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Combining deterministic and statistical approaches to compute two-dimensional turbulent flows with walls

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1 Research programme

'Although this may seem a paradox, all exact science is dominated by the idea of approximation' (Bertrand Russell).

We are searching for the best approximation to compute fully-developed two-dimensional turbulent flows. Fluid mechanics is governed by the Navier–Stokes equations, which are entirely deterministic. Fully-developed turbulence corresponds to very large Reynolds number flows (for which the micro-scale Reynolds number Re is larger than 10^4) and is the regime where the nonlinear advective term of Navier-Stokes equations strongly dominates the linear dissipative term. In this limit, the solutions to the Navier–Stokes equations are highly chaotic and we are unable to integrate them. Therefore, in order to compute fully-developed turbulent flows we need to combine a deterministic numerical integration with a statistical model. In this paper we propose a possible solution to this problem, based on the wavelet representation.

The classical method of computing fully-developed turbulent flows is based on Reynolds averaging: in each realization the flow is separated into a mean part and a fluctuating part using a suitable averaging procedure. Such a separation is necessary because the huge number of degrees of freedom in a high Reynolds number turbulent flow prohibits a direct numerical simulation (DNS). The goal is to calculate the evolution of the mean part in detail using a deterministic equation and to design a statistical model which simulates the effect of the fluctuating part on the mean. However, because the Navier–Stokes equations are nonlinear, we must address the closure problem, i.e. the equations for the nth order moment depend on the n+1th order moment. Thus this method requires that the statistics of the fluctuating part be known completely. The closure problem can be solved if the statistics are Gaussian, since in this case all higher-order even moments can be expressed in terms of the second order moments and all odd moments are zero. Therefore the fundamental difficulty in turbulence is to find an averaging technique that produces a fluctuating part with Gaussian statistics. Unfortunately, the classical averaging techniques (e.g. separation into large scale and small scale eddies used in Large Eddy Simulation) do not guarantee this. We will propose one such averaging procedure that overcomes the closure problem, based on the wavelet-representation of the vorticity field.

In this paper we focus on two-dimensional turbulence, but our results could be extended to three-dimensional turbulence. From a physical point of view the two-dimensional approximation is relevant for studying large-scale geophysical flows, due to the combined effect of stable stratification and Earth's rotation. From a mathematical point of view there are existence, uniqueness and regularity theorems for the two-dimensional Navier–Stokes equations, which are not yet available in three dimensions. These theorems are necessary to validate the numerical procedure we use to solve Navier–Stokes equations. Therefore, from a numerical analysis point of view, numerical integration is better justified for two-dimensional turbulent flows than for three-dimensional turbulent flows. Moreover, the existence of an inertial manifold has been proven for two-dimensional turbulent flows, and upper bounds for the dimension of the attractor have been given, but this is still an open problem for three-dimensional turbulent flows. A final argument justifying our interest in two-dimensional turbulent flows is that, according to the usual estimation the minimal number of degrees of freedom N necessary to compute fully-developed turbulent flows without a model, namely by DNS, scales as Re in two dimensions and as $Re^{9/4}$ in three dimensions. Therefore using DNS we are able to compute much larger Reynolds number flows in two dimensions than in three dimensions. For two-dimensional turbulence we have already reached the fully-developed turbulent regime without resorting to any *ad hoc* turbulence models, but this is not yet the case for fully-developed three-dimensional turbulence.

2 Coherent vortex eduction

Since 1984 we have proposed using the wavelet representation to analyze, model and compute fully-developed turbulent flows. We have shown that the strong wavelet coefficients correspond to the coherent vortices, while the weak wavelet coefficients correspond to the incoherent background flow (for 2D turbulence see Farge and Rabreau 1988, Farge and Sadourny 1989, Farge, Holschneider and Colonna 1990, and for 3D see Farge, Meneveau, Guezennec and Ho 1990, Farge 1992). Both components are multiscale and therefore cannot be separated by Fourier filtering. We have developed a method, inspired by Donoho's denoising technique (wavelet shrinkage see Donoho 1993), to separate coherent vortices from the background flow (Farge, Schneider and Kevlahan 97). To extract the coherent vortices we reconstruct the vorticity field from its wavelet coefficients, retaining only those larger than a threshold value $\tilde{\omega}_T = (2Z \log_{10} N)^{-1/2}$, which depends on Z the total enstrophy and N the resolution without any *ad hoc* adjustable parameter. Figure 1 shows an example of this separation applied to a vorticity field which has been computed at a resolution a 256^2. Only 0.7% of the wavelet coefficients correspond to the coherent vortices, which have the same non-Gaussian Probability Distribution Function (PDF) as the total field, while the remaining 99.3% weaker wavelet coefficients correspond to the incoherent background flow, which has a Gaussian PDF. Using the wavelet representation higher resolutions produce stronger compression ratios.

We separate each turbulent flow into:

- A low-dimensional dynamical system out of statistical equilibrium, which we can compute with as little approximation as possible, using a deterministic equation.

- A high-dimensional dynamical system which has reached a statistical equilibrium state, for which we can compute averages that can be modelled by a Gaussian stochastic process.

In order to perform this separation we assume that there are two kinds of elementary motions characteristic of two-dimensional turbulent flows:

- *Self-organized vortices*, which are dynamically stable and each vortex belongs to an elliptic region and is characterized by the existence of a monotonic coherence function $\omega = F(\psi)$. These vortices are out of statistical equilibrium because the encounters with other vortices are too rare to produce enough mixing to generate an equilibrium state. The one-point PDFs of the coherent vortices are non-Gaussian and their entropy evolves over time, which means that we cannot rely on the Central Limit Theorem to define averages. This means that we cannot discard the phase information of the vortices (i.e. their positions) and we should therefore compute their motions with as little approximation as possible.

- *Incoherent background flow*, made of one-dimensional vorticity filaments which are dynamically unstable (they belong to hyperbolic regions and are stretched in one direction and compressed in the orthogonal direction) and have reached statistical equilibrium (because the strain imposed by the coherent vortices on the background flow inhibits the formation of new coherent vortices and mixes the filaments). Because the one-point PDFs of the background flow are Gaussian and entropy has reached its maximum we can rely on the Central Limit Theorem to define averages and discard the phase information (namely the spatial distribution of the vorticity filaments). Only its mean and variance are necessary to describe the stochastic effect the background exerts on the coherent vortices.

We have shown that the incoherent background flow is slaved to the coherent vortices, due to their straining which inhibits the development of any nonlinear instability in the background flow (Kevlahan and Farge 1997). From this result we have conjectured that the number of coherent vortices may saturate to a constant number when the Reynolds number is sufficiently large to produce enough vortices in boundary layers in order to inhibit any instability which would otherwise form new vortices in the bulk of the flow. We have shown that both coherent vortices and the incoherent background are multiscale, and therefore we propose the wavelet representation to compute their evolution. This is because the wavelet basis is scale invariant (being based on the affine group), and is thus better suited to compute turbulent flows than the Fourier basis which is wavenumber invariant but not scale invariant (being based on the Weyl-Heisenberg group). Similarly, the wavelet basis is preferable to a pure grid point representation since a grid point basis is translation invariant, but not scale invariant.

3 Coherent Vortex Simulation

We are presently developing a new method to compute fully-developed two-dimensional turbulent flows, which is based on a wavelet phase-space segmentation (Farge et al. 1992, Farge et al. 1996, Farge et al. 1997) and which uses the wavelet-based Navier–Stokes solver designed by Jorgen Frölich and Kai Schneider (Frölich and Schneider 1996). This method computes the dynamics of the coherent vortices with a limited number of wavelet modes, keeping only the most excited ones which correspond to coherent vortices, and re-mapping the wavelet basis at each time step. We have compared simulations using these wavelet techniques with standard spectral simulations and nonlinearly filtered spectral simulations (Schneider, Kevlahan and Farge 1997). The results showed that the wavelet method is very accurate and require fewer active modes than spectral methods. Moreover, the aliasing errors remain localized to regions of strong gradients, in contrast to the Fourier representation which spreads aliasing error over the entire solution. One can always reduce aliasing errors by locally adding more wavelet modes to improve the resolution where needed.

We have shown that the number of active wavelet modes is approximately constant in time, even during intense nonlinear interactions, whereas the number of active Fourier modes peaks when the interactions are more intense and strongly excite small scale. The discarded coefficients, which correspond to the incoherent background flow and have a Gaussian one-point PDF for velocity and vorticity, should be modelled statistically in order to take into account their effect on the coherent vortices. We can either model them by a stochastic forcing having the same statistical behaviour, compute the linear equation characterizing their motions, or design a one-point turbulence model (such as Boussinesq, Smagorinsky or k-ϵ). The justification for this procedure is that the coherent vortices are not numerous enough and their encounters are too rare events to have reached a statistical equilibrium state, and therefore we have to compute their dynamics with a deterministic method. On the contrary, for the well-mixed background flow we can assume stationarity, homogeneity and ergodicity in order to define a statistical equilibrium state from which we can design an appropriate statistical model.

4 Wall effect

4.1 Mathematical analysis

In two space dimensions and for any finite time both the solutions of the incompressible Navier Stokes equation and of the incompressible Euler equations are well controlled for smooth initial data. This result is true for a solution defined either in the whole space, in a periodic box, in the interior of a vessel or around an obstacle. In the last two cases the fluid domain is denoted by Ω, the boundary of this domain is denoted $\partial\Omega$ and the outward unit normal to this boundary is denoted by $\vec{n}$. Some boundary conditions have to be assumed. The natural boundary condition for the Navier–Stokes equation is the viscous boundary condition

$$u_{\nu\,\partial\Omega} = 0 \tag{1}$$

and for the Euler equation it turns out to be

$$u_\nu \cdot \vec{n}, \tag{2}$$

which means that the fluid does not penetrate or leave the domain.

The viscous boundary condition is a natural approximation of the effect of a 'rough boundary'. This can be seen as follows. The action of a rough boundary can be described at the level of the Boltzmann equation with the introduction of a scattering kernel $R(v, v')$. Then an asymptotic analysis, with a Knudsen number and a Mach number both of the order of $\epsilon \to 0$ gives a boundary condition for the macroscopic equation of the form $u = O(\epsilon)$ on the boundary. For the Euler equation the tangential component of the velocity is not usually equal to zero. Therefore when the viscosity goes to zero the quantity

$$u_\nu \wedge \vec{n} \tag{3}$$

goes to a non-zero value. The solution cannot remain uniformly smooth near the boundary and this implies that some boundary layer must appear. Since the problem is nonlinear, in many unstable cases this layer of strong vorticity does not remain confined near the boundary, but instead moves into the interior of the domain. Such a situation corresponds to detachment of the boundary layer. This implies that the solution u_ν of the Navier Stokes equation which remains uniformly bounded in energy norm may not converge to a solution of the Euler Equation (Bardos and Ghidaglia 1998). The fact that the u_ν converges in the weak sense in the space

$$L^\infty_{loc}(\Re_t; L^2(\Omega)) \tag{4}$$

to a function u does not implies the relation

$$\lim_{\nu\to 0}(u_\nu \otimes u_\nu) = (u \otimes u) \tag{5}$$

Hilbert space theory shows that when the above relation is not valid one has

$$\lim_{\nu\to 0}(u^i_\nu \otimes u^j_\nu) - (u^i \otimes u^j) = R^{ij} \tag{6}$$

with R^{ij} denoting a symmetric positive tensor, and thus the limit of the Navier Stokes equation may not be the Euler equation, but rather the equation:

$$\frac{\partial u}{\partial t} + \nabla(u \otimes u) + \nabla R = -\nabla p, \ \nabla \cdot u = 0 \tag{7}$$

R plays the role of the Reynolds tensor and it is interesting to notice that in this case its appearance is due only to the fact u is a "weak limit" of solutions of the Navier Stokes equation with viscous boundary condition. No separation of scale and no introduction of a family of solutions nor of ensemble averaging is needed. It is possible that the region where R is not zero (which is not necessarily confined near the boundary) is the turbulent region.

With the divergence free condition the tensor R, which is called the 'defect measure', can be chosen to be symmetric and with zero trace. Such a property is also true for the strain tensor $S(u)$ therefore one can write

$$R = \nu_S(x,t)S(u) + \nu_T(x,t)T(u) \tag{8}$$

with $\nu_S(x,t)$ and $\nu_T(x,t)$ scalar functions and T orthogonal to S (for the canonical scalar product). Even if no space invariance is present, it may be possible that the defect measure, which represents a fluctuation, is rotationally invariant. In this case, the formula (8) should reduce to the formula

$$R = \nu_S(x,t)S(u) \tag{9}$$

leading in (7) to a turbulent diffusion provided that the function $\nu_S(x,t)$ is positive. Such a property should be true in 'reasonable' situations; it is not a direct consequence of the positivity of the tensor R. In fact at present there is no direct way of constructing this 'enhanced diffusion'. The above consideration should only be used as guideline for the study of two dimensional turbulent flows generated by viscous boundaries. Following Chorin (Chorin 1996) one could introduce a fractional step method with a diffusion equation, which would generate vorticity from the boundary, and a treatment of the incompressible Euler equation. To study the propagation of the vorticity this may be similar to the decomposition between the coherent vortices and the incoherent background flow we have proposed in this paper.

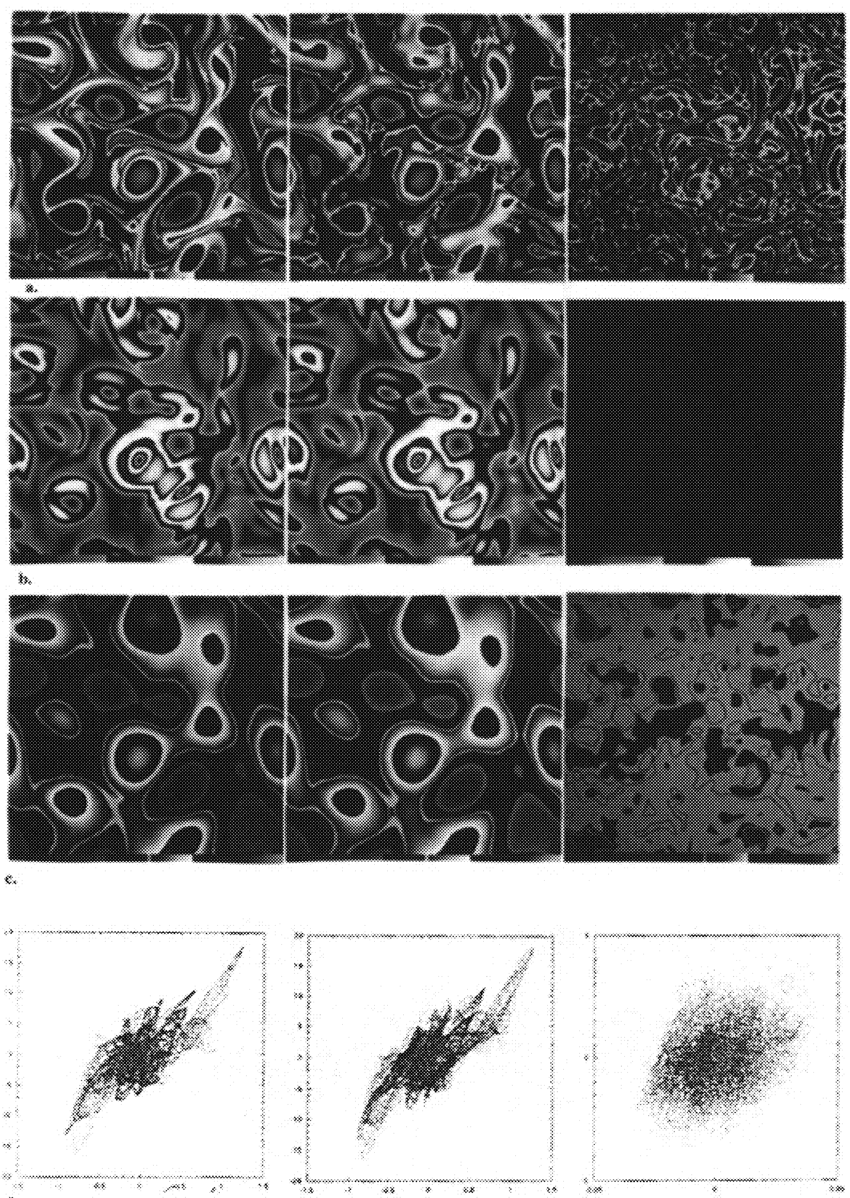
4.2 Physical modelling

Although DNS are the only methods able to compute turbulent flows without resorting to *ad hoc* models, they are currently only of academic interest and have not proved directly useful to engineers. This is due to two drawbacks. First, as mentioned above, such simulations cannot reach the high Reynolds numbers typical of most flows of interest to engineers. We hope that the combination of wavelet-based numerical methods for the Coherent Structure evolution and statistical modelling of the Gaussian background will overcome this limitation. Secondly, DNS are usually limited to simple boundary conditions (e.g. periodic boundary conditions) and geometries (e.g. a rectangular prism). The most realistic DNS manage to compute flow in a rectangular channel with periodic boundary conditions in the streamwise and spanwise directions and no slip conditions in the perpendicular direction (using Chebyshev polynomials). Obviously, a DNS which computes flow in a box with periodic boundary conditions is not useful for calculating the flow over a wing!

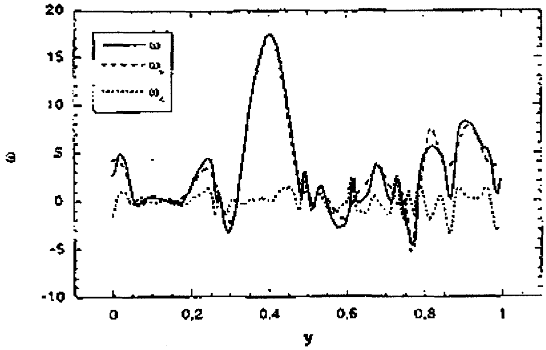
The problems of treating no-slip boundary conditions and complicated geometries at high Reynolds numbers are both theoretical and numerical in nature. The problem is difficult theoretically because the only mathematical theory available for flow over a wall (e.g. Prandtl's law of the wall) fails as soon as the boundary layer detaches. Unfortunately, the boundary layer detaches as soon as the flow becomes turbulent, which severely limits the use of boundary layer theory for describing fully turbulent flow over a wall. There are two main classes of numerical difficulties, the first related to matching the correct boundary conditions at the wall (or edge of the computational domain) and the second related to calculating the flow around complicated shapes. The turbulent boundary layer also demands very high resolution since the boundary layer thickness decreases with Reynolds number according to $\delta \propto Re^{-1/2}$.

Matching boundary conditions requires that the basis function used to represent the solution have the correct behaviour near the wall (e.g. that they are zero at the wall, as is the case for the Chebyshev polynomials mentioned above). This limits the choice of basis functions and means that a set of basis functions that is very efficient and accurate numerically may not be able to be used (this is the reason Chebyshev polynomials were used for the channel flow instead of the more efficient Fourier basis functions). Wavelets can be constructed so as to have the desired properties at the boundaries, and recently an efficient algorithm for constructing wavelets over an interval on an irregular grid has been developed (Sweldens 1996). The fact that wavelets can be easily constructed on irregular grids allows them to increase resolution near the wall in order to fully resolve the thin boundary layer of a fully-developed turbulent flow.

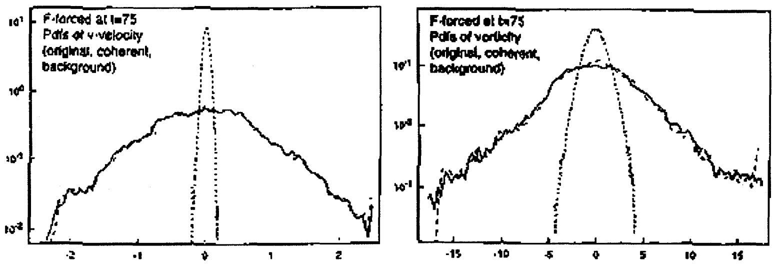
Figure 1. (Next two pages.)
Wavelet compression of vorticity: (a) The vorticity. (b) The modulus of velocity. (c) The stream function. (d) The coherence scatter plot. (e) Cut of vorticity. (f) PDFs of velocity and vorticity. (g) Energy spectrum. The solid lines correspond to the total vorticity ω, the dashed lines to the coherent part $\omega_>$, and the dotted lines to the incoherent part $\omega_<$. We observe that only 0.7% of the total number of wavelet coefficients are sufficient to represent all coherent vortices, while the remaining 99.3% correspond to the incoherent background flow, which is much weaker and homogeneous. The coherent vorticity $\omega_>$ contains 94.3% of the total enstrophy. Moreover, the velocity associated with the coherent vortices is quasi-identical to the total velocity and contains 99.2% of the total energy. As for the coherent stream function, $\psi_>$, it is identical to the total stream function ψ. The fact that the scatter plot of the background, $F_<$ such that $\omega_< = F_<(\psi_<)$, is isotropic proves that our method has extracted all coherent vortices. The PDFs of velocity and vorticity show that only 0.7% of the wavelet coefficients are sufficient to capture the non-Gaussian one-point statistical distributions of vorticity and velocity, while the remaining 99.3% correspond to a Gaussian distribution. The energy spectrum, on the contrary, is dominated at small scales by the incoherent background flow and therefore is insensitive to coherent vortices because they are too rare to affect the energy spectrum (which is the Fourier transform of the two-point correlation function).


a.
b.
c.
d.

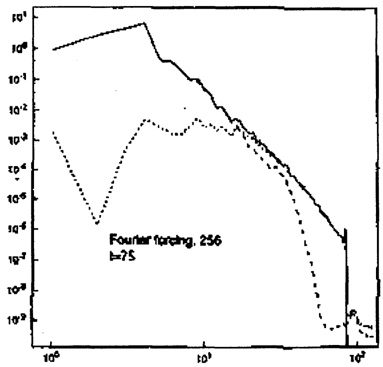
e. *Cut of vorticity*



f. *Velocity probability distribution* *Vorticity probability distribution*



g. *Energy spectrum*



The second problem, that of calculating flow around complicated objects, normally requires the calculation of a specially adapted grid or mesh, that follows the contours of the object and increases resolution where necessary (e.g. near sharp corners). Algorithms for generating such grids for finite element codes exist, but they are not suitable for use with DNS. Furthermore, such grid-based approach are impractical for objects which change their shape over time (e.g. a swimming fish, or a pumping heart). Grids adapted to complex geometries are clearly impossible for Fourier-based methods and are impractical for wavelet-based methods. Ideally, one would like to be able to calculate physically realistic boundary conditions for complicated (possibly time-dependent) geometries while still using a Cartesian grid to represent the solution. In this case the relevant equations are solved over a Cartesian 'virtual' domain which does not necessarily correspond to the geometry of the physical flow.

If the grid is fixed, the only alternative is to change the equations solved, and this is what penalisation methods do. The classic approach is to add an appropriate body force to simulate the presence of the obstacle. This approach has been used by Peskin (1977) and more recently by Goldstein, Handler and Sirovich (1993). The drawback of these methods is that the equations and solution method are rather complicated, and, more importantly, there is no mathematical estimation of the rate of convergence of the approximate solution to the exact solution. This means that one cannot easily estimate the error, or be confident that the solution will converge to the true solution.

A new approach introduced by Angot, Bruneau and Fabrie (1997), based on earlier work by Arquis and Caltagirone (1984) supposes that the fluid is a porous medium and is described by the D'Arcy equations. The interaction of a fluid with a solid is then approximated by letting the porosity go to infinity in the fluid and to zero in the solid. Practically, one solves the following equation

$$\frac{\partial u}{\partial t} + u \cdot \nabla u - \frac{1}{Re}\Delta u + \nabla p = \frac{H(x)u}{\epsilon Re} \tag{10}$$

where $H(x) = 1$ in the solid and $H(x) = 0$ in the fluid. It can be shown theoretically that the error in this approximation decreases like $\epsilon^{3/4}$ (in practice the error is found to be only about ϵ^1). This 'penalisation by artificial porosity' approach allows the flow around complicated objects (which may change shape over time) to be easily computed by changing the mask function $H(x)$. By combining this treatment of complicated geometries with the Coherent structure/random background simulation approach proposed above we hope to develop simulation methods for high Reynolds number flows that will be useful to engineers.

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