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Numerical simulation of a mixing layer in an adaptive wavelet basis

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Abstract. This note presents an adaptive wavelet method to compute two-dimensional turbulent flows. The Navier–Stokes equations in vorticity–velocity form are discretized using a Petrov–Galerkin scheme. The vorticity field is developed into an orthogonal wavelet series where only the most significant coefficients are retained. The testfunctions are adapted to the linear part of the equation so that the resulting stiffness matrix turns out to be the identity. The nonlinear term is evaluated on a locally refined grid in physical space. This numerical scheme is applied to simulate a temporally developing mixing layer. A comparison with a classical pseudo-spectral method is used for validation of the new method. The results show that the formation of Kelvin–Helmholtz vortices is well captured and all scales of the flow are well represented. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

wavelets / two-dimensional Navier–Stokes equation / mixing layer

Simulation numérique d'une couche de mélange en base d'ondelettes adaptative

Abstract. *Cette note présente une méthode de calcul des écoulements turbulents en base d'ondelettes adaptative. Les équations de Navier–Stokes, écrites dans la formulation vorticité-vitesse, sont discrétisées en utilisant un schéma de Petrov–Galerkin. Le champ de vorticité est développé en ondelettes orthogonales en ne retenant que les coefficients les plus significatifs. Les fonctions test sont adaptées à la partie linéaire de l'équation, de telle sorte que la matrice de raideur devienne l'identité. Le terme non linéaire est évalué sur un maillage raffiné localement de façon adaptative. Ce schéma numérique est utilisé pour calculer l'évolution d'une couche de mélange se développant en temps. Cette nouvelle méthode est validée en comparant les résultats obtenus à ceux calculés à l'aide d'une méthode pseudo-spectrale classique. La formation des tourbillons de Kelvin–Helmholtz est bien décrite et toutes les échelles excitées sont prises en compte. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS*

ondelettes / équation de Navier–Stokes bidimensionnelle / couche de mélange

Version française abrégée

Les ondelettes sont des fonctions bien localisées, à la fois dans l'espace physique et dans l'espace spectral. Elles permettent de construire de nouvelles bases orthogonales bien adaptées à l'analyse et à la simulation des écoulements turbulents [1,2]. Nous avons montré qu'un filtrage non linéaire effectué à partir des coefficients d'ondelettes du champ de vorticité permet de séparer les tourbillons cohérents de

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l'écoulement résiduel incohérent. Ainsi, à nombre de coefficients retenus égal, le filtrage non linéaire en ondelettes comprime-t-il mieux l'enstrophie et l'énergie que les filtrages linéaires utilisés classiquement, par exemple pour la méthode de simulation des grandes échelles (LES).

Cette note présente une méthode d'intégration en base d'ondelettes adaptative des équations de Navier–Stokes bidimensionnelles en formulation vorticité–vitesse (1). La discrétisation en temps est effectuée grâce à un schéma semi-implicite du second ordre de type Euler retardé pour le terme de dissipation, et Adams–Bashforth pour le terme de convection (2). Le champ de vorticité est developpé en série d'ondelettes orthogonales (3) en utilisant une méthode de Petrov–Galerkin (6). Les fonctions d'essai, appelées vaguelettes, sont solutions de la partie linéaire de l'équation (5). Ceci évite d'avoir à assembler et à résoudre la matrice de rigidité. A chaque pas de temps on ne retient que les coefficients d'ondelettes $\widetilde{\omega}$ dont la valeur est supérieure à un seuil choisi *a priori*, qui ne dépend que de l'enstrophie totale. On ajoute à l'ensemble des indices ainsi retenus ceux des coefficients d'ondelettes immédiatement voisins, en espace et en échelle. Ceci définit une nouvelle base qui anticipe ainsi le calcul de la solution au pas de temps suivant et s'adapte dynamiquement à l'évolution spatio-spectrale de la vorticité. Le terme non linéaire est évalué par collocation à l'aide d'un maillage localement raffiné, défini à partir des ondelettes retenues après seuillage. La vorticité est reconstruite dans l'espace physique à l'aide de l'algorithme de reconstruction adaptative [3]. En utilisant une décomposition en base de vaguelettes adaptatives, avec $\theta = (\nabla^2)^{-1}\psi$, on intègre $\nabla^2\,\mathbf{\Psi}^\star = \omega^\star$ pour obtenir $\widetilde{\mathbf{\Psi}}^\star$, puis reconstruire $\mathbf{\Psi}^\star$ sur le maillage localement raffiné (*fig. 2*). On calcule ensuite $\nabla\omega^\star$ et $\mathbf{v}^\star = (-\partial_y \mathbf{\Psi}^\star, \partial_x \mathbf{\Psi}^\star)$ à l'aide d'un schéma aux différences finies centrées du 4ième ordre. Finalement on évalue le terme non linéaire sur le maillage localement raffiné et on résout (6) en utilisant la décomposition en base de vaguelettes adaptative.

Nous avons testé cette méthode en étudiant une couche de mélange (*fig. 1*) se développant en temps à partir d'un profil de vitesse en tangente hyperbolique, perturbé par un bruit blanc Gaussien permettant d'induire l'instabilité de Kelvin–Helmholtz et ainsi la formation des tourbillons cohérents. Cette configuration de type couche de mélange se rencontre dans de nombreux écoulements turbulents. Elle a une dynamique fortement non linéaire, ce qui en fait un bon cas-test pour valider de nouvelles méthodes numériques telle celle présentée ici. Nous avons utilisé une résolution maximale $N = 256^2$ et choisi les ondelettes splines cubiques de Battle–Lemarié. Nous avons montré que les résultats obtenus sont semblables à ceux calculés avec une méthode pseudo-spectrale de même résolution. Avec ces deux méthodes on obtient le même nombre de tourbillons, les mêmes spectres (*fig. 2*) et la même structure spatiale de la vorticité (*fig. 3*), ceci tout au long de l'évolution. Seule l'enstrophie totale est légérement sous-estimée par la méthode adaptative en ondelettes (*fig. 4*), sans que cela n'ait cependant d'incidence notable sur la solution. Il est remarquable de constater que, à partir du moment où les tourbillons se sont formés (à l'instant $t = 7$ s), le nombre d'ondelettes nécessaires reste petit et quasi-constant tout au long du calcul. Le nombre de degrés de liberté utilisés reste compris entre 5300 et 4700, donc bien inférieur aux $N = 256^2 = 65536$ degrés de liberté correspondant au maillage homogène le plus fin nécessaire pour la méthode pseudo-spectrale. On constate que les 8% de coefficients retenus sont suffisants pour rendre compte de l'évolution des 90% de l'enstrophie totale.

Bien que la méthode numérique que nous proposons reste Eulérienne, car elle est basée sur une approche de type Galerkin, son caractère adaptatif, à la fois en espace et en échelle, lui permet de suivre les déplacements et les déformations des régions actives de l'écoulement comme le ferait une méthode Lagrangienne.

1. Introduction

Turbulent flows are characterized by their large number of active scales of motion, with increases with the Reynolds number. Hence, for the numerical simulation of fully developed turbulence the complexity has to be considerably reduced and turbulence models are thus unavoidable. In current approaches the fine scales of the flow are replaced by a subgrid scale model, e.g., in Large Eddy Simulation (LES), using a linear cut-off filter which therefore does not depend on the actual flow realization.

Wavelets are functions with simultaneous localization in physical and in Fourier space, which correspond to filters having a constant relative bandwidth. They allow adaptive filtering of signals and are well suited for investigating unsteady, inhomogeneous or intermittent phenomena like those encountered in turbulence.

In the past we have shown that wavelets are an efficient basis to represent turbulent vorticity fields, see, e.g., [1,2]. By means of nonlinear filtering of the wavelet coefficients of vorticity, we can separate the dynamical active part of the flow (i.e., the coherent vortices) from the incoherent background flow. This filtering technique is much more efficient than linear low pass filtering employed in LES, because it retains much more enstrophy and energy for the same number of modes.

From a numerical point of view wavelets constitute optimal bases to represent functions with inhomogeneous regularity, such as intermittent turbulent flow fields. Furthermore, the operators involved in the Navier–Stokes equations can be efficiently compressed in wavelet basis. The existence of fast pyramidal algorithms (with linear complexity), to transform the computed fields between lacunary wavelet coefficients and structured adaptive grids, allows to design efficient methods for solving nonlinear PDEs [3].

This note presents such a wavelet scheme to solve the two-dimensional Navier–Stokes equation in an adaptive wavelet basis, which dynamically adapts to the flow evolution. With respect to previous work [4] the nonlinear term is now evaluated adaptively, by partial collocation using a locally refined grid. This new algorithm is applied to compute a temporally growing mixing layer, which is a typical configuration encountered in many turbulent flows. The results are compared with those obtained with a classical Fourier pseudo-spectral method.

2. Adaptive wavelet method

For numerical simulation of two-dimensional turbulence we consider the Navier–Stokes equations written in velocity–vorticity formulation,

$$\partial_t \omega + \mathbf{v} \cdot \nabla \omega = \nu \nabla^2 \omega, \quad \nabla \cdot \mathbf{v} = 0, \tag{1}$$

with the velocity field $\mathbf{v} = (u, v)$, the vorticity $\omega = \nabla \times \mathbf{v}$ and the kinematic viscosity ν. We assume periodic boundary conditions in both directions.

For the time discretization we use finite differences with a semi-implicit scheme, i.e., Euler-backwards for the viscous term and Adams–Bashforth extrapolation for the nonlinear term, both of second order. We obtain

$$\left(\gamma I - \nu \nabla^2\right) \omega^{n+1} = \frac{4}{3} \gamma \omega^n - \frac{1}{3} \gamma \omega^{n-1} - \mathbf{v}^\star \cdot \nabla \omega^\star, \quad \text{where} \quad \omega^\star = 2\omega^n - \omega^{n-1} \tag{2}$$

with time step Δt, $\gamma = 3/(2\Delta t)$ and I representing the identity. For spatial discretization we use a Petrov–Galerkin scheme. Therefore the vorticity is developed into a set of trial functions and the minimization of the weighted residual of (2) requires that the projection onto a space of testfunctions vanishes.

As space of trial functions we use a two-dimensional multiresolution analysis (MRA) [1] and develop ω^n at time step n into an orthonormal wavelet series, from the largest scale $l_{\max} = 2^0$ to the smallest scale $l_{\min} = 2^{-J}$:

$$\omega^n(x, y) = \overline{\omega}^n_{0,0,0} \phi_{0,0,0}(x, y) + \sum_{j=0}^{J-1} \sum_{i_x=0}^{2^j-1} \sum_{i_y=0}^{2^j-1} \sum_{\mu=1}^{3} \widetilde{\omega}^{\mu,n}_{j,i_x,i_y} \psi^{\mu}_{j,i_x,i_y}(x, y), \tag{3}$$

with $\phi_{j,i_x,i_y}(x,y)=\phi_{j,i_x}(x)\,\phi_{j,i_y}(y)$, and:

$$\psi^{\mu}_{j,i_x,i_y}(x,y)=\begin{cases}\psi_{j,i_x}(x)\,\phi_{j,i_y}(y), & \mu=1\\ \phi_{j,i_x}(x)\,\psi_{j,i_y}(y), & \mu=2\\ \psi_{j,i_x}(x)\,\psi_{j,i_y}(y), & \mu=3\end{cases} \tag{4}$$

where $\phi_{j,i}$ and $\psi_{j,i}$ are the 2π-periodic one-dimensional scaling function and the corresponding wavelet, respectively. Due to orthogonality, the coefficients are given by $\overline{\omega}^{n}_{0,0,0}=\langle\omega^n,\phi_{0,0,0}\rangle$ and $\widetilde{\omega}^{\mu,n}_{j,i_x,i_y}=\langle\omega^n,\psi^{\mu}_{j,i_x,i_y}\rangle$, where $\langle\cdot\,,\cdot\rangle$ denotes the inner product.

The testfunctions θ^{μ}_{j,i_x,i_y} are then defined as solutions of the linear part of equation (2):

$$\big(\gamma I-\nu\nabla^2\big)\theta^{\mu}_{j,i_x,i_y}=\psi^{\mu}_{j,i_x,i_y} \tag{5}$$

This avoids assembling the stiffness matrix and solving a linear equation at each time step. The functions θ, called vaguelettes, are explicitely calculated in Fourier space and have similar localization properties as wavelets.

The solution of (2) therewith reduces to a simple change of basis,

$$\widetilde{\omega}^{\mu,n+1}_{j,i_x,i_y}=\big\langle\omega^{n+1},\psi^{\mu}_{j,i_x,i_y}\big\rangle=\left\langle\frac{4}{3}\gamma\omega^n-\frac{1}{3}\gamma\omega^{n-1}-\mathbf{v}^\star\cdot\nabla\omega^\star,\theta^{\mu}_{j,i_x,i_y}\right\rangle \tag{6}$$

An adaptive discretization is obtained by applying at each time step a nonlinear wavelet thresholding technique, retaining only wavelet coefficients $\widetilde{\omega}^{\mu,n}_{j,i_x,i_y}$ with absolute value above a given threshold $\epsilon=\epsilon_0\sqrt{Z}$, where $Z=\frac{1}{2}\int\omega^2\,\mathrm{d}x$. For the next time step the index coefficient set (which addresses each coefficient in wavelet space) is determined by adding neighbours to the retained wavelet coefficients. Consequently only those coefficients $\widetilde{\omega}$ in (6), belonging to this extrapolated index set, are computed using the adaptive vaguelette decomposition [3].

The nonlinear term $\mathbf{v}^\star\cdot\nabla\omega^\star$ is evaluated by collocation on the locally refined grid. The vorticity $\omega^\star$ is reconstructed in physical space on an adaptive grid from its wavelet coefficients $\widetilde{\omega}^\star$, using the adaptive wavelet reconstruction algorithm [3]. Using the adaptive vaguelette decomposition with $\theta=(\nabla^2)^{-1}\psi$, we solve $\nabla^2\mathbf{\Psi}^\star=\omega^\star$, get $\widetilde{\mathbf{\Psi}}^\star$ and finally reconstruct $\mathbf{\Psi}^\star$ on the refined grid. By means of centered finite differences of 4th order we finally compute $\nabla\omega^\star$ and $\mathbf{v}^\star=(-\partial_y\mathbf{\Psi}^\star,\partial_x\mathbf{\Psi}^\star)$ on the adaptive grid and we evaluate the nonlinear term pointwise. Subsequently (6) can be solved using the adaptive vaguelette decomposition.

Let us mention that the total complexity of the algorithm is of order $O(N_{\mathrm{ad}})$, where N_{ad} denotes the number of wavelet coefficients retained in the adaptive basis. As the present implementation has not yet been optimized, the computing time is as effective as a classical pseudo-spectral method for the resolution used here ($N=256^2$).

3. Numerical results for a mixing layer

As application we consider a temporally developing mixing layer [6], schematically sketched in *fig. 1*. The initial velocity has a hyperbolic-tangent profile $u(y)=U\tanh(2y/\delta_0)$ which implies a vorticity thickness $\delta_0=2U/(\mathrm{d}u/\mathrm{d}y)|_{y=0}$. From linear stability analysis the mixing layer is known to be inviscidly unstable. A perturbation leads to the formation of vortices by Kelvin–Helmholtz instability, where the most amplified mode corresponds to a longitudinal wavelength $\lambda=7\delta_0$ [5]. To trigger the instability we have superimposed a weak white noise in the rotational region.

For the numerical simulation we use a maximal resolution of $N=256^2=65536$, which corresponds to $J=8$ in (3) and cubic spline wavelets of Battle–Lemarié type. The viscosity is $\nu=5\cdot10^{-5}\ \mathrm{m^2s^{-1}}$ and the

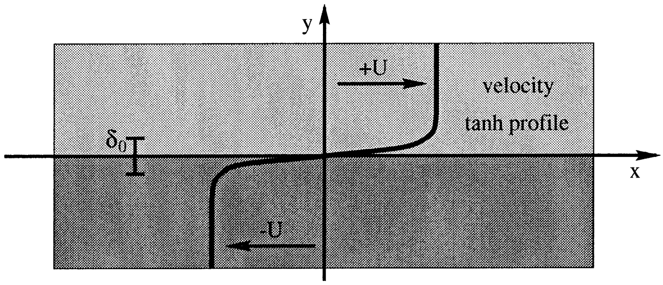
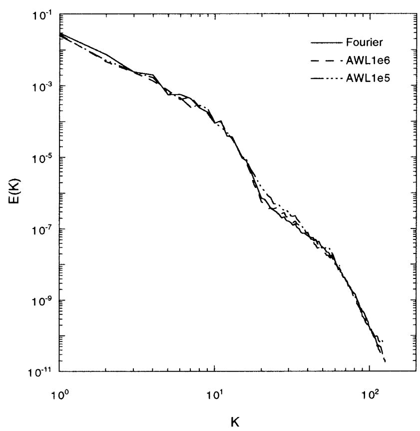


Figure 1. Scheme of initial configuration for the mixing layer.

Figure 1. *Schéma décrivant la condition initiale pour la couche de mélange.*

time step $\Delta t = 2.5 \cdot 10^{-3}$ s. The initial vorticity thickness δ_0 is chosen such that 10 vortices should develop in the numerical domain of size $[0, 2\pi]^2$. In *fig. 2* (left) we compare the energy spectrum at $t = 37.5$ s for a reference run, computed using a classical pseudo-spectral method, and for two wavelet runs computed using different thresholds ($\epsilon_0 = 10^{-6}$ and 10^{-5}). *Fig. 2* (left) shows that all scales of the flow are well resolved for both thresholds. The underlying grid (corresponding to the centers of active wavelets) of the run with $\epsilon_0 = 10^{-6}$ at $t = 37.5$ s shows a local refinement in regions of strong gradients, where dissipation is most active. In *fig. 3* (top) we show the evolution of the vorticity field for the adaptive wavelet simulation with threshold $\epsilon_0 = 10^{-6}$ and for the reference pseudo-spectral run. In both simulations, as predicted by the linear theory, 10 vortices are formed, which subsequently undergo successive mergings. In *fig. 3* (bottom) the active wavelet coefficients (gray entries) are plotted using a logarithmic scale. The coefficients $\widetilde{\omega}^{\mu}_{j,i_x,i_y}$ are placed at position $\{x = 2^j(1 - \delta_{\mu,1}) + i_x,\ y = 2^j(1 - \delta_{\mu,2}) + i_y\}$ with the origin in the lower left corner and the y-coordinate oriented upwards, from coarser to finer scales. We observe that, during the computation, the basis dynamically adapts to the flow evolution, with only $8\,\%$ of the coefficients being used. We observe that in the wavelet simulation the formation and evolution of vortices are well captured, although we find that at later times a slight phase shift appears with respect to the reference run. This is due to the fact that the retained wavelet coefficients contains $94\,\%$ of the total enstrophy, as observed on *fig. 4* (left) which shows the time evolution of the total enstrophy using the different thresholds. This $6\,\%$ loss of enstrophy comes from the fact that in the wavelet simulations we have not modelled the effect of



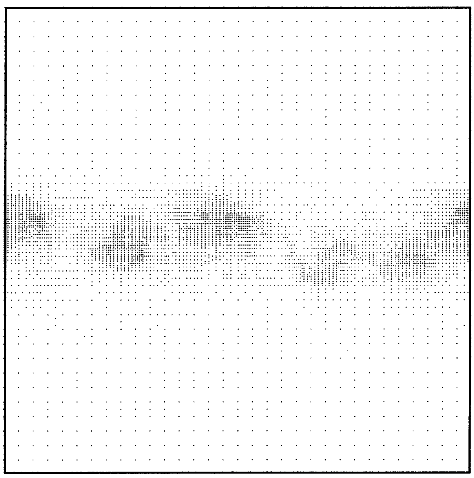


Figure 2. Left: energy spectra for the Fourier reference run and for the adaptive wavelet simulations with thresholds $\epsilon_0 = 10^{-6}, 10^{-5}$. Right: adaptive grid reconstructed from the index set of the retained wavelet coefficients. Both at time $t = 37.5$ s.

Figure 2. *Spectres d'énergie et grille adaptative à* $t = 37{,}5$ s.

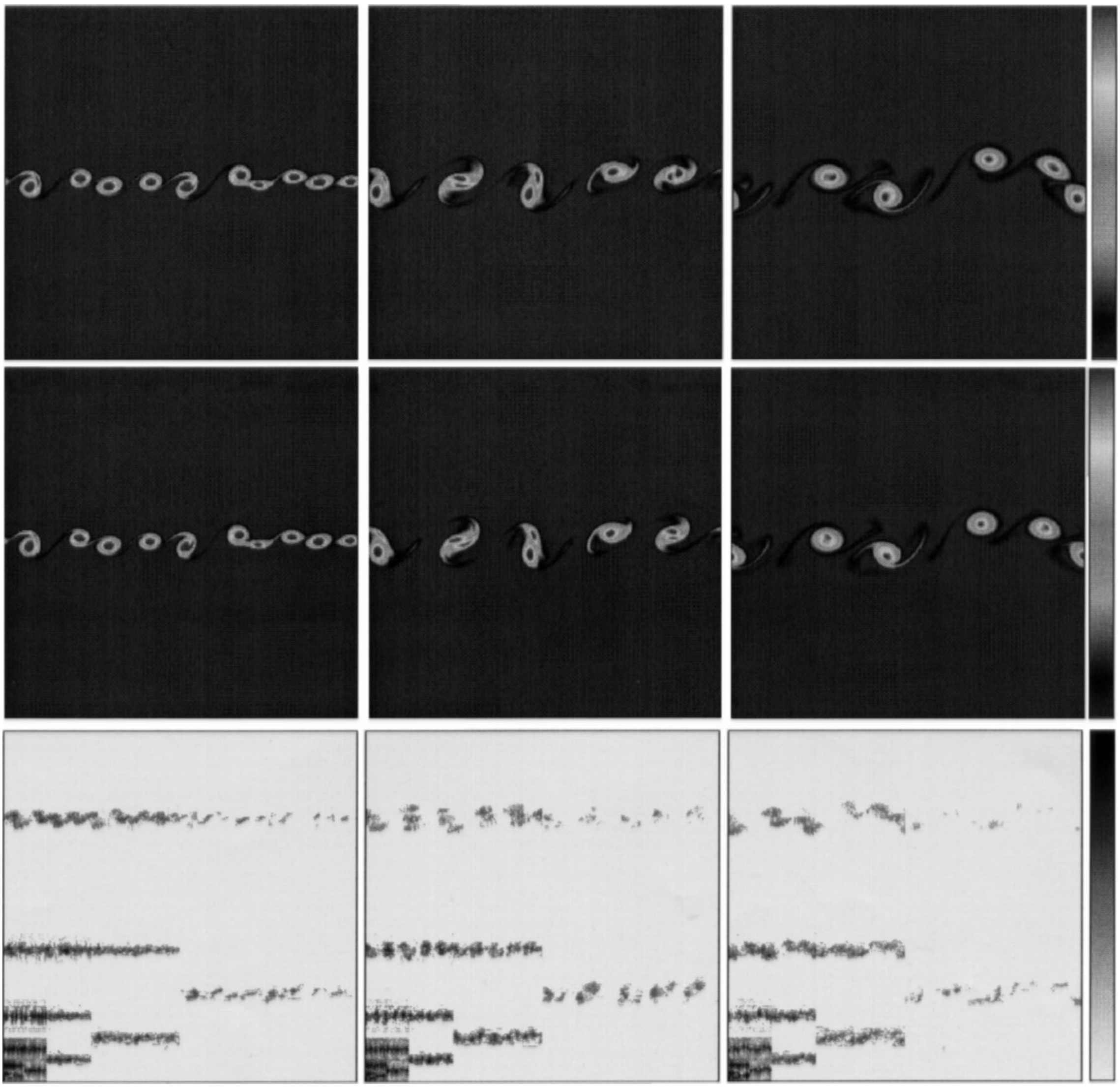


Figure 3. Vorticity field at $t = 12.5,\ 25,\ 37.5$ s. Top: spectral method. Middle: adaptive wavelet method with threshold $\epsilon_0 = 10^{-6}$. Bottom: corresponding wavelet coefficients.

Figure 3. *Champ de vorticité à* $t = 12{,}5,\ 25,\ 37{,}5$ s. *En haut: méthode spectrale. Au milieu: méthode en base d'ondelettes adaptative pour le seuil* $\epsilon_0 = 10^{-6}$. *En bas: coefficients d'ondelettes correspondants.*

the discarded modes onto the retained ones, similarly to the subgrid scale model used in LES. This will be considered in future work, where the enstrophy of the discarded wavelets will be reinjected into the coherent vortices using the wavelet forcing method we have proposed [7]. Finally, we plot the time evolution of the number of degrees of freedom for the two wavelet runs in *fig. 4* (right). First, we observe an initial phase, up to $t = 7$ s, where there is a strong reduction in the number of active modes, which corresponds to the formation of the coherent vortices. Then, the number of active modes remains almost constant, which represents a significant reduction of the number of modes, with $N_{\rm ad} = 5000$ for $\epsilon_0 = 10^{-6}$ and $N_{\rm ad} = 2000$ for $\epsilon_0 = 10^{-5}$ out of $N = 65536$ initial modes.

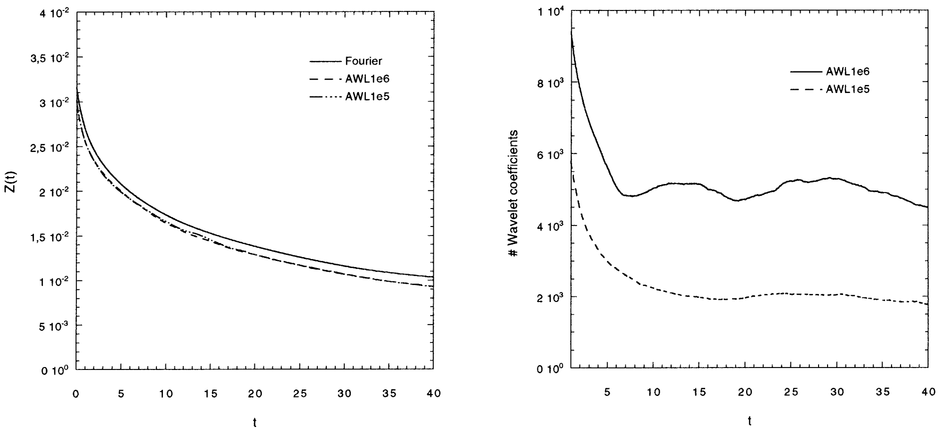


Figure 4. Left: Evolution of enstrophy. Right: Evolution of active wavelet coefficients. For the adaptive wavelet method with $\epsilon_0 = 10^{-6}, 10^{-5}$.

Figure 4. *Evolution de l'enstrophie et du nombre de coefficients d'ondelettes.*

In conclusion, although the numerical method we propose is Eulerian, based on a Galerkin scheme, its adaptive character, in both space and scale, allows us to track the displacements and deformations of active flow regions, as Lagrangian methods would do. This method can also be applied to compute other flow configurations, such as jets or wakes, where coherent vortices play an important dynamical role.

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