

# Extraction of coherent vortex tubes in a 3D turbulent mixing layer using orthogonal wavelets

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**Abstract** We present a new technique to extract coherent vortex tubes out of turbulent flows. The method is based on an orthogonal vector-valued wavelet decomposition of the vorticity field using the fast wavelet transform. A nonlinear thresholding

use the wavelet representation of the vorticity field to analyze (Farge (1992)), to extract (Farge, Schneider & Kevlahan 1999) and to compute them (Schneider & Farge 2000). Recently, we generalized the vortex extraction technique for three-dimensional flows using a vector-valued wavelet decomposition (Farge, Pellegrino & Schneider 2001). Orthogonal wavelet bases are well suited for these tasks, because they are made of self-similar functions well localized in both physical and spectral spaces (Daubechies (1992), Farge (1992)) leading to an efficient representation of intermittent data, such as turbulent flow fields.

The vortex extraction method is based on an orthogonal wavelet decomposition of the vorticity field, a subsequent thresholding of the wavelet coefficients and a reconstruction from those whose modulus is above the given threshold. Its value is motivated by mathematical theorems yielding optimal min-max estimators for denoising of intermittent data (Donoho 1993). It depends on the enstrophy of flow and the Reynolds number only. In (Farge, Schneider & Kevlahan 1999) we showed that few strong wavelet coefficients represent the organized part of the flow, *i.e.*, the coherent vortices. The remaining many weak wavelet coefficients represent the background flow which is structureless and may be modelled by some stochastic process.

After a short presentation of the wavelet based vortex extraction algorithm, we demonstrate its efficiency by applying it to a turbulent mixing layer computed by means of high resolution direct numerical simulation. Finally, we conclude and give some perspectives for modelling turbulent flows.

## 2. Wavelet algorithm for vortex extraction

We consider the vorticity field  $\vec{\omega} = \nabla \times \vec{v}$  at a given time  $t$ ,  $\vec{v}$  being the velocity field, computed at resolution  $N = 2^{3J}$ , where  $N$  is the number of grid points and  $J$  the number of dyadic scales. We use a three-dimensional vector-valued Multi-Resolution Analysis (MRA) of  $(L^2(\mathbb{R}^3))^3$ , *i.e.*, a set of nested subspaces  $\vec{V}_j \subset \vec{V}_{j+1}$  for  $j = 0, \dots, J-1$ , representing the flow at different scales  $l = 2^{-j}$ . Considering the orthogonal complement spaces  $\vec{W}_j = \vec{V}_{j+1} - \vec{V}_j$ , we obtain a wavelet representation. Therefore the vorticity vector is developed into an orthogonal wavelet series,

$$\vec{\omega}(\vec{x}) = \vec{\omega}_{0,0,0} \phi_{0,0,0}(\vec{x}) + \sum_{j=0}^{J-1} \sum_{i_x=0}^{2^j-1} \sum_{i_y=0}^{2^j-1} \sum_{i_z=0}^{2^j-1} \sum_{\mu=1}^{2^n-1} \vec{\omega}_{j,i_x,i_y,i_z}^{\mu} \psi_{j,i_x,i_y,i_z}^{\mu}(\vec{x}), \quad (1)$$

with  $\phi_{j,i_x,i_y,i_z}(\vec{x}) = \phi_{j,i_x}(x) \phi_{j,i_y}(y) \phi_{j,i_z}(z)$ , and

$$\psi_{j,i_x,i_y,i_z}^\mu(\vec{x}) = \begin{cases} \psi_{j,i_x}(x) \phi_{j,i_y}(y) \phi_{j,i_z}(z) & ; \mu = 1 \\ \phi_{j,i_x}(x) \psi_{j,i_y}(y) \phi_{j,i_z}(z) & ; \mu = 2 \\ \phi_{j,i_x}(x) \phi_{j,i_y}(y) \psi_{j,i_z}(z) & ; \mu = 3 \\ \psi_{j,i_x}(x) \phi_{j,i_y}(y) \psi_{j,i_z}(z) & ; \mu = 4 \\ \psi_{j,i_x}(x) \psi_{j,i_y}(y) \phi_{j,i_z}(z) & ; \mu = 5 \\ \phi_{j,i_x}(x) \psi_{j,i_y}(y) \psi_{j,i_z}(z) & ; \mu = 6 \\ \psi_{j,i_x}(x) \psi_{j,i_y}(y) \psi_{j,i_z}(z) & ; \mu = 7 \end{cases} \quad (2)$$

where  $\phi_{j,i}$  and  $\psi_{j,i}$  are the one-dimensional scaling function and the corresponding wavelet, respectively. Due to orthogonality, the scaling coefficients are given by  $\bar{\omega}_{0,0,0} = \langle \vec{\omega}, \phi_{0,0,0} \rangle$  and the wavelet coefficients are given by  $\tilde{\omega}_{j,i_x,i_y,i_z}^\mu = \langle \vec{\omega}, \psi_{j,i_x,i_y,i_z}^\mu \rangle$ , where  $\langle \cdot, \cdot \rangle$  denotes the  $L^2$ -inner product.

The extraction algorithm can be summarized as follows:

- given  $\vec{\omega}(\vec{x})$ , sampled on a grid  $(x_i, y_j, z_k)$  for  $i, j, k = 0, N-1$ , and the total enstrophy  $Z = \frac{1}{2} \int |\vec{\omega}|^2 d\vec{x}$ ,
- perform the three-dimensional wavelet decomposition (*i.e.*, apply the Fast Wavelet Transform to each component of  $\vec{\omega}$ ) to obtain  $\tilde{\omega}_{j,i_x,i_y,i_z}^\mu$  for  $j = 0, J-1, i_x, i_y, i_z = 0, 2^{J-1}-1, \mu = 1, \dots, 7$ ,
- compute the threshold  $\epsilon_T = (4/3Z \log_e N)^{1/2}$  and threshold the coefficients  $\tilde{\omega}$  to obtain

$$\tilde{\omega}_C = \begin{cases} \tilde{\omega} & \text{for } |\tilde{\omega}| > \epsilon_T \\ 0 & \text{for else} \end{cases} \quad \tilde{\omega}_I = \begin{cases} \tilde{\omega} & \text{for } |\tilde{\omega}| \leq \epsilon_T \\ 0 & \text{for else,} \end{cases} \quad (3)$$

- perform the three-dimensional wavelet reconstruction (*i.e.*, apply the inverse Fast Wavelet Transform) to compute  $\vec{\omega}_C$  and  $\vec{\omega}_I$  from  $\tilde{\omega}_C$  and  $\tilde{\omega}_I$ , respectively,
- use Biot-Savart's relation  $\vec{V} = (\nabla \times)^{-1} \vec{\omega}$  to reconstruct the coherent and incoherent velocity fields from the coherent and incoherent vorticity fields, respectively.

Note that the decomposition of  $\vec{\omega} = \vec{\omega}_C + \vec{\omega}_I$  is orthogonal and hence it follows that  $Z = Z_C + Z_I$ .

The complexity of the Fast Wavelet Transform (FWT) is of  $O(N)$ , where  $N$  denotes the total number of grid points.

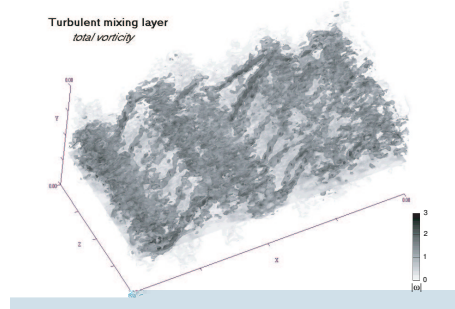


Figure 1. Isosurfaces of total vorticity of the forced 3D mixing layer.

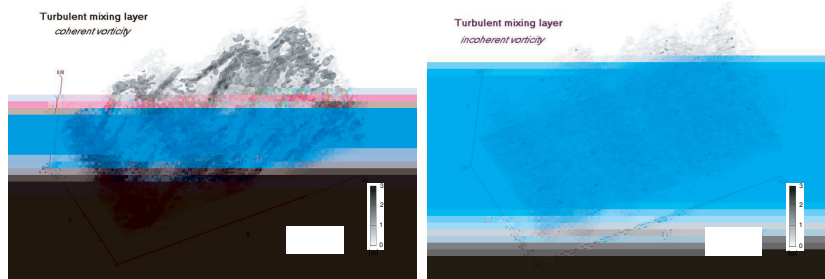


Figure 2. Forced mixing layer: coherent vorticity (left) and incoherent vorticity (right), using the same grey scale as the previous figure.

### 3. Application to a turbulent mixing layer

Here we apply the above algorithm to a forced three-dimensional turbulent mixing layer calculated by high-resolution direct numerical simulation (Rogers & Moser 1994) evolved for 40 eddy turnover times. The modulus of vorticity for the total flow is shown in Fig. 1. We observe four transverse rolls, which are produced by the 2D Kelvin-Helmholtz instability, together with well-pronounced longitudinal vortex tubes, called ribs, resulting from the three-dimensional instability. In Fig. 2 (left) we plot the coherent vorticity, reconstructed from 3% of the total number  $N$  of wavenumber coefficients. The incoherent vorticity (Fig. 2, right), reconstructed from 97% of the wavenumber coefficients, contains 1% of the turbulent kinetic energy and 82% of the enstrophy. It is nearly homogeneous with very weak amplitude and contains no structure.

Now we focus on the extraction of two vortex tubes, *i.e.*, ribs in between two rolls. In Fig. 3 we plot the isosurfaces of the vorticity mod-

ulus for the total flow, the coherent part (3% of the wavelet coefficients) and the incoherent part (97% of the wavelet coefficients). We observe that the coherent flow almost perfectly preserves the vortex tube present in the total flow while the incoherent flow is structureless

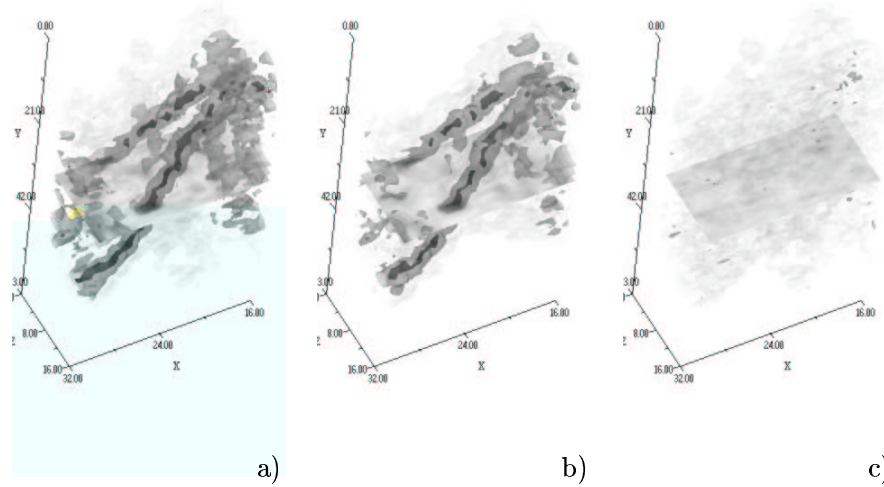


Figure 3. Extraction of two ribs (Zoom of Fig.1 and 2). Isosurfaces of vorticity modulus for a) total flow, b) coherent part, c) incoherent part using the same grey scale.

The corresponding longitudinal energy spectra in Fig. 4 show that the coherent part exhibits the same power-law behaviour as the total flow over the whole inertial range. In contrast, the incoherent flow has a flat energy spectrum, *i.e.*, an equipartition of energy, which means in other words that it is decorrelated. The PDFs of vorticity show that the coherent part preserves the same strongly non Gaussian behaviour as the total flow, in particular its extreme values. For the incoherent flow we observe a strongly reduced variance of the vorticity PDF which shows an exponential behaviour.

#### 4. Perspectives

These results give the motivation to develop a new turbulence model, called CVS (Coherent Vortex Simulation), where the evolution of the coherent part of the flow (vortex tubes) is deterministically computed in an adaptive wavelet basis, while the the influence of the incoherent flow onto the coherent one is statistically modelled (Farge, Schneider & Kevlahan 1999; Schneider & Farge 2000; Farge & Schneider 2001).

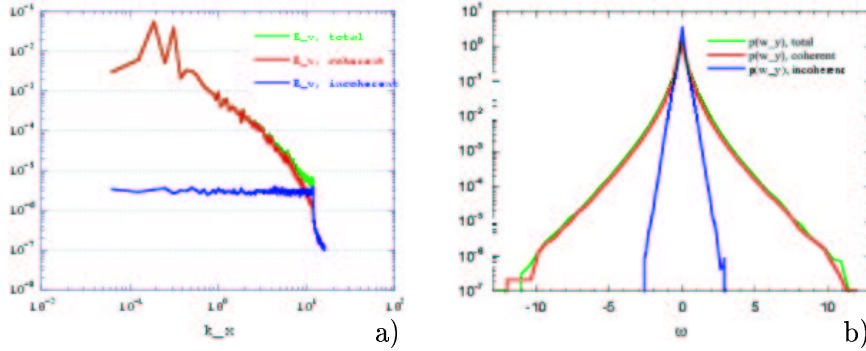


Figure 4. a) Energy spectra in the longitudinal direction and b) vorticity PDFs.

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