^{-1}$. For the wavelet decomposition we use cubic spline wavelets of Battle–Lemarié type. The wavelet compression is realized by cancelling all wavelet coefficients smaller than the threshold $\epsilon = 10^{-4}\,\mathrm{s}^{-1}$.

Figure 1 shows that both energy and enstrophy are kept steady during more than 60 eddy turn over times. *Figure 2* displays the vorticity field in a stationary regime at $t = 0$, 6 and 12 s; namely neither the energy spectrum (*fig. 3*) nor the PDF of vorticity (*fig. 5*) change significantly in time. The vortices

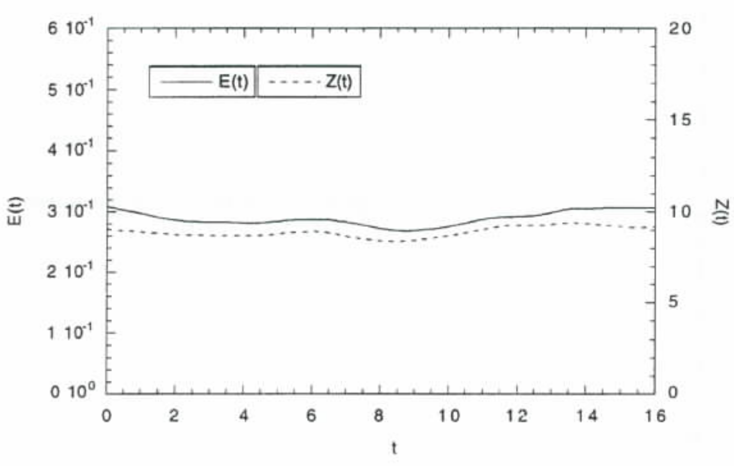


Fig. 1. – Evolution of energy E and enstrophy Z.

Fig. 1. – Évolution temporelle de l'énergie E *et de l'enstrophie* Z.

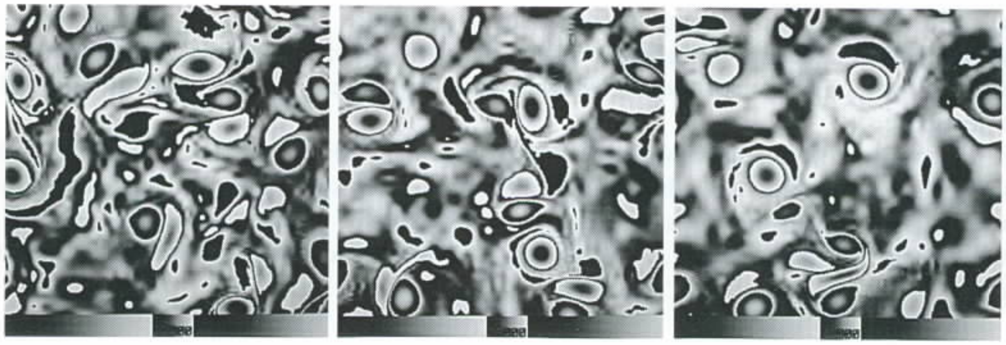
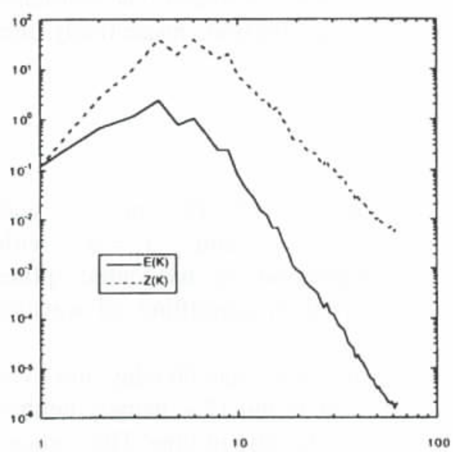
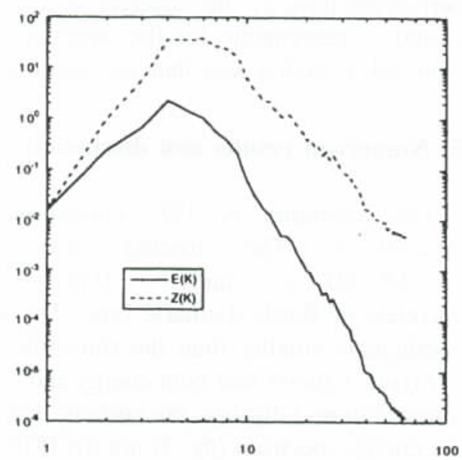


Fig. 2. – Vorticity at $t = 0$, 6 and 12 s.

Fig. 2. – Vorticité à t = *0, 6 et 12 s.*





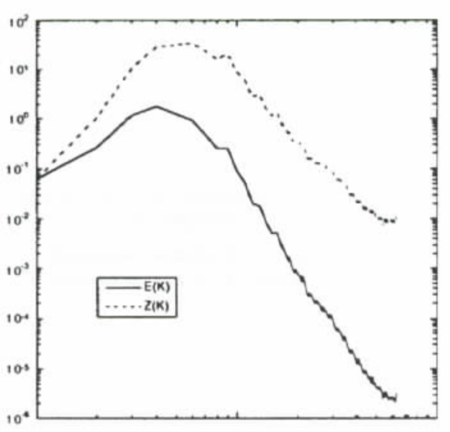


Fig. 3. – Energy $E(k)$ and enstrophy $Z(k)$ spectra at $t = 0$, 6 and 12 s.

Fig. 3. – Spectres d'énergie E(k) *et d'enstrophie* Z(k) *à* t = *0, 6 et 12 s.*

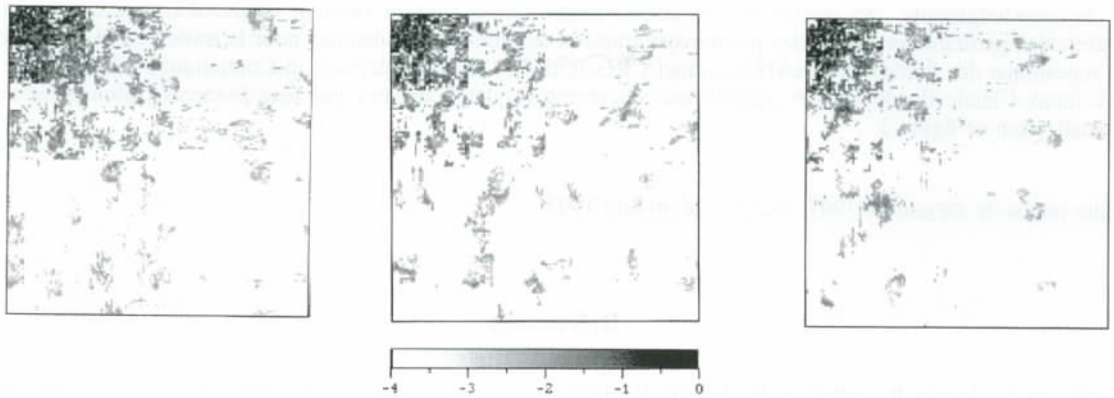


Fig. 4. – Active wavelet coefficients at t = 0, 6 and 12 s.

Fig. 4. – Coefficients d'ondelettes actifs à t = *0, 6 et 12 s.*

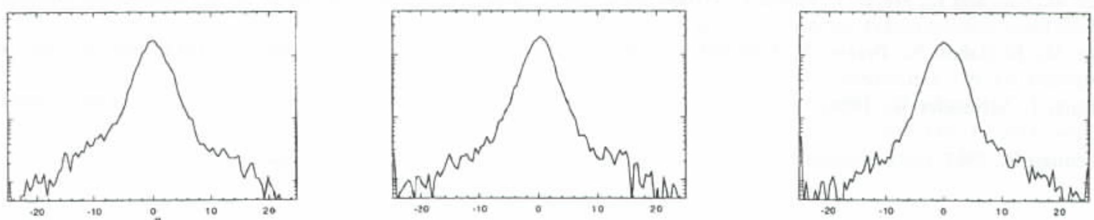


Fig. 5. – Probability distribution function of vorticity at t = 0, 6 and 12 s.

Fig. 5. – Fonction de distribution de probabilité de la vorticité à t = *0, 6 et 12 s.*

present in the initial condition become more circular and well isolated during the flow evolution, because they are better able to stand mutual strain owing to the additional enstrophy injected into them. We observe that the slopes of the spectra (see *fig. 3*) are much steeper (close to k^{-6}) than the k^{-3} law predicted by the statistical theory of homogeneous turbulence (Kraichnan, 1967). This discrepancy, as observed for other types of forcing (Basdevant *et al.*, 1981), confirms the fact that the spectral behaviour of two-dimensional turbulent flows is not universal and depends on the forcing. We also observe that there is no inverse energy cascade, because the maximum of energy remains localized around wavenumber $k_f = 4$. This is different from what is predicted by the statistical theory of homogeneous turbulence (Kraichnan, 1967). In *figure 4* we observe that the spatial support of the active wavelet coefficients decreases with the scale, which reveals a strong intermittency of the flow. Consequently the vorticity field is efficiently compressed in a wavelet basis, because only about 20% of the 128^2 coefficients are needed to represent the flow dynamics. We also show that the PDF of vorticity (*fig. 5*) is Gaussian for the weak values, corresponding to the background flow, and presents non-Gaussian tails for the strong values, corresponding to the vortices.

In the work presented here, we only excite the vortices produced by the flow's nonlinear dynamics. We can also use the same wavelet forcing to create new vortices by injecting enstrophy locally in the regions of the backgound flow where the strain becomes weaker than the vorticity, in order to simulate the formation of new vortices by instabilities, such as Kelvin–Helmholtz instability. We will perform further numerical experiments to explore this new kind of wavelet forcing.

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