Coherent Vortex Extraction from Three-Dimensional Homogeneous Isotropic Turbulence : Comparison of Wavelet and Fourier Nonlinear Filtering Methods

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Two nonlinear filtering methods for extracting coherent structures from direct numerical simulation data of isotropic turbulence are compared. One is based on the orthonormal wavelet decomposition, called coherent vortex extraction (CVE) method, and the other is based on the Fourier decomposition, called Fourier nonlinear filtering (FNF) method. We examine contributions of structures extracted by applying these methods to turbulent flows. Coherent structures extracted by using the CVE method, reconstructed from only 3.6% of the degrees of freedom, preserve the vortex tubes as well as statistical quantities of turbulence. In contrast, the structures extracted by using the FNF method, represented by 7.3% of the degrees of freedom, only well preserve low-order statistical quantities, e.g., energy, enstrophy and the energy spectrum in the inertial range, but smooth out the vortex tubes.

1. INTRODUCTION

Extraction and characterization of coherent structures in turbulence, e.g., vortex tubes, is a long lasting problem. A sound understanding of their dynamics is a prerequisite for accurate and physically-based modeling. The structures exhibit strong intermittency in physical space. The intermittency can be defined as the lacunarity of their fine scale activity, which means that the spatial support of the active regions decreases with scale.

Wavelet analysis is a powerful tool which keeps track of both location and scale of intermittent fields and allows their sparse representation. A wavelet nonlinear filtering method for extracting coherent vortices out of hydrodynamic turbulent fields, called coherent vortex extraction (CVE) method, was proposed[1,2]. It is based on orthonormal wavelets which yield a non-redundant representation and for which fast transformation algorithms are available. The CVE method allows us to divide the vorticity field into two orthogonal parts, coherent and incoherent vorticity. The coherent vorticity reconstructed from few wavelet coefficients of vorticity, whose moduli are above a threshold, which is motivated from denoising theory[3], contains the coherent vortex tubes, and exhibits statistics similar to those of the total vorticity. The incoherent vorticity reconstructed from the remaining large majority of the wavelet coefficients corresponds to an almost uncorrelated random background flow with quasi-Gaussian one-point statistics. The CVE method was applied to direct numerical simulation (DNS) data of three-dimensional homogeneous isotropic turbulence in a periodic box[2,4,5], which is one of the most canonical turbulent flows to explore universality of small scale turbulence and to develop turbulent modeling. It was shown that only about 3% of the wavelet coefficients of vorticity represent the coherent vortex tubes, and that they well preserve statistics of the turbulent flow.

Fields satisfying periodic boundary conditions can also be represented by Fourier series. As done in CVE with the wavelet coefficients, we can split the Fourier coefficients of vorticity into two sets, i.e., weak and strong coefficients depending on their magnitude. Subsequent reconstruction of both of the coefficient sets thus yields two orthogonal fields in physical space. Therewith, coherent vortices can be extracted by the reconstruction of the strong coefficients. Hereafter, we call this method Fourier nonlinear filtering (FNF) method. Note that, in spectral space, we perfectly keep information of wavenumber but lose that of location. Therefore, the FNF method is less suited to extraction of coherent structures from turbulence. Fourier linear filtering was compared with CVE in Farge *et. al.* [4]. Note that a filtering method is linear if a filter operator F satisfies $F[\boldsymbol{g}(\boldsymbol{x})+\boldsymbol{h}(\boldsymbol{x})]=F[\boldsymbol{g}(\boldsymbol{x})]+F[\boldsymbol{h}(\boldsymbol{x})]$ and $F[\xi\boldsymbol{g}(\boldsymbol{x})]=\xi F[\boldsymbol{g}(\boldsymbol{x})]$, where $\boldsymbol{g}(\boldsymbol{x})$ and $\boldsymbol{h}(\boldsymbol{x})$ are any three-dimensional vector fields, and ξ is any scalar. On the other hand, a nonlinear filtering method, e.g. CVE or FNF, does not satisfy the above relations.

In this paper, we compare the two nonlinear filtering methods and quantitatively show the advantage of CVE with respect to FNF. These methods are applied to DNS data[6] of three-dimensional homogeneous isotropic turbulence computed in a periodic box with 256^3 grid points and a Taylor micro-scale Reynolds number of R_λ=167. Contributions of the extracted structures to turbulent statistics are examined through their visualizations and assessment of their statistical properties.

2. EXTRACTION OF COHERENT VORTICES FROM TURBULENCE

In this section, we briefly summarize the three-dimensional orthonormal wavelet decomposition used in CVE, and describe the FNF method.

2.1 ORTHONORMAL WAVELET ANALYSIS

Let us consider a three-dimensional vector field $\boldsymbol{v}(\boldsymbol{x})=(v_1(\boldsymbol{x}),v_2(\boldsymbol{x}),v_3(\boldsymbol{x}))$ with $\boldsymbol{x}\in\boldsymbol{T}^3=[0,2\pi]^3$ where $\boldsymbol{x}=(x_1,x_2,x_3)$. Three-dimensional orthonormal wavelet analysis unfolds $\boldsymbol{v}$ into scale, positions and directions, using a mother wavelet $\Psi_m(\boldsymbol{x})$ which is constructed as $\Psi_m(\boldsymbol{x})=\psi_\alpha(x_1)\psi_\beta(x_2)\psi_\gamma(x_3)$ ($\alpha,\beta,\gamma=0,1$, $m=\alpha+2\beta+4\gamma$ and $m=1,2,\cdots,7$) by tensor product of a one-dimensional scaling function $\psi_0(x)$ and a mother wavelet $\psi_1(x)$.

Each component $v_\ell(\boldsymbol{x})$ of the field sampled on 2^J equidistant grid points in each space direction of the Cartesian coordinates can thus be decomposed into an orthogonal wavelet series:

$$v_\ell(\boldsymbol{x})=\overline{v_\ell}+\sum_{\varsigma=1}^{J}\sum_{\iota_1,\iota_2,\iota_3=0}^{2^{\varsigma-1}-1}W_{m,\boldsymbol{\iota}}^{\varsigma}[v_\ell]\Psi_{m,\boldsymbol{\iota}}^{\varsigma}(\boldsymbol{x}), \tag{1}$$

where $\overline{v_\ell}=\int_{T^3}v_\ell(\boldsymbol{x})\mathrm{d}\boldsymbol{x}/(2\pi)^3$, $W_{m,\boldsymbol{\iota}}^{\varsigma}[v_\ell]=\int_{T^3}v_\ell(\boldsymbol{x})\Psi_{m,\boldsymbol{\iota}}^{\varsigma}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$, $\Psi_{m,\boldsymbol{\iota}}^{\varsigma}(\boldsymbol{x})=2^{3\varsigma/2}\Psi_m(2^\varsigma\boldsymbol{x}-2\pi\,\boldsymbol{\iota})$, and $\boldsymbol{\iota}=(\iota_1,\iota_2,\iota_3)$. Einstein's summation convention is used for repeated indices except for the Greek indices. We use the Coiflet 12, which is compactly supported, quasi-symmetric, defined with a filter of length 12 and has four vanishing moments. Readers interested in details on the orthogonal wavelet transform may refer to, e.g., Mallat [7].

2.2 CVE METHOD

A wavelet-based nonlinear filtering method for extracting coherent vortices from three-dimensional turbulent flows, the CVE method, has been proposed[2]. The coherent vortices are defined as '*what remains after denoising*'.

An orthogonal wavelet decomposition is applied to the vorticity field $\boldsymbol{\omega}(\boldsymbol{x})$. The wavelet coefficients of vorticity are split into two sets by a threshold based on denoising theory[3]. The coherent vorticity $\boldsymbol{\omega}_C(\boldsymbol{x})$ is reconstructed from few wavelet coefficients whose moduli are larger than the given threshold depending on the enstrophy and grid points of the field. Thus, we obtain two orthogonal fields: the coherent vorticity $\boldsymbol{\omega}_C$

and the incoherent vorticity $\boldsymbol{\omega}_I(\boldsymbol{x})$, the latter can be obtained by subtraction, $\boldsymbol{\omega}_I = \boldsymbol{\omega} - \boldsymbol{\omega}_C$. The coherent and incoherent velocity fields are obtained from the coherent and incoherent vorticity fields by applying the Biot-Savart operator, respectively. Hereafter, we denote the coherent and incoherent vorticity by WC and WI, respectively. Readers interested in details of CVE may refer to the original articles[1,2,4,5,8].

2.3 FNF METHOD

In the FNF method, vorticity $\boldsymbol{\omega}(\boldsymbol{x})$ is first decomposed into Fourier series. The vorticity $\boldsymbol{\omega}_C^F(\boldsymbol{x})$ is then obtained by retaining only the Fourier coefficients whose moduli are larger than a threshold. The remaining vorticity $\boldsymbol{\omega}_I^F(\boldsymbol{x})$ is reconstructed from Fourier coefficients whose moduli are smaller than the threshold, and could be also simply obtained by $\boldsymbol{\omega}_I^F = \boldsymbol{\omega} - \boldsymbol{\omega}_C^F$. Here, we determine the threshold so that a compression rate C[%], i.e., the percentage of the coefficients whose moduli are larger than a threshold, for FNF is either almost the same as that for CVE or about twice as large as that for CVE. Hereafter, the former case is denoted by FNF1 and the latter by FNF2. The vorticity structures extracted by the use of FNF1 (FNF2) are denoted by FC1 (FC2), and the remaining vorticity by FI1 (FI2).

3. NUMERICAL RESULTS

Here, we apply both coherent vortex extraction algorithms, CVE and FNF, to flow data of homogeneous isotropic turbulence computed by DNS[6] with 256^3 grid points and R_λ=167.

Table 1 Statistical properties of CVE, FNF1 and FNF2 for the coherent contributions WC, FC1 and FC2. The percentages of the retained coefficients C, energy E and enstrophy Z with respect to the total flow are given.

	Basis	C[%]	E[%]	Z[%]
CVE: WC	Coiflet 12	3.6	99.1	81.1
FNF1: FC1	Fourier	3.7	99.4	76.1
FNF2: FC2	Fourier	7.3	99.7	85.1

3.1 STRUCTURE OBTAINED BY THE USE OF CVE

Figure 1 shows isosurfaces of total, coherent and incoherent vorticity corresponding to $|\boldsymbol{\omega}| = \Omega$, $|\boldsymbol{\omega}_C| = \Omega$ and $|\boldsymbol{\omega}_I| = 0.4\Omega$, respectively, where $\Omega = \langle \omega_t \rangle + 4\sigma_\omega$. Here, $\langle \omega_t \rangle$ and σ_ω are the mean value and the standard deviation of the modulus of the total vorticity. We observe that the coherent vorticity $\boldsymbol{\omega}_C$, represented by 3.6 % of the wavelet coefficients, retains the same vortex tubes as those present in the total vorticity. The coherent flow contains most of the energy and about 80 % of enstrophy of the total field (see Table 1). In contrast, the incoherent vorticity $\boldsymbol{\omega}_I$ is almost structureless.

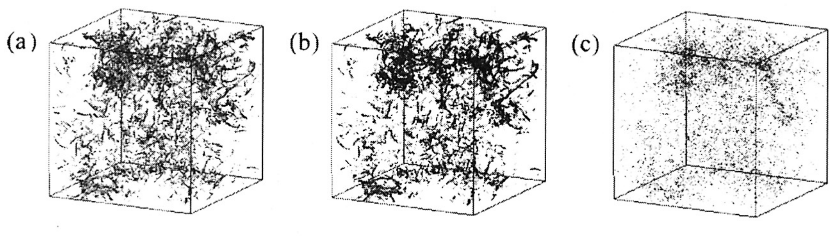

Fig. 1 Isosurfaces of modulus of vorticity for (a) the total field, (b) WC and (c) WI. Cubes of size 256^3 are visualized.

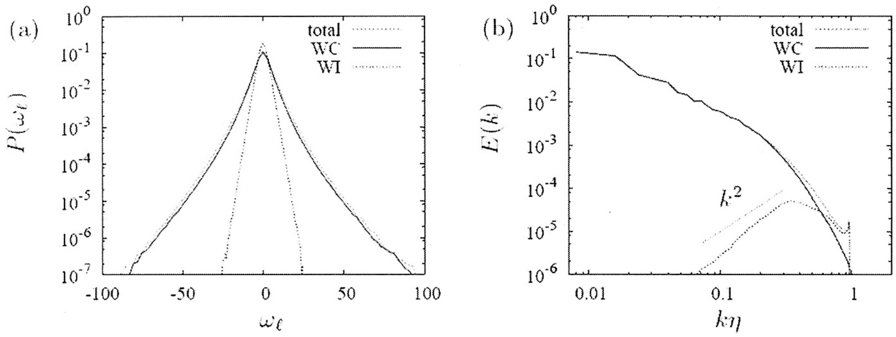

Fig.2 (a) PDFs of ℓth vorticity components ω_ℓ for $\boldsymbol{\omega}$, $\boldsymbol{\omega}_C$ and $\boldsymbol{\omega}_I$, and (b) energy spectra $E(k)$ for the case of CVE. Here $k=|\boldsymbol{k}|$ is the wavenumber and η is the Kolmogorov length scale.

The probability density functions (PDFs) of the ℓth components of $\boldsymbol{\omega}$, $\boldsymbol{\omega}_C$ and $\boldsymbol{\omega}_I$ are depicted in Fig. 2(a). The PDFs of the ℓth components of $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_C$ almost overlap and show stretched exponential tails. The PDF of the ℓth component of $\boldsymbol{\omega}_I$ has an exponential shape with a reduced variance compared to that of $\boldsymbol{\omega}$.

Figure 2(b) shows that the energy spectrum of the coherent velocity is identical to that of the total velocity all along the inertial range. In the dissipative range, i.e. for $k\eta > 0.3$, the spectrum of the coherent velocity differs from that of the total velocity, although there is still a significant contribution of the coherent vortices for $k\eta > 0.3$. Concerning the incoherent flow, the scaling of the incoherent energy spectrum is close to k^2, which corresponds to an equipartition of incoherent energy between all wavenumbers $\boldsymbol{k}$. The incoherent velocity is therefore spatially decorrelated, which is consistent with the observation that the incoherent vorticity is almost structureless (see Fig. 1(c)). These results in this subsection are consistent with the previous ones[2,4,5].

3.2 STRUCTURE OBTAINED BY THE USE OF FNF

Figures 3(a) and (b) show that the extracted structures FC1 and FC2, represented by 3.7 % and 7.3 % of the Fourier coefficients, respectively, retain less vortex tubes than WC. We also observe that FC2 contains more vortices than FC1. Table 1 shows that FC1 has less enstrophy than WC, but FC1 retains slightly more energy. As expected, FC2 has more energy and enstrophy than FC1. Figures 3(c) and (d) show isosurfaces of vorticity for FI1 and FI2. Both look more coherent-like in contrast to WI (see Fig. 1(c)).

Figure 4(a) shows that the PDF of $\boldsymbol{\omega}_C^F$ for FNF1 does not preserve the PDF of the total field, which is in contrast with the PDF of $\boldsymbol{\omega}_C$ obtained by CVE (see Fig. 2 (a)). For FNF2, the total PDF is better preserved than in FNF1, as shown in Figs. 4(a) and (b). For FNF1, the standard deviation of the ℓth component of $\boldsymbol{\omega}_I^F$ is comparable to that of $\boldsymbol{\omega}_C^F$, which is consistent with the observations of Figs. 3(a) and (c). For FNF2, the standard deviation of the ℓth component of $\boldsymbol{\omega}_I^F$ is about half of that of $\boldsymbol{\omega}_C^F$, but it is still larger than that of $\boldsymbol{\omega}_I$.

Figures 5(a) and (b) show the corresponding energy spectra $E(k)$. The spectra for FC1 and FC2 well preserve the energy spectrum of the total field all along the inertial range. However, in the dissipation range, both spectra for FC1 and FC2 rapidly drop with increasing $k\eta$. The spectra for FI1 and FI2 increase with increasing $k\eta$ more rapidly than the spectrum for WI does. We find wavenumber ranges where the spectra $E(k)$ for FI1 and FI2 increase approximately as $E(k) \propto k^4$ and $E(k) \propto k^6$, respectively.

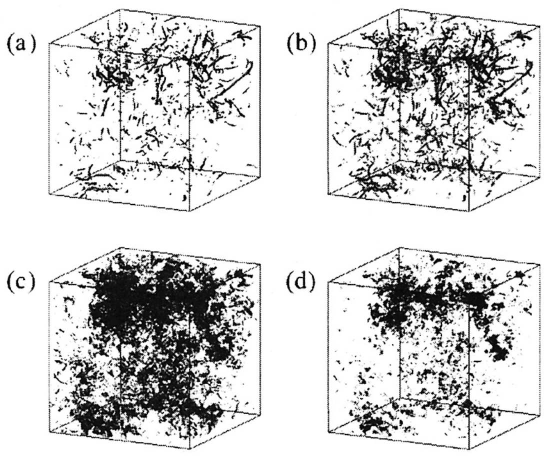

Fig.3 Isosurfaces of modulus of vorticity for (a) FC1, (b) FC2, (c) FI1 and (d) FI2. The isosurface values for FC1 and FC2 are the same as those for WC and the total field. For FI1 and FI2, the isosurface values agree with that of WI. Cubes of size 256^3 are visualized.

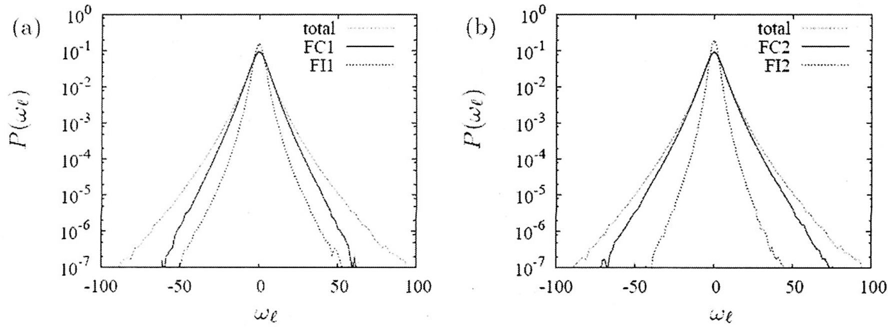

Fig.4 PDFs of ω_ℓ for (a) FNF1 and (b) FNF2.

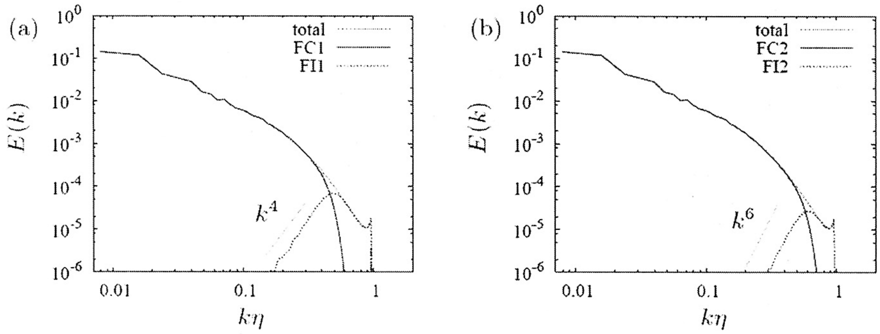

Fig.5 $E(k)$ vs. $k\eta$ for (a) FNF1 and (b) FNF2.

4. CONCLUSION

We have compared two nonlinear filtering methods for extracting coherent vortices out of three-dimensional homogeneous turbulence in a periodic box. One is the CVE method based on orthonormal wavelets, and the other is the FNF method based on the Fourier decomposition. We have applied these methods to DNS data of homogeneous isotropic turbulence with 256^3 grid points and R_λ=167. It is confirmed that coherent structures extracted by the use of CVE, reconstructed from 3.6 % of the wavelet coefficients, preserve the coherent vortices as well as the statistical quantities of turbulence. We have found that the structures extracted by means of FNF, represented only by 3.7 % of the Fourier coefficients, retain low-order statistical quantities, such as energy, enstrophy and the energy spectrum in the inertial range. However, even if the compression rate for FNF is about twice as large as that for CVE, the structures extracted by using FNF do not preserve the vortex tubes as it is the case for CVE.

The above advantages of CVE motivate further developments of the coherent vortex simulation (CVS) method proposed in Farge and Schneider[9]. It is based on a deterministic computation of the time evolution of the coherent flow using an adaptive wavelet basis, while the influence of the incoherent flow onto the coherent flow is neglected or statistically modeled. First results of CVS were obtained in three-dimensional turbulent mixing layers[10]. Recently, CVS has been applied to three-dimensional homogeneous isotropic turbulence[11]. The statistical predictability of turbulent flows is well preserved by CVS, where the influence of the incoherent flow is neglected. Thus discarding the incoherent flow is sufficient to turbulent dissipation. For a review on wavelet methods for computational fluid dynamics we refer to Schneider and Vasilyev[12].

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REFERENCES

1) Farge, M., Schneider, K. and Kevlahan N., "Non-Gaussianity and coherent vortex simulation for two-dimensional turbulence using an adaptive orthonormal wavelet basis", *Phys. Fluids* Vol.11,(1999), pp.2187-2201.
2) Farge, M., Pellegrino, G. and Schneider, K., "Coherent vortex extraction in 3d turbulent flows using orthogonal wavelets", *Phys. Rev. Lett.* Vol.87,(2001), 054501.
3) Donoho, D., "De-noising by soft thresholding", *IEEE Transaction on Information Theory* Vol.41,(1995), pp.613-627.
4) Farge, M., Schneider, K., Pellegrino, G., Wray, A., and Rogallo, R., "Coherent vortex extraction in three-dimensional homogeneous turbulence: comparison between CVS-wavelet and POD-Fourier decompositions", *Phys. Fluids* Vol.15,(2003), pp.2886-2896.
5) Okamoto, N., Yoshimatsu, K., Schneider, K., Farge, M. and Kaneda, Y., "Coherent vortices in high resolution direct numerical simulation of homogeneous isotropic turbulence: A wavelet viewpoint", *Phys. Fluids*. Vol.19,(2007), 115109.
6) Kaneda, Y., Ishihara, T., Yokokawa, M., Itakura, K. and Uno, A., "Energy dissipation rate and energy spectrum in high resolution direct numerical simulations of turbulence in a periodic box", *Phys. Fluids* Vol.15,(2003), L21-L24.
7) Mallat , S., A wavelet tour of signal processing. (Academic Press, 1998).
8) Azzalini, A., Farge, M. and Schneider, K., "Nonlinear wavelet thresholding: A recursive method to determine the optimal denoising threshold", *Appl. Comput. Harm. Anal.*Vol.18,(2005), pp.177-185.
9) Farge, M., and Schneider, K., "Coherent vortex simulation (CVS), a semideterministic turbulence model using wavelets", *Flow, Turbul. Combust.* Vol. 66,(2001), pp.393-426.
10) Schneider, K., Farge, M., Pellegrino, G., and Rogers, M., "Coherent vortex simulation of 3D turbulent mixing layers using orthogonal wavelets", *J. Fluid Mech.* Vol. 534,(2005), pp.39-66.

11) Okamoto, N., Yoshimatsu, K., Schneider, K., Farge, M. and Kaneda, Y., "Coherent vortex simulation: application to 3D homogeneous isotropic turbulence", Advances in Turbulence XII Proceedings of the 12th Euromech European Turbulence Conference, September 7-10, 2009, Marburg, Germany; ed. B. Eckhardt, Springer, (2009), pp.759-762.
12) Schneider, K., and Vasilyev, O. V., "Wavelet methods in computational fluid dynamics", *Annu. Rev. Fluid Mech.* Vol. 42,(2010), pp.473-503.