Extraction of coherent vortex tubes in a 3D turbulent mixing layer using orthogonal wavelets

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Abstract We present a new technique to extract coherent vortex tubes out of turbulent flows. The method is based on an orthogonal vector-valued wavelet decomposition of the vorticity field using the fast wavelet transform. A nonlinear thresholding of the wavelet coefficients is applied, where the threshold depends on the Reynolds number and on the total enstrophy of the flow, only. The coherent vortex tubes are reconstructed from the strong wavelet coefficients while the remaining weak coefficients correspond to an incoherent background flow. As example we present an application of this method to a turbulent mixing layer computed by high resolution direct numerical simulation. We find that only few wavelet coefficients are neccesary to represent the coherent vortex tubes of the flow. The incoherent background flow reconstructed from the remaining weak coefficients is structureless and exhibits an energy equipartition.

Of wavelets I'm an adherent;
Though some people think me quite errant,
Like the structures extracted
By techniques compacted,
My lecture's entirely coherent.
—H.K. Moffatt, 2001

1. Introduction

Many turbulent flows exhibit organized structures (*e.g.*, Jimenez & Wray (1993)) evolving in an unorganized random background. A separation of the flow into these two components is a prerequisite for a sound physical modelling of turbulence. Since these coherent vortices are well localized and excited on a wide range of scales, we have proposed to use the wavelet representation of the vorticity field to analyze (Farge (1992)), to extract (Farge, Schneider & Kevlahan 1999) and to compute them (Schneider & Farge 2000). Recently, we generalized the vortex extraction technique for three-dimensional flows using a vector-valued wavelet decomposition (Farge, Pellegrino & Schneider 2001). Orthogonal wavelet bases are well suited for these tasks, because they are made of self-similar functions well localized in both physical and spectral spaces (Daubechies (1992), Farge (1992)) leading to an efficient representation of intermittent data, such as turbulent flow fields.

The vortex extraction method is based on an orthogonal wavelet decomposition of the vorticity field, a subsequent thresholding of the wavelet coefficients and a reconstruction from those whose modulus is above the given threshold. Its value is motivated by mathematical theorems yielding optimal min-max estimators for denoising of intermittent data (Donoho 1993). It depends on the enstrophy of flow and the Reynolds number only. In (Farge, Schneider & Kevlahan 1999) we showed that few strong wavelet coefficients represent the organized part of the flow, *i.e.*, the coherent vortices. The remaining many weak wavelet coefficients represent the background flow which is structureless and may be modelled by some stochastic process.

After a short presentation of the wavelet based vortex extraction algorithm, we demonstrate its efficieny by applying it to a turbulent mixing layer computed my means of high resolution direct numerical simulation. Finally, we conclude and give some perspectives for modelling turbulent flows.

2. Wavelet algorithm for vortex extraction

We consider the vorticity field $\vec{\omega} = \nabla \times \vec{v}$ at a given time t, $\vec{v}$ being the velocity field, computed at resolution $N = 2^{3J}$, where N is the number of grid points and J the number of dyadic scales. We use a three-dimensional vector-valued Multi-Resolution Analysis (MRA) of $\left(L^2(R^3)\right)^3$, *i.e.*, a set of nested subspaces $\vec{V}_j \subset \vec{V}_{j+1}$ for $j = 0, ... J-1$, representating the flow at different scales $l = 2^{-j}$. Considering the orthogonal complement spaces $\vec{W}_j = \vec{V}_{j+1} - \vec{V}_j$, we obtain a wavelet representation. Therefore the vorticity vector is developed into an orthogonal wavelet series,

$$\vec{\omega}(\vec{x}) \;=\; \vec{\omega}_{0,0,0}\,\phi_{0,0,0}(\vec{x}) \;+\; \sum_{j=0}^{J-1}\sum_{i_x=0}^{2^j-1}\sum_{i_y=0}^{2^j-1}\sum_{i_z=0}^{2^j-1}\sum_{\mu=1}^{2^n-1} \vec{\omega}^{\mu}_{j,i_x,i_y,i_z}\,\psi^{\mu}_{j,i_x,i_y,i_z}(\vec{x})\,, \tag{1}$$

with $\phi_{j,i_x,i_y i,i_z}(\vec{x}) = \phi_{j,i_x}(x)\ \phi_{j,i_y}(y)\ \phi_{j,i_z}(z)$, and

$$\psi^{\mu}_{j,i_x,i_y,i_z}(\vec{x}) = \begin{cases} \psi_{j,i_x}(x)\ \phi_{j,i_y}(y)\ \phi_{j,i_z}(z) & ;\ \mu=1 \quad , \\ \phi_{j,i_x}(x)\ \psi_{j,i_y}(y)\ \phi_{j,i_z}(z) & ;\ \mu=2 \quad , \\ \phi_{j,i_x}(x)\ \phi_{j,i_y}(y)\ \psi_{j,i_z}(z) & ;\ \mu=3 \quad , \\ \psi_{j,i_x}(x)\ \phi_{j,i_y}(y)\ \psi_{j,i_z}(z) & ;\ \mu=4 \quad , \\ \psi_{j,i_x}(x)\ \psi_{j,i_y}(y)\ \phi_{j,i_z}(z) & ;\ \mu=5 \quad , \\ \phi_{j,i_x}(x)\ \psi_{j,i_y}(y)\ \psi_{j,i_z}(z) & ;\ \mu=6 \quad , \\ \psi_{j,i_x}(x)\ \psi_{j,i_y}(y)\ \psi_{j,i_z}(z) & ;\ \mu=7 \quad , \end{cases} \tag{2}$$

where $\phi_{j,i}$ and $\psi_{j,i}$ are the one-dimensional scaling function and the corresponding wavelet, respectively. Due to orthogonality, the scaling coefficients are given by $\bar{\vec{\omega}}_{0,0,0} = \langle \vec{\omega}\,,\ \phi_{0,0,0} \rangle$ and the wavelet coefficients are given by $\tilde{\vec{\omega}}^{\mu}_{j,i_x,i_y,i_z} = \langle \vec{\omega}\,,\ \psi^{\mu}_{j,i_x,i_y,i_z} \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the L^2-inner product.

The extraction algorithm can be summarized as follows:

- given $\vec{\omega}(\vec{x})$, sampled on a grid (x_i, y_j, z_k) for $i,j,k = 0, N-1$, and the total enstrophy $Z = \frac{1}{2}\int |\vec{\omega}|^2 d\vec{x}$,

- perform the three-dimensional wavelet decomposition (*i.e.*, apply the Fast Wavelet Transform to each component of $\vec{\omega}$) to obtain $\tilde{\vec{\omega}}^{\mu}_{j,i_x,i_y,i_z}$ for $j = 0, J-1, i_x, i_y, i_z = 0, 2^{J-1}-1, \mu = 1, ..., 7$,

- compute the threshold $\epsilon_T = (4/3 Z \log_e N)^{1/2}$ and threshold the coefficients $\tilde{\vec{\omega}}$ to obtain

$$\tilde{\vec{\omega}}_C = \begin{cases} \tilde{\vec{\omega}} & \text{for } |\tilde{\vec{\omega}}| > \epsilon_T \\ 0 & \text{for else} \end{cases} \qquad \tilde{\vec{\omega}}_I = \begin{cases} \tilde{\vec{\omega}} & \text{for } |\tilde{\vec{\omega}}| \le \epsilon_T \\ 0 & \text{for else,} \end{cases} \tag{3}$$

- perform the three-dimensional wavelet reconstruction (*i.e.*, apply the inverse Fast Wavelet Transform) to compute $\vec{\omega}_C$ and $\vec{\omega}_I$ from $\tilde{\vec{\omega}}_C$ and $\tilde{\vec{\omega}}_I$, respectively,

- use Biot-Savart's relation $\vec{V} = (\nabla\times)^{-1}\vec{\omega}$ to reconstruct the coherent and incoherent velocity fields from the coherent and incoherent vorticity fields, respectively.

Note that the decomposition of $\vec{\omega} = \vec{\omega}_C + \vec{\omega}_I$ is orthogonal and hence it follows that $Z = Z_C + Z_C$.

The complexity of the Fast Wavelet Transform (FWT) is of $O(N)$, where N denotes the total number of grid points.

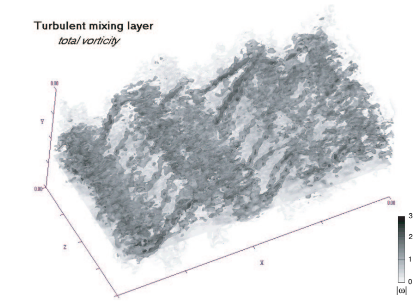


Figure 1. Isosurfaces of total vorticity of the forced 3D mixing layer.

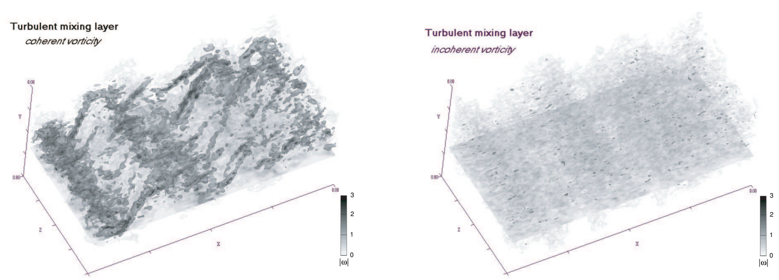
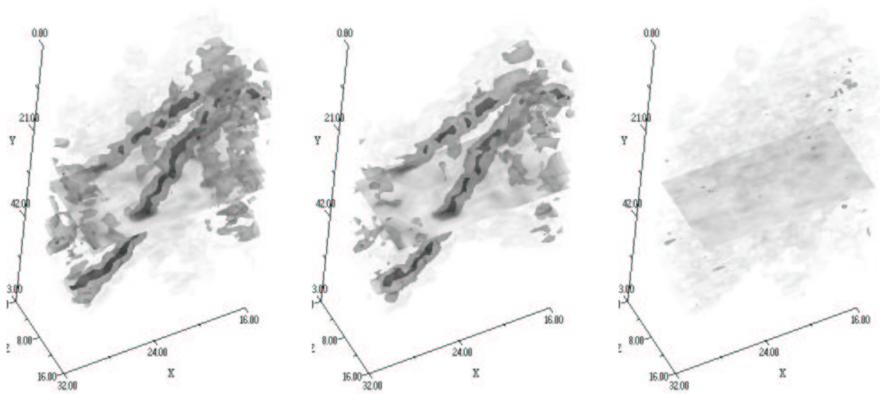


Figure 2. Forced mixing layer: coherent vorticity (left) and incoherent vorticity (right), using the same grey scale as the previous figure.

3. Application to a turbulent mixing layer

Here we apply the above algorithm to a forced three dimensional turbulent mixing layer calculated by high resolution direct numerical simulation (Rogers & Moser 1994) evolved for 40 eddy turn over times. The modulus of vorticity for the total flow is shown in Fig. 1. We observe four transverse rollers, which are produced by the the 2D Kelvin–Helmholtz instability, together with well pronounced longitudinal vortex tubes, called ribs, resulting from three-dimensional instability. In Fig. 2 (left) we plot the coherent vorticity, reconstructed from 3% of the total number N of wavelet coefficients. The incoherent vorticity (Fig. 2, right), reconstructed from 97% of the wavelet coefficients, contains 1% of the turbulent kinetic energy and 82% of the enstrophy. It is nearly homogeneous with very weak amplitude and contains no structure.

Now we focus on the extraction of two vortex tubes, *i.e.*, ribs in between two rollers. In Fig. 3 we plot the isosurfaces of the vorticity modulus for the total flow, the coherent part (3% of the wavelet coefficients) and the incoherent part (97% of the wavelet coefficients). We observe that the coherent flow almost perfectly preserves the vortex tube present in the total flow, while the incoherent flow is structureless.



a) b) c)

Figure 3. Extraction of two ribs (Zoom of Fig.1 and 2). Isosurfaces of vorticity modulus for a) total flow, b) coherent part, c) incoherent part using the same grey scale.

The corresponding longitudinal energy spectra in Fig. 4 show that the coherent part exhibits the same power-law behaviour as the total flow over the whole inertial range. In contrast, the incoherent flow has a flat energy spectrum, *i.e.*, an equipartition of energy, which means in other words that it is decorrelated. The PDFs of vorticity show that the coherent part preserves the same strongly non Gaussian behaviour as the total flow, in particular its extreme values. For the incoherent flow we observe a strongly reduced variance of the vorticity PDF which shows an exponential behaviour.

4. Perspectives

These results give the motivation to develop a new turbulence model, called CVS (Coherent Vortex Simulation), where the evolution of the coherent part of the flow (vortex tubes) is deterministically computed in an adaptive wavelet basis, while the the influence of the incoherent flow onto the coherent one is statistically modelled (Farge, Schneider & Kevlahan 1999; Schneider & Farge 2000; Farge & Schneider 2001).

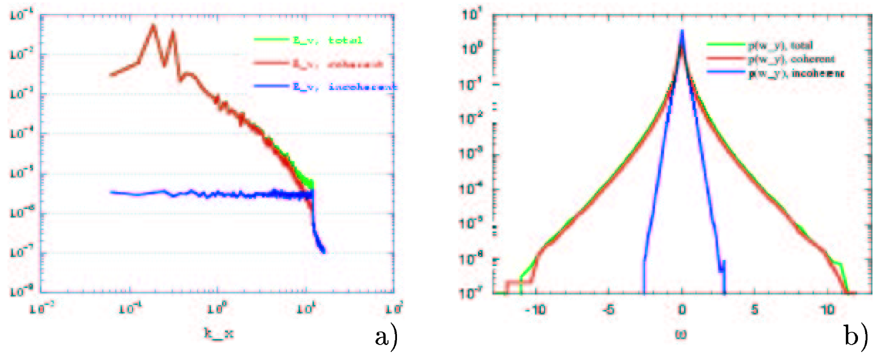


Figure 4. a) Energy spectra in the longitudinal direction and b) vorticity PDFs.

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