A Philosophical and Historical Journey through Mixing and Fully-developed Turbulence

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Ten years ago, the first Advanced Institute held in Cargèse considered the topic of *"Mixing and Disorder"* and gathered physicists, mechanicians and chemical engineers to examine common features in the manifestations and applications of mixing. A similar project motivated the organisation of a new Institute in view of recent developments, some of which were induced by the first meeting. Some subjects which were very active at the time of the first meeting, such as hydrodynamic dispersion in porous media (soils, catalytic beds), have since matured enough to be excluded from the new Institute. Other, such as chaotic advection, have emerged. The recent developments emphasize the role of dynamical features, such as coherent vortices, in promoting mixing. We acknowledge modernity of the new treatments of turbulence and mixing, but at the same time we recognize their permanence in philosophy, science and technology from the early times of humanity.

The philosophical and historical reflexions we present in this paper are necessarily biased by our background as physicists and our lack of philosophical and historical expertise. What we attempt here is to look at our own research in physics from different perspectives, by changing the point of view from operational and technical, which is the standard style of scientific papers, towards critical and reflective aimed at clarifying the essential questions we should address to improve our understanding of mixing and turbulence. Clarifying the questions we address is an essential part of research, because questions are our guidelines while techniques are merely the tools used to find the answers. Nowadays techniques evolve on shorter time scale than fundamental questions and therefore there is a risk that research will be limited to technical improvements if we forget the underlying questions. It often happens that the right questions were asked a long time ago, but could not be answered at that time, due to lack of technical tools (lack of computers and efficient numerical methods for instance). This is why one has to dig in history to recover the thread of thought to guide our present research.

Our paper is organized in three parts: the first, written by Etienne Guyon, is on mixing, the second and the third, written by Marie Farge, are on fully-developed turbulence. We will restrict ourselves to incompressible Newtonian fluids and fully-developed (i.e. highly nonlinear) turbulent flows, without addressing the question of the transition towards a turbulent state.

1. Mixing

1.1 Philosophical journey

Mixing is present everywhere. It is observed in a large number of natural phenomena involving fluid as well as solid granular substances. It has been used throughout the history of mankind in a wide range of applications. Moreover, it often takes place at the scale of direct observation. Thus we should not be surprised that it has been studied in science and philosophy from the beginning of humanity. In the classical Greek philosophy in particular, it is found as a subject of direct investigation as well as metaphorically. On the one hand Democritus and Epicurus introduced atomism and the notion of random motion of atoms leading to mixing and separation. On the other hand Plato, Aristotle and their followers made constant reference to the pure and the mixed in the reflections on the self as well as on the social and the political. Rather than following an historical approach, which would lead us finally to contemporary philosophers, let us consider a few basic concepts related to mixing which have been analysed throughout the twenty five centuries of philosophy. Mixing is often described in terms of oppositions such as: pure/impure, mix/disperse, near/away, simple (*stoicheia*)/complex, elemental/composite or reversible/irreversible. In doing so, we follow Heraclitus who claimed, starting from similar pairs of opposites, that the association of the contraries is a necessary condition in order to reach the unique. This dialectics has been the object of many philosophical studies which we will briefly review.

The dialectics of *'Pure and impure'* has been recently analysed by the contemporary philosopher Vladimir Jankélévitch in a beautifully written essay. One of the basic questions of philosophy is indeed that of unity and purity, which leads to the definition of the self. Purification is the result of distillation, of an elimination of foreign elements. But this operation repeated to the extreme makes it impossible to define the pure which would have lost all its added ingredients. In some philosophies the mere reference to the pure is enough to render it impure. Pure is for Plato the essence of matter. In the quest for an organized universe it is of the same perfect nature as the classical polyhedron shapes and it relates to rationality. The impure and composite is the irrational and thus depreciated. The symbolism of water as described by G. Bachelard in *'L'eau et les rêves'* associates water with symbols of purification. However, water is seldom pure. It accepts many elements by dissolution. For the stoicians, a single drop of wine in the ocean makes it alcoholized. We also know that dust particles or material heterogeneities are necessary to mark and follow the flow of water. We are indeed condemned to a mixed life. We make full use of impurities throughout our life to characterize, by contrast, what could be pure, just as the crystallographer uses the topology of defects to characterize the symmetry group of a perfect crystal. Life in society itself is impure. In Plato's *'Politics'*, couple and family are compared to the interwoven threads (another form of mixed texture) which associate dissimilar beings in a permanent combination. This is further applied to threads of social life, which combine and hold contraries. There is a clear need to go beyond these contradictions. Rather than trying to define impure and composite structures starting from association of pure elements, it is more realistic to start from impure and see purity as an ultimate renunciation to impure.

Dual approaches tend to oppose and discriminate: good and bad, black and white. The separation of the wheat from the chaff or that of the saved from the damned in the last judgement leaves little chance for an intermediate permanent state. Many natural processes operate similar discriminations: oil separates naturally from vinegar, active transport in biological membranes leads to an increase of ion concentration gradients. Granular flows of large and small grains paradoxically also lead to segregation. In all the above dual operations, we obtain purity by separation. However, most of the time we are faced with a plural approach in which gradations are present on a multiplicity of scales. Part of the popular success of fractal concepts comes clearly from the fact that it has filled a gap in our unconscious perception of the infinite obtained by iteration over an infinite range of scales. "Multiple" accepts impure and diversity through this pluralist view. Complexity which has recently become fashionable recently aims at a description of composite multiple-scale structures. It has a long tradition in philosophy in its attempt of a definition of the pure and unique in the world of the multiple.

Philosophers, as well as scientists, search for a unifying mechanism which would hold together the diverse elements of a whole. Bergson often uses the example of an orange colour. It should be considered as pure because our perception of it is pure despite the fact that it could have been obtained by amalgamation of several colours. Thus, from this perceptual point of view, the decomposition of orange into elementary colours is artificial. The unity of our being, even that of body and soul, is to be found throughout the diversity of its elements. In the science of composite materials in particular, we are aiming for such global descriptions which lead to a new kind of purity, that of homogeneized structures. The mixture of dielectric grains is replaced by an equivalent and homogeneous composite from the early works of Clausius and Mossotti in the mid 18th century. Another example of homogeneization is given by Albert Einstein's calculation of the viscosity of a dilute suspension which gives it the status of a pure liquid slightly more viscous than the suspending one. We propose to employ the term "small disorder" to describe mixtures to which homogeneization treatments can be applied. Fractal structures which could only be homogeneized at the upper scale of similarity correspond to a situation of "large disorder".

We should not, however, overlook the scale of the original heterogeneities which have been averaged out in the homogeneisation process. They are essential, in particular when we deal with chemical reaction. For stoicians, the distinction between *fusion* and *juxtaposition* was as crucial as it is today for a physical chemist. The operation of melting is a way to discriminate between the two kinds of composite structures. In the melting of an alloy, the temperature varies continuously during heating. On the other hand, in a composite material temperature shows steps corresponding to the melting of the individual grains of the various phases. The original nature of the elements has been lost forever. In such a juxtaposition process, the individual elements can be separated again in principle. The stoician view is not far from the distinction, amply discussed throughout the Cargèse school, between *macromixing*, obtained in stirring, and *micromixing* which is the ultimate step resulting from diffusion and which is necessary for a chemical reaction to take place. Between fusion and juxtaposition so defined, a third type of mixing is introduced, *true mixing* (*krasis*), that of a drop of wine in the ocean, in which the elements are coextensive and present down to the smallest scale while retaining their identity and being, in principle, separable again. For the stoicians, this last kind of mixing is that of the body and the soul.

The pre-scientific (from the atomic physicists to Descartes and Newton) images of reactivity resulting from mixing and close associations of elements are varied: mixing is seen via images of branching, fingering, interweaving, with only loose connection with observations. In this view, the reaction occurs through attachments of elements induced by their various shapes, local glue when elements are in contact or, again, by involving forces acting at a distance. Only at very small distance - the approach being achieved by diffusion - will the reaction occur. This is clearly not sufficient for proper understanding of reactions, as chemical reactions involve definite ratios of the constituants. In his study of *'The Mixed'*, Pierre Duhem emphasizes the opposition between this geometrical view of the reaction and that proposed by the peripathetician school of Aristotle, which envisions the product of the reaction as an entity distinct from the elements which have been involved in the process. One crucial element will decide in favor of this view. When Lavoisier writes *'Water is composed of oxygen and of an inflammable gas in proportion of 83 parts to 15'*, he illustrates by this particular example the law of definite proportions. The product of the reaction is a new entity with a rigorously defined composition. There is also clearly a limiting solubility in the association of sugar and water, but it is a function of temperature and of the presence of other elements. One should, however, moderate the victory of Aristotle over Epicurus as there exist reactions or associations controled by shapes, as in biochemistry, or weak glue as in colloids. In the various modes of reaction, the common element is the necessity of a close approach of reacting elements which is usually achieved by molecular diffusion. The relative efficiency of the processes of this close approach and of the reaction itself is measured in terms of a Damkohler number which is the ratio of transport time to reaction time. The atomist philosophers anticipated this fact through their description of the continuous agitation of molecules. This is illustrated in this text taken from the fundamental questions in physics posed by Epicurus to Herodote: *'Atoms move continuously for all eternity. They collide and, further, separate from each other. Conversely, other atoms are set into vibration as soon as they happen to be bound by interweaving or when they are surrounded by other atoms with which they bind'.* The description clearly encompasses the main ideas of the kinetic theory of gases and has also been used to describe turbulence. In addition it includes the concept of reaction resulting from the close approach of elements.

We have just been invited by Epicurus to take the dynamic view of mixing. The question of the processes taking place during mixing and, more generally, of its dynamics must be raised at this point, as, up to now, we have only had momentary views of the composite obtained in the mixing process. Indeed, mixing closely associates spatial and temporal descriptions, introduced by the philosophers dealing with atomism. Democritus also discusses the random motion of individual atoms which approach, separate and react. This kinetic theoretical view is developed in Lucretius' *'De Natura Rerum'* and brings in the notion of chance in the famous image of the dust particles moving permanently as seen in a ray of light with the unpredictable pattern of the individual particle. There is necessity in the global behaviour - once again homogenized as measured by the thermodynamic temperature or pressure. These philosophers make use of a similar atomistic description in order to describe the unpredictable turbulent motions such as those that could be observed by Lucretius in the flow of the Arno river or in the fumes above mount Etna. The description makes much use of the multiple turbulent structures (*turba* means crowd) undergoing irregular motions as those of the atomic grains mentioned before. The *clinamen* describes small variations of a deterministic regular flow, which we call laminar, and which is amplified by turbulence. It is clearly the ancestor of the sensitivity to initial conditions, and the amplification of the initial differential deviation speaks to us as a Lyapounouv exponent. The concept of bifurcation embedded in this description appears throughout the philosophical studies of free choice. In the medieval scholastics, the philosophers Abelard and Buridan showed how autonomy and individual freedom results from the repeated and necessary free choice. The mule of Buridan who is equally hungry and thirsty is at an equal distance from buckets filled with water and grain. Will it starve to death? Clearly no if the drive is strong enough. In the scientific view of broken symmetry, like that found in the buckling of a compressed bar above a certain threshold pressing force, the initial symmetry will be lost and - again as a result of the sensitivity to initial conditions - the bar will buckle in one of the equally possible deformed shapes. The mule will also choose if it is in sufficient need and is likely to survive! In doing so, it will express its free will.

1.2 Historical journey

An understanding of the multiplicity of turbulent scales and their influence on the turbulent drag was already present in this quotation of Saint Venant in 1851: *'If Newton's hypothesis as reproduced by Navier and Poisson, which consists in taking internal friction proportional to the velocity of sliding filaments, can be applied to different points of the same fluid section, all known facts lead us to infer that the coefficient of proportionality must increase with the transverse dimension of the section; this can be explained, up to a certain point, by noting that the filaments do not move parallel to each other and that ruptures, eddies and other forms of complicated or oblique movements must greatly influence the value of friction and that they form and develop more in large sections!'* The notion of scale similarity is introduced in the text as well as the need for the introduction of variable drag coefficient which will lead T. Von Karman to define many years later turbulence as the *'science of variable constants'* /Boussinesq 1877/. In his *'Théorie des eaux courantes'*, from which Saint Venant's quotation has been extracted, Bousssinesq defines an eddy viscosity τ from the ratio of wall stress ϵ to the average velocity gradient across the wall: $\tau = \epsilon U/y$.

The study of mixing of a passive scalar became, due to Osborne Reynolds, intimately related to that of turbulence and also (with some precautions, particularly when the Prandtl number is large compared with unity) as a marker to the flow. Reynolds expresses his interest in mixing by writing : *'The general idea of mixing is so familiar to us that the vast generalization to which these ideas afford the key remain unnoticed'*. His famous experiment of the onset of turbulence in a tube makes use of a narrow filament of injected dye which visualizes flow lines and whose spreading is used as an indicator of turbulence. In an evening talk at the Royal Society with the title : *'The study of fluid motion by means of coloured bands'*, Reynolds generalizes the use of filaments of dyes made in the Poiseuille flow experiment. His work anticipated the many turbulent and chaotic flow visualizations presented during this meeting. Reynolds introduced a statistical treatment of turbulence by separating from the mean flow U_o a fluctuating component U_{fl}. In the classical case of a two-dimensional flow parallel to a plane perpendicular to y this separation is: $U(y) = U_0(y)+U_{fl}$ with $U_{fl} = \{u, v, 0\}$ and the mean of U_{fl} being zero. The exchange of momentum discussed by Saint Venant is made explicit here as the convective flux of the fluctuating momentum ρu is carried convectively with the velocity component v. Thus, the so called Reynold stress τ which gives the added friction caused by turbulent eddies is given by $\tau = \rho uv$. The expression can be extended to the case of the transport of a passive scalar $C(y)$ whose fluctuating component $c(y)$ is convectively transported, thus giving rise to an additional turbulent diffusivity.

The next main contributor to the field of turbulent mixing is G. I. Taylor who, from his first work on the subject in 1915 and for over half a century, retained an interest in many manifestations of mixing in turbulent, laminar and unstable flows. He made full use of the time he spent as a sailor since his youth and all along his life from the initial observations he made on the boat "The Scotia" which was sent to Labrador after the sinking of the "Titanic" to evaluate the risk of encountering icebergs. In this expedition he used kites, boat chimney smoke and fog to study the phenomenon of dispersion. He later used the marine current in the Irish sea as a natural hydrodynamic channel. He also developed an interest in aeronautics as a pilot during the First World War. But he is best known for his elegant model experiments emphasizing the interplay between flow geometry and convective as well as diffusive mixing. Starting from his *'Diffusion by continuous movements'* in 1921, he was a major contributor to the statistical concepts to turbulence. Around 1920, simultaneously but independently from L. Prandtl, he introduced the mixing length which makes use of the Reynolds stress tensor. In a turbulent two-dimensional gradient flow, the model assumes that the eddies convect the fluctuation of velocity on a scale l, the *mixing length*. Thus, the fluctuating component u can be related to the mean velocity profile $U_0(y)$ by $u = l\frac{dU_0}{dy}$. The fluctuating component v should also be of the order of u as can be envisioned from the picture of an eddy of radius l. Thus, the Reynolds stress is: $\tau = \rho uv = \rho l^2(\frac{dU_0}{dy})^2$. This gives a microscopic meaning to the transport coefficient ϵ introduced by Boussinesq as: $\epsilon = \rho l^2 \frac{dU_0}{dy} \propto \rho \lambda v$. In this last form, the analogy with the kinetic theory which is at the origin of the statistical treatment of turbulence becomes explicit, λ plays the role of the mean free path of the kinetic theory model and v that of the average thermal velocity. Rather than thinking of transport of momentum by eddies as in the work of Reynolds and later Prandtl, Taylor considered that of the vorticity ω, eliminating pressure gradient terms from the evolution equation for ω. The analysis in terms of a kinetic theory of vortices was pursued one step further by Von Karman. Using the similarity hypothesis and neglecting the effect of viscosity, he obtained in particular the logarithmic velocity profile of a shear flow.

A parallel can be drawn between G. I. Taylor and his contemporary L. F. Richardson, who was also a very fine observer (he also published an article on theory of turbulence from the observation of chimney smoke!) and whose intuitions had considerable impact on the further development of the turbulence and mixing. In particular his experimental study of *'Atmospheric diffusion shown on a distance neighbor graph'* in 1926 makes use of the observations of pairs of markers injected in turbulent flows starting from an initial separation l of a few centimeters to several kilometers. He found that the rate of separation increases with l, and concluded from a log plot of the results a power law increase, now called the Richardson diffusion: $\frac{dl^2}{dt} \propto D(l) l^{4/3}$. The diffusivity starts from a value equal to twice the molecular diffusivity at small separations as expected from independent walks of two tracers. It increases with separation as more and more eddy of sizes smaller than l contribute to separation, in the spirit of the similarity hypothesis.

2. Fully-developed Turbulence

2.1 Philosophical journey

When we study turbulent flows we are overwhelmed by their apparent complexity, which leads us to describe them as "disordered" or "random". But, as already stated four centuries ago by Spinoza, it is important to understand that the notion of "order" is subjective and that we call "disordered" a system whose dynamics appears to us too complicated to be described in details: *'Because those who do not understand the nature of things, but only imagine them, affirm nothing concerning things, and take the imagination for the intellect, they firmly believe, in their ignorance of things and of their own nature, that there is an order in things. For when things are so disposed that, when they are presented to us through the senses, we can easily imagine them, and so can easily remember them, we say that they are well-ordered; but if the opposite is true, we say that they are badly ordered, or confused. And since those things we can easily imagine are especially pleasing to us, men prefer order to confusion, as if order were anything in Nature more than a relation to our imagination' /Spinoza/.*

James Clerk Maxwell, in the article on *'Diffusion'* that he wrote for the 5th edition of the Encyclopedia Britannica, published in 1877, also emphasized the fact that the terms "order", "disorder" and "dissipation" are subjective notions: *'Confusion, like the correlative term order, is not a property of material things in themselves, but only in relation to the mind which perceives them. The notion of dissipated energy could not occur to a being who could trace the motion of every molecule and size it at the right moment. It is only to a being in the intermediate stage, who can hold of some forms of energy while others elude his grasp, that energy appears to be passing inevitably from the available to the dissipated state' /Maxwell 1877/.* This is why in the theory of turbulence we should be aware that the observer plays a role, and that we should take into account his scale of observation and the question he poses concerning the turbulent flow. For instance, we would not model turbulence in the same way if we evaluate the drag coefficient of an aircraft in order to compute the power of its engines, or if we study trailing vortices in order to estimate the safe distance between landing airplanes. When calculating the drag coefficient it is sufficient to consider only low-order averaged quantities (such as two-point correlation) based on frequent events which correspond to the center of the probability distribution function (PDF) of velocity or vorticity, while to find the safe distance between airplanes we must consider rare events and departure from the low-order averages, corresponding to the tails of the same PDFs.

The aim of the theory of turbulence is to define quantities whose evolution we can predict from quantities we are going to model without tracking their detailed dynamics. This programme was already clearly stated 67 years ago by Richardson in a paper entitled *'Diffusion regarded as a compensation for smoothing'* where he wrote that: *'By an arbitrary choice we try to divide motions into two classes: (a) those which we treat in detail, (b) those which we smooth away by some process of averaging. Unfortunately these two classes are not always mutually exclusive. [...] Diffusion is a compensation for neglect of detail. [...] The form of the law of diffusion depends entirely upon the arbitrarily chosen method of averaging, which is always implied when diffusion or viscosity are mentioned. This calls attention to the desirability of making much more explicit statements about smoothing operations than has hitherto been the custom' /Richardson and Gaunt 1930/.* Here Richardson emphasizes a key question, still requiring discussion in turbulence theory today, which is the definition of the averaged quantities we want to measure and predict in order to describe turbulent flows. The best discussion we know of defining appropriate averages in turbulence is found in a paper written in 1956 by Kampé de Fériet entitled *'The notion of average in turbulence theory'* /Kampé de Fériet.

The flow of a fluid is called turbulent when it exhibits an unsteady, chaotic and random behaviour; in this regime the flow is sensitive to initial conditions and therefore unpredictable. Such flows mix transported quantities, like momentum, heat or concentration of passive scalar, much more efficiently than laminar flows where only molecular diffusion is involved. This leads to enhanced diffusion, which is called "turbulent diffusion". These properties are displayed by both two- and three-dimensional turbulent flows, although some of their dynamical properties are different. In two dimensions, vorticity is a scalar advected by the fluid and enstrophy (L^2-norm of vorticity) cannot grow. In three dimensions, vorticity is a vector and enstrophy can then be generated by vortex stretching.

We should be aware that the definition of turbulence we have given here depends on our views on this phenomenon, and that the notion of turbulence has evolved in time and is still changing. Turbulence has always been a problem faced by engineers working in hydraulics and in aerodynamics, but since the beginning of the 20th century, it has also caught interest of theoretical physicists and mathematicians. This has lead to new questions such as: is there a universal statistical equilibrium state typical for turbulence which would play a role similar to Maxwell's distribution in the kinetic theory of gases? Are the solutions of Navier-Stokes equations unique and smooth for all times or do they develop singularities in finite time? We should be aware that our definition of turbulence depends on the conceptual tools we have at our disposal to describe them. If one relies on the theory of dynamical systems, one will see turbulent flows as a collection of vortices having chaotic dynamics in space and time which is characterized by a strange attractor in phase-space. If one relies on stochastic tools, one will emphasize ther randomness of turbulent flow, which can be characterized by the Probability Distribution Function (PDF) of a large ensemble of different realizations of the same flow.

Our definition of turbulence also depends on the technical tools we have at hand. For instance, the development of hot wire anemometry in the 50s has allowed experimentalists to obtain pointwise measurements of many flow realizations and therefore to calculate reliable statistics, such as energy spectra and PDFs, but this technique does not give meaningful information (for instance the instantaneous spatial distribution of velocity, pressure or temperature field) about an individual flow realization. It is the generalization of computers, beginning in the 80s, for both laboratory (data acquisition and processing) and numerical experiments (direct numerical simulation), which changed our views on turbulent flows by allowing multi-point (generally in the form of grid point sampling) measurements and by giving access to the detailed spatio-temporal structure of the different fields of interest (velocity, vorticity, pressure, temperature, concentration, ...). The change of viewpoint allowed by the computer has been advocated by experimentalists such as Günther Ahlers who testifies that: *'I believe that the most important experimental development of the 1970's was the advent of the computer in the laboratory'* [...]. Data acquisition and processing *'did not only provide us a new tool but they also gave us completely new perspectives on what types of experiments to do'* /Aubin 1997/.

In the historical review that follows, which deliberately adopts a personal and therefore limited point of view, we will focus on fully-developed turbulent flows and restrict ourselves to incompressible Newtonian fluids. We will distinguish three different perspectives to look at turbulent flows: the statistical kinetic approach (from Boussinesq to Taylor), the statistical probabilistic approach (from Kolmogorov to intermittency models) and the deterministic approach (from Helmholtz to dynamical system methods). The first approach, inspired by the kinetic theory of gases (developed by Maxwell, Boltzmann and Einstein), separates flows into mean and fluctuating motions assuming their temporal or spatial scales to be sufficiently separated (scale separation hypothesis), which leads to turbulent viscosity (Boussinesq and Reynolds) or mixing length (Prandtl) models. The second approach relies on the theory of stochastic processes (developed by Wiener, Khinchin and Kolmogorov), which involves random functions and probability measures and predicts the scaling law of the energy spectrum. The third approach focuses on the vorticity field of individual flow realizations and tries to understand the formation and interaction of coherent vortices in order to identify a low order dynamical system presenting the same chaotic behaviour as the complete flow.

2.2 Historical journey

2.2.1 The statistical kinetic approach

In order to try to master the complexity of turbulent flows, the first statistical approach was to decompose the velocity field into its mean value and fluctuations, by analogy with the kinetic theory of gases which distinguishes the mean motion from the fluctuating (thermal) motion of molecules. This statistical kinetic approach was proposed by Saint-Venant and Boussinesq supposing the existence of "fluid molecules" /Boussinesq 1877/, then by Reynolds in 1894 /Reynolds 1894/ and by Lorentz in 1896 /Lorentz 1896/. After decomposing the velocity field into a mean contribution plus fluctuations, one rewrites the Navier-Stokes equation to predict the evolution of the mean velocity as a function of fluctuations. This procedure yields the Reynolds equation. However, one encounters difficulties due to the nonlinear term $(\vec{V}\cdot\nabla\vec{V})$ of the Navier-Stokes equation: the second-order moment of the velocity fluctuations, called the Reynolds stress tensor, depends on the third-order moment, which depends on the fourth-order moment, and so on ad infinitum. At each order one considers there are more unknowns than equations and one faces a closure problem. In order to close the hierarchy of Reynolds equations the usual strategy is to add another equation, or system of equations, chosen from some *a priori* phenomenological hypotheses. In particular, one must suppose that there exists a scale separation, namely that fluctuating motions are sufficiently decoupled from the mean motions to guarantee that the average of the product (coming from the nonlinear term of Navier-Stokes equation) is equal to the product of averages (5th Reynolds postulate /Monin and Yaglom 1965/).

To close the hierarchy of Reynolds equations, in 1925 Prandtl introduced a scale, called the "mixing length", characteristic of the velocity fluctuations (introduced in paragraph 1.2). Following the hypothesis proposed by Boussinesq /Boussinesq 1877/, and by analogy with molecular diffusion which regularizes velocity gradients for scales smaller than the molecular mean free path, Prandtl supposed that there exists turbulent diffusion, which smoothes the velocity fields at scales smaller than the mixing length; he then rewrote the Reynolds stress tensor as a turbulent diffusion term. As soon as one considers large Reynolds number flows, the mixing length hypothesis fails because the analogy with the kinetic theory of gases does not work. Molecular motions can be modelled by a diffusion equation (linear Laplacian operator $\nabla^2\vec{V}$ applied to a mean velocity field with viscosity as transport coefficient) because they are decoupled from the large scale motions. But this is not possible for fully-developed turbulent motions because the nonlinear term $(\vec{V}\cdot\nabla\vec{V})$ of the Navier-Stokes equation, which dominates the linear dissipation term $(\nabla^2\vec{V})$ in this case, involves all scales, and thus there is no scale separation to decouple large scale motions from small-scale motions. This is a major obstacle in any attempt to model turbulence using moment equations and therefore the closure problem is still open. An important direction of research is to find a new representation of turbulent flows, in which there may exist such a separation, not based on scales, but on decoupling motions out of statistical equilibrium from well thermalized motions, which can then be modelled. Such a separation may be possible with a nonlinear closure procedure, based on wavelet representation and using conditional averages which depend on the local behaviour of each flow realization /Farge et al. 1992/, /Farge et al. 1997/.

Turbulence modelling should rely on a careful statistical analysis of turbulent flows, observed in the laboratory or in numerical experiments. Traditionally one studies the long time velocity correlation, supposing the turbulent flow has reached a statistically steady state, so the time covariance no longer evolves. The use of the covariance and cross-correlation functions was first proposed by Einstein, in a paper written in French and published in 1914 in Switzerland, to study the statistics of long time series of fluctuating quantities /Einstein 1914/, /Yaglom 1986/. In 1935 Taylor /Taylor 1935/ formulated, in addition to statistical stationarity, the hypothesis of statistical isotropy (and consequently homogeneity), namely he supposed that averages are invariant under both translation and rotation of the fields. Taylor, using the statistical tools introduced by Wiener (himself inspired by the earlier work of Taylor on Brownian motion /Taylor 1921/), proposed to study isotropic turbulence by measuring its energy spectrum, which is the modulus of the Fourier transform of the two-point correlation of the velocity increments /Taylor 1935/. Taylor also supposed that the spatial distribution of velocity changes slowly as it is carried past the point at which the frequency spectrum (the Fourier transform of the time correlation) is measured, which in this case gives the same energy spectrum for both time and space correlations. This is known as "Taylor's hypothesis" and is commonly used in most laboratory experiments to interpret time correlations as space correlations. These new statistical tools introduced by Taylor to analyze turbulent flows have brought about a change in turbulence research and are still in use today. Although Taylor in his work on turbulent diffusion /Taylor 1921/ was the first to introduce stochastic tools in turbulence, he always adopted a dynamical point of view. This explains why he never showed a strong interest for Kolmogorov's theory. In his famous paper, entitled *'The Spectrum of Turbulence'* /Taylor 1938/, Taylor expressed his intuition that the energy dissipation is not densely distributed in space, namely that turbulent flows are intermittent. He stated the hypothesis that this intermittency is related to the spottiness of the spatial distribution of vorticity: *'The fact that small quantities of very high frequency disturbances appear, and increase as the speed increases, seems to confirm the view frequently put forward by the author (himself) that the dissipation of energy is due chiefly to the formation of very small regions where the vorticity is very high'* /Taylor 1938/. Understanding intermittency is still a very important open problem, perhaps the essential one, to solve before we can build a satisfactory theory of turbulence.

2.2.2 The statistical probabilistic approach

In order to overcome the difficulty with closure, due to the fact that there is no scale separation to decouple large scale motions from small scale motions in turbulent flows, another statistical approach has been put forward. We replace the observation of individual flow realizations by the measure of the correlation between the velocities in a large number of flow realizations. We construct ensemble averages, assuming that we know the probability law of the process governing turbulent flows, and thus all its moments. This probabilist approach was initiated by Gebelein in 1935 and developed by many scientists, often independently from one another, among them Kampé de Fériet /Kampé de Fériet 1939/, Millionshchikov /Millionshchikov 1939/, Kolmogorov /Kolmogorov 1941/, Obukhov /Obukhov 1941/, Onsager /Onsager 1949/, Heisenberg and Von Weizsäcker /Heisenberg 1948/.

Since Gibbs, such a probabilistic approach has become standard in statistical physics, but the difficulty in applying it in turbulence arises from the fact that turbulent flows are open thermodynamical systems due to the injection of energy by external forces and its dissipation by viscous frictional forces. To resolve this difficulty Kolmogorov supposed the existence of an energy cascade, based on the hypotheses that external forces, which inject energy into the flow, act only on the large scales, while frictional forces, which dissipate energy, act only on the small scales. In the limit of very large Reynolds numbers, Kolmogorov supposed that there exists an intermediate range of wavenumbers, called the inertial range, for which energy is conserved and only transferred from low to high wavenumber modes, at a constant rate ϵ. As a consequence of the cascade hypothesis, Kolmogorov assumed that the skewness of the velocity increment probability distribution is non zero and does not vary with the scale, which implies that turbulent flows are non Gaussian and non intermittent. All these hypotheses led him to the prediction that the modulus of the energy spectrum scales according to a power-law $\epsilon^{2/3}k^{-5/3}$, k being the modulus of the wavenumber.

In fact, the hypothesis of a turbulent cascade and the resulting scaling of the energy spectrum can only be true for ensemble averages. If one considers each flow realization, the enstrophy (L^2-norm of vorticity) and energy are produced very locally in physical space, at locations where there are boundaries or internal shear layers, and therefore very non-locally in wavenumber space (this from the definition of the Fourier transform and the resulting Heisenberg's uncertainty principle). Likewise, the spatial support of dissipation is highly spotty for a given flow realization and therefore also non-local in wavenumber space. These observations are in contradiction with the hypothesis of a low-wavenumber injection and a high-wavenumber dissipation of energy necessary to maintain an inertial range. Such observations have already been made more than 400 years ago by Leonardo da Vinci when he wrote: *'Where turbulence of water is generated, where turbulence of water maintains for long, where turbulence of water comes to rest'* /Frisch 1995/. Obviously da Vinci was describing one turbulent flow realization in physical space and not in wavenumber space, one being dual of the other. His remark has therefore nothing to do with the notion of turbulent cascade which is a concept formulated for ensemble averages and which says nothing about individual realization. Vinci was observing the local production of vortices in the boundary layers and internal shear layers, their advection by the global velocity field and their dissipation resulting from their mutual nonlinear interactions. The possible confusion between physical space observations and the cascade concept has been discussed 23 years ago by Kraichnan in a paper entitled *'On Kolmogorov's inertial-range theories'* /Kraichnan 1974/, where he remarked: *'The terms "scale of motion" or "eddy of size l" appear repeatedly in the treatment of the inertial range. One gets an impression of little, randomly distributed whirls in the fluid, with the fission of the whirls into smaller ones, after the fashion of Richardson's poem. This picture seems to be drastically in conflict with what can be inferred about the qualitative structures of high Reynolds numbers turbulence from laboratory visualization techniques and from plausible application of the Kelvin's circulation theorem'.* Incidently we should notice that Richardson's poem described the formation of smaller and smaller structures at the interface of clouds and not in the bulk of turbulent flows /Richardson 1922/. Unfortunately the picture Richardson has proposed has been misinterpreted since then as the dynamical process responsible for the energy cascade although the resulting breaking of whirls into smaller ones does not seem to be mechanically possible in the bulk of an incompressible turbulent flow.

Due to observational evidence of small-scale intermittency introduced by Townsend in 1951 /Townsend 1951/ and following a criticism of Landau who pointed out that the dissipation rate ϵ should fluctuate, Kolmogorov proposed a new theory /Kolmogorov 1962/ which added an intermittency correction μ to the energy spectrum scaling, such that

$E(k) \propto k^{-5/3-\mu}$. Kolmogorov's 1962 paper opened a debate, which is still very lively today. Twenty three years ago Kraichnan wrote /Kraichnan 1974/: *'The 1941 theory is by no means logically disqualified merely because the dissipation rate fluctuates. On the contrary, we find that at the level of crude dimensional analysis and eddy-mitosis picture the 1941 theory is as sound a candidate as the 1962 theory. This does not imply that we espouse the 1941 theory. On the contrary, the theory is made implausible by the basic physics of vortex stretching. The point is that this question cannot be decided a priori; some kind of non-trivial use must be made of the Navier-Stokes equation'.* Kraichnan, although he is a master of the statistical approach, claims that one needs to understand first the generic dynamics of the Navier-Stokes equation before being able to construct a statistical theory taking into account intermittency: *'If the Kolmogorov law $E(k) \propto k^{-5/3-\mu}$ is asymptotically valid, it is argued that the value μ depends on the details of the nonlinear interaction embodied in the Navier-Stokes equations and cannot be deduced from overall symmetries, invariances and dimensionality'* /Kraichnan 1974/.

Pauli liked to say that there exist true theories, false theories and theories which are neither true nor false, namely, following Popper's terminology, theories which are unfalsifiable. In order to fit observations unfalsifiable theories prefer to add new parameters rather than to change their hypotheses. Consequently these theories can be very accurate and optimal for describing existing data, but they are not able to predict new facts and they lack predictability power. A typical example is the geocentric theory of planetary motions which was able to describe the motions of all known planets with a reduced number of epicycles. When new observations became available new epicycles were added in order to preserve the corpus of the theory. The theory was very useful in practice but was unable to predict the existence of an unknown planet as Leverrier did for Neptune. Unfalsifiable theories are poor from an epistemological point of view but they produce a large number of publications because each new refinement step in adding a new parameter leads to new models and new experimental checks. This may be what Kraichnan had in mind when he wrote: *'Once the 1941 theory is abandoned, a Pandora's box of possibilities is open. The 1962 theory of Kolmogorov seems arbitrary, from an a priori viewpoint [...]. We make the point that even in the general framework of some kind of self-similar cascade, and of intermittency which increases with the number of cascade steps, the 1962 theory is only one of many possibilities'.* One can also add a new class to Pauli's picture: "too true" theories, namely theories one cannot falsify even in situations where their hypotheses are not valid, which is illustrated by Kraichnan's comment on Kolmogorov 41's theory: *'Kolmogorov's 1941 theory has achieved an embarrassement of success. The $-5/3$ spectrum has been found not only where it reasonably could be expected, but also at Reynolds numbers too small for a distinct inertial range to exist as in boundary layers and shear flows where there are substantial departures from isotropy, and such strong effects from the mean shearing motion that the stepwise cascade appealed to by Kolmogorov is dubious'* /Kraichnan 1974/. The robustness of the energy spectrum is not entirely surprising because it is the Fourier transform of the two-point correlation of the velocity increments and therefore it is unsensitive to rare events, such as coherent vortices which are produced in shear layers and in boundary layers. But this robustness of Kolmogorov's theory is lost as soon as one considers higher-order statistics, as we will discuss in the following paragraph.

2.2.3 The deterministic approach

In parallel to the statistical probabilistic approach based on ensemble averages, there has also be the tendency to analyze each flow realization separately. Both tendencies, probabilistic and deterministic, were well represented by the participants to the 1st International Conference on Turbulence held in Marseille in 1961 /CNRS 1961/. The probabilistic approach was advocated by Kolmogorov, Obukhov, Millionshchikov, Yaglom, Batchelor, Favre, Kampé de Fériet, Kraichnan and Lumley, while the deterministic approach was defended by Liepmann, Roshko, Laufer, Von Karman, Taylor, Saffman and Moffatt. Actually, Townsend had been the first (in 1951) to suggest that there exist organized structures, called coherent structures, which play an essential role in the transport properties of turbulent flows and which may be responsible for their chaotic and intermittent behaviour. In 1955 Theodorsen in a paper on *'The Structures of Turbulence'* /Theodorsen 1955/ formulated the hypothesis that coherent structures (in the form of bended vortex tubes), such as horseshoes or hairpins vortices, are responsible for the eddy motions of 3D turbulent flows. The presence of such coherent vortices in fully-developed turbulent flows has been observed in both laboratory and numerical experiments. For instance, many numerical experiments have shown that, when two-dimensional or three-dimensional turbulent flows are initialized with an homogeneous random distribution, vorticity tends to concentrate in a finite number of coherent vortices, which are formed during the flow evolution and seem to have their own dynamics and interaction laws quasi-independent of the background flow where they evolve. Although there is not yet a universal definition of coherent vortices (we prefer the term "vortices" rather than "structures" which is too general) on which everyone agrees, we will define them as "condensates" of the vorticity field where rotation dominates strain. They survive on time-scales much longer than the eddy turnover time characteristic of velocity fluctuations, although they can be rapidly destabilized and destroyed by strong strain. They exhibit phase-correlation over a spatial and/or temporal range significantly larger than the smallest scales of the flow. They emerge out of random initial conditions, or are created in boundary layers by instabilities. They have their own dynamics, such as same-sign vortex merging (also called vortex pairing), opposite-sign vortex binding and emission of unidimensional vorticity filaments when they are strongly strained. In two dimensions, due to the orthogonality between vorticity and velocity gradient vectors, there is no vortex stretching, while in three dimensions coherent vortices, such as vortex tubes (often called "filaments" although "tubes" may be more adequate), can be stretched by velocity gradients. This leads to either vorticity production during vortex stretching or vorticity dissipation during vortex breakdown.

For two-dimensional flows it is possible to characterize coherent vortices from a topological point of view in studying the stress (velocity gradient) tensor. Its symmetric part corresponds to the irrotational strain while its antisymmetric part corresponds to the rotation undergone by a fluid element. The eigenvalues of the stress tensor allow for the flow to be separated into different regions where the Lagrangian dynamics is different. In two dimensions there are two types of regions which are either elliptic or hyperbolic /Weiss 1992/. Imaginary eigenvalues correspond to elliptic regions where rotation dominates strain and for which fluid trajectories remain close, which characterizes coherent vortices. Real eigenvalues correspond to hyperbolic regions where strain dominates rotation and for which fluid trajectories separate exponentially, which characterizes the hyperbolic stagnation points of the background flow. In three-dimensional flows where vortex stretching plays a key role, there are more types of topological regions. Unfortunately the classical theory of turbulence is myopic to the presence of coherent vortices because they are advected by the flow in a homogeneous and isotropic random fashion. They are also highly unstable and their temporal and spatial support may be very small for three-dimensional turbulence. Consequently their presence only affects the high-order velocity structure functions (namely the high-order statistical moments of the velocity increments), which have been measured only recently. These measurements contradict Kolmogorov's theory which predicts a linear dependence (with slope 1/3) between the scaling exponent of the velocity structure functions and their order. In addition it has been found that there are actually two distinct nonlinear dependences for odd and for even order structure functions /Van der Water 1993/.

One should not forget that cascade is only a hypothesis of Kolmogorov's theory and that the observed scaling of the energy spectrum of turbulent flows can be explained without this hypothesis. An alternative viewpoint, that we will call "dynamical", is based on the assumption that the nonlinear dynamics of turbulent flows tends to form quasi-singularities in physical space, such as point vortices for 2D flows or vortex filaments for 3D flows. This dynamical view was already advocated by Kraichnan in a paper writtten in 1974 where he explained: *'The stretching mechanism has led a number of authors to conjecture that the small-scale structure should consist typically of extensive thin sheets or ribbons of vorticity, drawn out by the strirring action of their own shear field (Townsend 1951, Batchelor 1953, Kraichnan 1959, Corrsin 1962, Saffman 1968, Tennekes 1968). In this picture, the randomness lies in the distribution of thickness and extension of the thin sheets and ribbons, and in the way they are folded and tangled through the fluid. A typical small-scale structure is thought to be small in one or two dimensions only, not in the third'* /Kraichnan 1974/. This last point means that the energy spectrum of only one realization averaged over angles already exhibits a broad-band energy distribution similar to the one observed in the inertial range for ensemble averages. If each flow realization exhibits a power-law spectrum, this is *a fortiori* true for an ensemble of realizations.

This dynamical interpretation of the energy spectrum has been then developed by Saffman /Saffman 1971/ for two-dimensional turbulence and by Lundgren /Lundgren/ for three-dimensional turbulence. Saffman supposes that the nonlinear dynamics of two-dimensional turbulent flows tend to form vorticity discontinuities along lines, that we will call one-dimensional vorticity filaments, and lead to a k^{-4} energy spectrum. Lundgren supposes that the nonlinear dynamics of three-dimensional turbulent flows tend to form vortex tubes, that we will call two-dimensional vorticity filaments, which roll up into spirals in such a way that the energy spectrum scales as $k^{-5/3}$. These ideas were later developed by Gilbert /Gilbert 1988/ and Moffatt /Moffatt 1993/, who suggested that the power-law scaling of two-dimensional turbulent flows can be explained by the rolling up of vorticity filaments into spirals which, by accumulation of singularity at the center of the spiral, lead to a k^{-3} scaling of the energy spectrum. In 1991 Farge and Holschneider /Farge and Holschneider 1991/ proposed, instead of one-dimensional vorticity filaments, the formation of two-dimensional cusp-like axisymmetric coherent vortices, for which vorticity remains bounded in the core due to viscous effects. This introduces the core radius as a new length scale and the larger the Reynolds number, the smaller the core radius will be. Farge and Holschneider conjectured that these cusp-like vortices are formed from initially random vorticity field by an inviscid instability, similar to Kelvin-Helmholtz instability, which accretes vorticity onto the strongest singularities of the initial random distribution (which correspond to the tails of the vorticity PDF). The same mechanism may explain the formation of vortices at the wall. They have shown that the cusp-like quasi-singularities remain stable under Navier-Stokes evolution /Farge, Holschneider and Philipovitch 1992/ and that the strain they impose on the background flow organizes it, and inhibits further instability that otherwise could develop in their vicinity /Kevlahan and Farge 1997/. Actual singularities may only exist for very large or infinite Reynolds numbers, but this is still an important open question. In 1982 Cafarelli, Kohn and Nirenberg /Cafarelli 1982/ proved that, if finite time singularities exist for three-dimensional Navier-Stokes equations, they will be on a space-time support of measure one. Therefore, if singularities or quasi-singularities exist, they can only be rare events (rare because if their time support is one their space support is then zero), which can hardly be detected using standard statistical methods or two-point correlations, since low-order statistics are insensitive to rare events.

3. Which are the essential questions for turbulence and mixing?

After more than a century of turbulence research/Reynolds 1883/, no single convincing theoretical explanation has given rise to a consensus among physicists. In fact, there exists a large number of *ad hoc* models, called "phenomenological", that are widely used by fluid mechanists to compute industrial applications where turbulence plays a role, but many parameters of those turbulence models cannot be derived from first principles and must be determined by performing experiments in wind tunnels or water tanks. In fact it is still not known whether fully-developed turbulence has the universal behaviour (independent of initial and boundary conditions) assumed for it in the limit of infinitely large Reynolds numbers and infinitely small scales. Our understanding of turbulence is impaired by the fact that we do not yet know which are the "right questions" to ask. We have not yet identified the "right objects", namely the structures and elementary interactions from which it would be possible to construct a satisfying statistical mechanics (or kinetic theory) of fully-developed turbulence. Ignorance of the elementary physical mechanisms at work in turbulent flows arises in part from the fact that we ignore coherent vortices because we use two-point correlation functions, we think in terms of Fourier modes that are delocalized and that we consider L^2-norm, such as energy and enstrophy, instead of higher-order norms.

Our present lack of understanding of the dynamics of coherent vortices arises from several reasons:

-1. We focus on the velocity field and not on the vorticity field.

-2. We study the flow evolution in an Eulerian frame of reference, instead of a Lagrangian frame attached to each fluid particule. It is more appropriate to follow the evolution of vorticity or of circulation, because vortex tubes are advected by the flow and even conserved in absence of dissipation (Helmholtz laws and Kelvin's circulation theorem).

-3. Classically we perform point measurements and two-point correlations which are insensitive to rare events such as coherent vortices.
-4. The infinite time limit which is usual in statistical physics is not interesting for many applications of turbulence where we are interested in studying the transient evolution and need methods, for instance, to accelerate mixing. In particular, the infinite time limit is irrelevant to many meteorological situations where one cannot guarantee the statistical stationarity and the time decorrelation (Markov's process hypothesis) of the external forcing.
-5. "Real life" problems are bounded in space, time and scale. Therefore we should focus more on statistically unsteady and inhomogeneous turbulent flows, although theoreticians and mathematicians seldom consider spatial boundaries, or initial conditions, or scale cut-offs, because they destroy the homogeneity, stationarity and self-similarity they prefer to assume.
-6. We consider only the modulus of the energy spectrum and not its phase, therefore we completely lose track of spatial coherence.
-7. When we develop numerical codes (combinations of a Navier-Stokes solver and a turbulence model) to compute fully-turbulent flows we address neither the non-linear problem *per se* nor the fact that one does not have a statistical equilibrium. On the contrary, one supposes either a linear behaviour of the subgrid scale motions (in the case of Direct Numerical Simulation, DNS) or the existence of a scale separation, which supposes a statistical equilibrium for the subgrid scale motions (in the case of Large Eddy Simulation, LES). This last point was already noticed 24 years ago by Kraichnan when he wrote: *'Our basic point is that the inertial-range cascade represents strong statistical disequilibrium. This carries two implications. First, that analogies with equilibrium and near-equilibrium phenomena are unjustified. Second, that the structure of the inertial range depends on the actual magnitude of the coefficients coupling the degrees of freedom and not just on their overall symmetry and invariance properties. This is because cascade is a transport process and the coefficient magnitudes affect the rate of transport'* /Kraichnan 1974/.

An open question to address is: what is the importance of coherent vortices in mixing and turbulence? Do they play an essential role which should be taken into account in our models or can we neglect them? We should get rid of the present misconception which relates coherent vortices to the small wavenumbers (large eddies) of turbulent flows. This false view comes from the fact that one tries to recover some dynamical picture from averaged quantities which have already lost track of the spatial and time coherence which characterizes coherent vortices. In particular the energy spectrum in the inertial range is dominated by the background and not by the coherent structure, because their spatial and temporal support is too small to have a sufficient weight in the integral when one computes second-order structure function (see the energy spectra in figure g). This is not true for high-order structure functions and actually the presence of coherent vortices may explain the departure from Kolmogorov's prediction one observes for high-order structure functions. On the contrary, when one considers the probability distribution function (PDF) of vorticity, one finds that its non-Gaussian shape comes from the coherent vortices, which are responsible for its heavy tails (see the PDF in figure f). This is due to the fact that, although coherent vortices are quite rare in space and time, they are present in any realization of a turbulent flow. The formation of coherent vortices is probably a deep consequence of the incompressible Navier-Stokes dynamics one should try to clarify. To answer the previous question concerning the role of coherent vortices in turbulent flows, we need a clear definition, still lacking of what is a coherent vortex and we need an appropriate method to extract them. For two-dimensional flows it has been proposed /Farge et al. 1992/, /Farge et al. 1997/ to use the wavelet representation and nonlinear filtering to extract coherent vortices from the incoherent background flow (see figures a, b, c), showing that large wavelet coefficients correspond to the coherent vortices while small wavelet coefficients correspond to the background flow. The coherent vortices thus extracted have the same coherence (characterized by a functional relation between vorticity and streamfunction) as the overall vorticity field, while the background is incoherent (see the coherence function on figure d). Both regions, coherent as well as incoherent, are multiscale, and therefore cannot be separated by Fourier filtering, although they exhibit different scaling (see figure g). This programme has been extended to three-dimensional turbulent flows. It has shown /Farge et al.1990/ that coherent vortices present in mixing layer and channel flows are responsible for the flow intermittency, and that the wavelet representation can be used to detect the quasi-singularities which develop in such inhomogeneous flows. The wavelet representation has also been used to compute two-dimensional Navier-Stokes equations following only the dynamics of coherent vortices in remapping the basis functions at each time step in order to optimally follow their motion in space and scale /Schneider, Kevlahan and Farge 1997/.

The statistical probabilistic approach may be too abstract and its links with experimental observations are difficult to ascertain in most cases. In order to compare its predictions with laboratory or numerical experiments, one has to guarantee that ensemble averages converge to time or space averages, and therefore satisfy the ergodic hypothesis. One supposes that there is only one attractor which satisfies Sinai-Bowen-Ruelle conditions (for almost all initial conditions the time average exists and is the same) and that the observed turbulent flow has visited all possible phase-space configurations compatible with this attractor. Therefore, due to our limited understanding of turbulence, it has become more and more important to perform many well controlled experiments to try to get better insight and propose new models to describe the behaviour of high Reynolds number turbulent flows. There are two kinds of experimental approach, each having its own limitations. First, laboratory experiments, where it is easy to measure one- or two- (or even more) point correlations and accumulate long time statistics, but where one cannot measure in many locations and at the same time the instantaneous spatial distribution of velocity and vorticity. Secondly, numerical experiments where it is easy to measure the time evolution and spatial distribution of velocity and vorticity fields for one given flow realization, but where it is out of computational reach to compute large ensemble averages. The two approaches are in fact complementary, laboratory experiments allow us to perform statistical analysis, while numerical experiments allow us to perform dynamical analysis. Unfortunately, these two approaches are also dual, and it is therefore difficult to compare them. For instance, the statistical analysis deals with averages, describes turbulent flows in terms of fluctuations (departure from the mean field) expressed in the statistical notion of turbulent eddies, and uses random functions and probability measures, while the dynamical analysis considers each flow realization *per se*, describes the flow in terms of isolated coherent vortices, and deals with non-random functions or distributions. A lot of confusion in our understanding of turbulence is due to the fact that we try to retrieve dynamical insight out of statistical averages and statistical information out of only one flow realization. Moreover the statistical analysis relies on the Fourier representation, while the dynamical analysis relies on the spatial representation. We must be aware that we cannot reconcile these two representations, unless we use basis functions which are localized in both physical space and wavenumber space, such as wavelets or wavelet packets /Farge 1992/.

Kolmogorov's statistical theory is the simplest possible universal theory (simple in the sense of "Occam's razor" or Aristotle's logical simplicity principle). It is very well verified for second-order moments (or two-point correlation), but it fails to correctly predict higher-order moments. We believe that coherent vortices may explain this discrepancy and are essential for the understanding of turbulence. Therefore we need to find another theoretical setting, which should actually take into account the existence of coherent vortices, as characteristic features of turbulent flows. We would like to construct a statistical mechanics of turbulent flows based on the dynamics of coherent vortices, but we still do not know what should be the appropriate invariant measure for this. In the limit of infinite Reynolds number, Kolmogorov's prediction for the two-point correlation (L^2-norm) will always be verified, because the contribution of coherent vortices tends to disappear from the integral measure of their spatial support tending to zero in this limit. However, his prediction will not prove true as one measures higher-order moments (L^p-norm with p the order), because in this case the weight of coherent vortices in the integral will become significant. In this picture dissipation results from the strong nonlinear interactions between coherent vortices. The larger the Reynolds number, the more and more local in physical space, and therefore nonlocal in wavenumbers, dissipation will be. The Kolmogorov's dissipative scale is an averaged quantity and we have conjectured /Farge et al. 1990/ that its variance in space is large and depends on the flow intermittency. In this picture universality seems to be lost because the density of coherent vortices depends on the initial conditions and on the forcing. But there may be a universal way of describing coherent vortices in terms of internal degrees of freedom having a quantified (discrete) amount of enstrophy, while the background incoherent flow can be seen as a thermal bath which only affects the coupling between coherent vortices. The prediction of Kolmogorov's theory may be verified only for the incoherent background flow, which is homogeneous, Gaussian and well-mixed.

Today we can again try to construct a kinetic theory of turbulence, at least in the case of two-dimensional turbulence where high-resolution direct numerical simulations of the Navier-Stokes equation at large Reynolds numbers (still inaccessible for three-dimensional simulations) have given us insight into the dynamically important features of turbulent flows. These simulations, and also high resolution visualizations of laboratory experiments (such as those performed by Dimotakis and presented in this book), reveal to us the shape and elementary interactions of coherent vortices. But in order to develop a statistical kinetic theory of turbulent flows we need to identify a gap between mean and fluctuating motions, namely a sufficient decoupling between the active and passive components of motion. We know this decoupling does nor exist when we do the separation in Fourier basis, but we have shown that we have more chances to find it in a space-scale representation such as wavelet bases /Farge et al. 1992/, /Farge et al. 1997/. Moreover we think that the theory of fully-developed turbulence is in a pre-scientific phase, because we do not yet have an equation, nor a set of equations, that could be used to compute efficiently turbulent flows from first principles. The Navier-Stokes equation, which is the fundamental equation of fluid mechanics, is appropriate to study laminar flows and transition to turbulence. But it is not the right one to compute fully-developed turbulent flows because its computational complexity becomes intractable when the Reynolds number becomes too large. For example, the number of degrees of freedom necessary to compute a turbulent flow by direct numerical simulation (DNS) is proportional to $Re^{9/4}$, Re being the Reynolds number and therefore, for aerodynamics applications where typically $Re = 10^7$, it requires the solution of a linear system of 10^{16} equations. However, for large Reynolds number flows it should be possible to define averaged quantities, like in statistical mechanics, and find the corresponding transport equations to compute the evolution of these new quantities which would be the appropriate observables to describe turbulence. Like the Navier-Stokes equation that can be derived from Boltzmann's equation by considering appropriate limits (Knudsen and Mach numbers tending to zero), appropriate averaging procedures to define new coarse-grained variables (velocity and pressure) and associated transport coefficients (viscosity and density), the turbulence equations may be derived as a step further in this hierarchy of embedded approximations. Unfortunately the appropriate parameters are easier to define when we go from Boltzmann equation to Navier-Stokes equation than from the Navier-Stokes to turbulence equation. In the first case only a linear averaging procedure, namely a coarse-graining based on a known statistical equilibrium distribution of velocities, is needed. In the second case we have to find an appropriate nonlinear procedure, namely some conditional averaging yet to be found. For this we have to identify the dynamically active structures constitutive of fully-developed turbulent flows, to describe their elementary interactions and to characterize their dynamics. We should then find a nonlinear procedure to extract those elementary structures out of turbulent flows. But we may not be able to apply our statistical methods to them because they are out of statistical equilibrium and their statistics are not stationary. On the contrary, the remaining background flow is sufficiently mixed to guarantee ergodicity, stationarity and homogeneity required by Kolmogorov's theory, which can then be used to model the incoherent background flow. In conclusion, we think that the future of turbulence research will be a combination of both deterministic and statistical approaches. The deterministic approach will be necessary to compute the evolution of the low-dimensional dynamical system corresponding to the elementary structures out of statistical equilibrium, namely the coherent vortices. The statistical approach will be needed to model the incoherent background flow using an equivalent stochastic process having the same statistics.

Hans Liepman, former Von Karman professor at Caltech, likes to point out that in turbulence research we are like the drunk man who has lost his keys in a dark alley but who finds it easier to search for them under a street light. For as long as we have been studying turbulence we know that it has to do with vortex production in wall shear layers and vortex interactions. But, because we do not have a good theoretical grasp of their structure and dynamics when there are many vortices, we prefer to use the statistical formalism which, assuming we accept several hypotheses (ergodicity, low-wavenumber forcing and high-wavenumber dissipation, stationarity, homogeneity and isotropy of the statistics), gives us technical tools to predict the average spectral distribution of energy in the inertial range for an ensemble of flow realizations. But this is not the key we are looking for, because it does not enable us to grasp the elementary dynamics and compute the evolution of a given turbulent flow realization from first principles, neither to understand the near-wall dynamics. In the proceedings of the 1st International Conference on Turbulence, held in 1961 in Marseille, Hans Liepmann was already making the following remark: *'It is clear that the essence of turbulent motion is vortex interactions. In the particular case of homogeneous isotropic turbulence this fact is largely masked, since the vorticity fluctuations appear as simple derivatives of the velocity fluctuations. In general this is not the case, and a Fourier representation is probably not the ultimate answer. The proposed detailed models of an eddy structure represent, I believe, a groping for an eventual representation of a stochastic rotational field, but none of the models proposed so far has proven useful except in the description of a single process'* /Liepmann 1962/. Twenty-five years later Hans Liepmann gave us the following guidelines: *'Kolmogorov's theory has been counter-productive. It is OK for light or sound scattering by turbulent flows, but it is not useful for the main lines of turbulence. [...] In turbulence you have long range forces, and it is difficult to extrapolate from Boltzmann's gas, which has short range forces. Therefore I am uneasy about Reynolds equations. [...] As long as we are not able to predict the drag on a sphere or the pressure drop in a pipe from continuous, incompressible and Newtonian assumptions without any other complications (namely from first principles), we will not have made it!'* /Liepmann 1997/.

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Legend of the figures

Wavelet compression of vorticity.
(a) The vorticity. (b) The modulus of velocity. (c) The stream function. (d) The coherence scatter plot. (e) Cut of vorticity. (f) PDFs of velocity and vorticity. (f) Energy spectrum. The solid lines correspond to the total vorticity ω, the dashed lines to the coherent part $\omega_>$, and the dotted lines to the incoherent part $\omega_<$.
We observe that only 0.7% of the total number of wavelet coefficients are sufficient to represent all coherent structures, while the remaining 99.3%correspond to the incoherent background flow, which is much weaker and homogeneous. The coherent vorticity $\omega_>$ contains

94.3% of the total enstrophy. Moreover, the velocity associated with the coherent structures is quasi-identical to the total velocity and contains 99.2% of the total energy. As for the coherent stream function, $\psi_>$ is perfectly identical to the total stream function ψ. The fact that the scatter plot of the background, $F_<$ such that $\omega_< = F_<(\psi_<)$, is isotropic proves that our method has extracted all coherent structures. The PDFs of velocity and vorticity show that only 0.7% of the wavelet coefficients are sufficient to capture the non-Gaussian one-point statistical distributions of vorticity and velocity, while the remaining 99.3% correspond to Gaussian distributions. The energy spectrum, on the contrary, is dominated at small scales by the incoherent background flow and therefore is insensitive to coherent structures because they are too rare to affect the energy spectrum (which is the Fourier transform of the two-point correlation function).

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