Wavelet Analysis of the Conditional Vorticity Budget in Fully Developed Homogeneous Isotropic Turbulence

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Abstract. We study the conditional balance of vortex stretching and vorticity diffusion of fully developed three-dimensional homogeneous isotropic turbulence with respect to coherent and incoherent flow contributions. This decomposition is achieved by the Coherent Vorticity Extraction based on orthogonal wavelets. The analysis allows to discriminate coherent and incoherent contributions to the different terms arising in the conditional balance equation and yields insights into the interaction of coherent and incoherent flow contributions.

1. Introduction

The challenge of understanding the statistical properties of fully developed turbulence is closely related to the presence of coherent structures in the flow, which goes along with strong statistical correlations. For example, already the single-point vorticity statistics displays a highly non-Gaussian shape, which indicates pronounced spatial correlations of the vorticity field. In this respect one of the most interesting problems in turbulence research is to understand the relation between the coherent structures and their implications for statistical properties of the flow.

In this context it is particularly interesting to study dynamical statistical relations such as balance equations, of which maybe the most fundamental one related to vorticity is the balance of enstrophy production and dissipation. Deriving an equation for the vorticity probability density function (PDF) is even more informative and allows to study the conditional budget of enstrophy production and dissipation, or equivalently the balance of conditional vortex stretching and diffusion, where the ordinary budget equation is contained as a special case. This budget equation was introduced by Novikov and has been studied in a number of publications [1, 2, 3]. The conditional vorticity budget allows to quantify vortex stretching and vorticity diffusion as a function of vorticity magnitude and consequently to statistically discriminate strong vorticity regions in the flow from weak ones. On the level of the conditional balance of vortex stretching and vorticity diffusion it is furthermore possible to decompose the resulting quantities in terms of coherent and incoherent contributions with the help of the orthogonal wavelet decomposition. Farge et al. [4] proposed a method to extract the coherent structures out of turbulent flows, called Coherent Vorticity Extraction (CVE). This technique is based on denoising of vorticity in wavelet space. It was shown in [5] that the CVE method is more efficient than Fourier filtering and in [6] that fewer wavelet coefficients are necessary to reconstruct the coherent structures with increasing Reynolds number, which means that the CVE method becomes more attractive as the flow becomes more intermittent. This method was applied to study the vortical structures in sheared and rotating turbulence in [7] and mixing layers in [8]. For all investigated flows, it was shown that the coherent vortices are well represented with few wavelet coefficients and the statistics of the remaining background flow exhibit more Gaussian-like behavior.

Should we also cite Farge_1999 here?

The application of the CVE procedure in the present work is motivated to get further insights into the conditional vorticity budget by discriminating coherent and incoherent contributions, e.g., to the vortex stretching term and to give a quantitative view on the interaction of coherent and incoherent contributions.

In the remainder of this article, we will first review the theoretical background for the conditional vorticity budget as well as the orthogonal wavelet decomposition and the technique of Coherent Vorticity Extraction. Then we will present and discuss the numerically obtained results, before we conclude. The present proceedings paper represents the short version of a more extended work including a comparison to Fourier filtering, which will be published elsewhere. REF

2. Conditional Vorticity Budget and Coherent Vorticity Extraction

The dynamics of incompressible flows ($\nabla \cdot \boldsymbol{u}(\boldsymbol{x},t) = 0$) can be described in terms of the vorticity field $\boldsymbol{\omega}(\boldsymbol{x},t) = \nabla \times \boldsymbol{u}(\boldsymbol{x},t)$, defined as the curl of the velocity field. Its evolution equation takes the form

$$\frac{\partial}{\partial t}\boldsymbol{\omega} + \boldsymbol{u} \cdot \nabla\boldsymbol{\omega} = \mathrm{S}\boldsymbol{\omega} + \nu\Delta\boldsymbol{\omega} + \nabla \times \boldsymbol{F}, \tag{1}$$

where $\mathrm{S}(\boldsymbol{x},t) = \frac{1}{2}\left[\nabla\boldsymbol{u}(\boldsymbol{x},t) + (\nabla\boldsymbol{u}(\boldsymbol{x},t))^T\right]$ denotes the rate-of-strain tensor, ν denotes the kinematic viscosity, and $\boldsymbol{F}(\boldsymbol{x},t)$ represents an external large-scale forcing applied to the flow in order to maintain statistical stationarity. If one is now interested in the single-point statistics of the vorticity, a comprehensive characterization can be obtained by studying the evolution equation of the vorticity probability density function. As we are considering homogeneous isotropic turbulence, this PDF does not depend on the spatial coordinate. Furthermore, the single-point statistics is fully determined by the PDF of the magnitude of the vorticity, in the following denoted by $\check{f}(\Omega;t)$. Standard PDF methods [9, 10, 11, 12] and the exploitation of the given statistical symmetries [13] lead to the evolution equation of this PDF reading

$$\frac{\partial}{\partial t}\check{f}(\Omega;t) = -\frac{\partial}{\partial\Omega}\left[s(\Omega,t) + d(\Omega,t) + e(\Omega,t)\right]\check{f}(\Omega;t). \tag{2}$$

The terms on the right-hand side are related to the conditional average of the vortex-stretching term, the conditional diffusion term and the conditional forcing term. They are given by

$$\left\langle \mathrm{S}\boldsymbol{\omega}\middle|\boldsymbol{\Omega}\right\rangle = s(\Omega,t)\,\widehat{\boldsymbol{\Omega}} \qquad s(\Omega,t) = \left\langle \widehat{\boldsymbol{\omega}}\cdot\mathrm{S}\boldsymbol{\omega}\middle|\Omega,t\right\rangle \tag{3}$$

$$\left\langle \nu\Delta\boldsymbol{\omega}\middle|\boldsymbol{\Omega}\right\rangle = d(\Omega,t)\,\widehat{\boldsymbol{\Omega}} \qquad d(\Omega,t) = \left\langle \nu\widehat{\boldsymbol{\omega}}\cdot\Delta\boldsymbol{\omega}\middle|\Omega,t\right\rangle \tag{4}$$

$$\left\langle \nabla\times\boldsymbol{F}\middle|\boldsymbol{\Omega}\right\rangle = e(\Omega,t)\,\widehat{\boldsymbol{\Omega}} \qquad e(\Omega,t) = \left\langle \widehat{\boldsymbol{\omega}}\cdot(\nabla\times\boldsymbol{F})\middle|\Omega,t\right\rangle, \tag{5}$$

i.e., as the terms of the conditionally averaged right-hand side of the vorticity equation (1) projected on the direction of the vorticity, $\widehat{\boldsymbol{\omega}} = \frac{\boldsymbol{\omega}}{\omega}$, where $\omega = \|\boldsymbol{\omega}\|$. Hence, the closure problem of turbulence in this formulation comes in terms of of the unknown conditional averages. If we additionally consider stationary turbulence, the PDF as well as the conditional averages become independent of time. This also implies

$$s(\Omega) + d(\Omega) + e(\Omega) = 0 \tag{6}$$

as the probability current for stationary one-dimensional problems has to vanish. Furthermore, it has been shown in a number of publications [1, 2, 12] that the conditional vortex stretching and diffusive term balance at sufficiently high Reynolds numbers. Compared to those terms, the external forcing has a negligible effect, such that we obtain the approximation

$$s(\Omega) + d(\Omega) \approx 0, \tag{7}$$

or equivalently

$$[s(\Omega) + d(\Omega)]\, \widehat{\boldsymbol{\Omega}} = \langle \mathrm{S}\boldsymbol{\omega} | \boldsymbol{\Omega} \rangle + \langle \nu \Delta \boldsymbol{\omega} | \boldsymbol{\Omega} \rangle \approx \mathbf{0}. \tag{8}$$

This central relation states that vortex stretching and vorticity diffusion tend to cancel for a fixed magnitude of vorticity on the statistical average. This balance has been extensively discussed by Novikov [2]. Recently in [12], its influence with respect to the shape and evolution of the vorticity PDF was investigated.

The conditional balance is much more informative than the ordinary enstrophy balance as we, for example, can discuss the results as a function of vorticity magnitude highlighting possible correlations. Of course, the average enstrophy balance (discussed, e.g., in [14]) is obtained from the terms in equation (7) according to

$$\langle \boldsymbol{\omega} \cdot \mathrm{S}\boldsymbol{\omega} \rangle + \langle \nu \boldsymbol{\omega} \cdot \Delta \boldsymbol{\omega} \rangle = \int_0^\infty \Omega\, [s(\Omega) + d(\Omega)]\, \check{f}(\Omega)\, \mathrm{d}\Omega \approx 0. \tag{9}$$

Higher-order moment relations can also be obtained in the same manner, demonstrating the amount of information contained in conditional averages.

If we now want to establish a connection between this conditional balance and the coherent vorticity structures present in the flow, we have to discriminate coherent from incoherent contributions to the vorticity field. The challenge in this context is to define what exactly a coherent structure is, and many different approaches have been introduced in recent years [15]. A conceptually different way to determine the coherent contributions is to analyze the vorticity field in terms of CVE introduced in [16, 4] which is based on a denoising approach.

Is [15] a review or should we add more references?

To this end, we decompose the vorticity field into coherent and incoherent contributions according to

$$\boldsymbol{\omega}(\boldsymbol{x}, t) = \boldsymbol{\omega}^c(\boldsymbol{x}, t) + \boldsymbol{\omega}^i(\boldsymbol{x}, t) \tag{10}$$

using an orthogonal wavelet decomposition. In this study we use the Coiflet 30 wavelet [17], which has ten vanishing moments. The wavelet coefficients are calculated as

$$\widetilde{\boldsymbol{\omega}}_{\boldsymbol{\lambda}} = \int_{[0,2\pi]^3} \boldsymbol{\omega}(\boldsymbol{x}) \psi_{\boldsymbol{\lambda}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \tag{11}$$

where $\psi_{\boldsymbol{\lambda}}$ denotes the wavelet specified by a multi-index characterizing the scale of the wavelet as well as its spatial direction. From this set of wavelet coefficients the ones constituting the coherent part are defined as

Should we write $\|\boldsymbol{\omega}_{\boldsymbol{\lambda}}\|$?

$$\widetilde{\boldsymbol{\omega}}^c_{\boldsymbol{\lambda}} = \begin{cases} \widetilde{\boldsymbol{\omega}}_{\boldsymbol{\lambda}} & \text{if} \quad |\widetilde{\boldsymbol{\omega}}_{\boldsymbol{\lambda}}| > \varepsilon = \sqrt{2\sigma \ln N} \\ \mathbf{0} & \text{else} \end{cases} \tag{12}$$

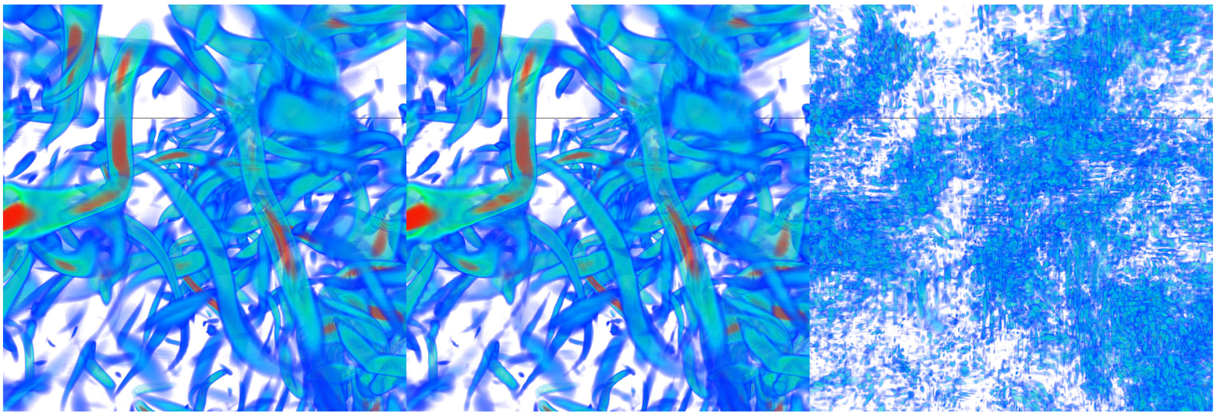

Figure 1. Close-up volume visualization of a wavelet-decomposed vorticity field (shown are the magnitudes). From left to right: total field, coherent contribution, incoherent contribution. While the color scale is the same for the total and coherent contribution, it has been adjusted for the visulaization of the incoherent contribution due to a largely reduced amplitude of this field. The visualizations have been produced with the free software package VAPOR (`www.vapor.ucar.edu`).

and those of the incoherent part as the remainder. Here $\sigma = \sqrt{\langle \boldsymbol{\omega}^2 \rangle /3}$ is the standard deviation of the vorticity and N the total number of grid points of the data from the direct numerical simulation. We refer the reader to [4] for more details on the wavelet decomposition.

Once the fields are decomposed into coherent and incoherent, we can also calculate coherent and incoherent contributions to the conditional vorticity budget (8). By this decomposition, we obtain for the conditional diffusive term

$$\left\langle \nu\Delta\boldsymbol{\omega} \middle| \boldsymbol{\Omega} \right\rangle = \left\langle \nu\Delta\boldsymbol{\omega}^c \middle| \boldsymbol{\Omega} \right\rangle + \left\langle \nu\Delta\boldsymbol{\omega}^i \middle| \boldsymbol{\Omega} \right\rangle = \left[d^c(\Omega) + d^i(\Omega) \right] \widehat{\boldsymbol{\Omega}}, \tag{13}$$

such that we get two separate contributions from the coherent and incoherent parts of all fields of the ensemble. This decomposition, for instance, allows to quantify how much enstrophy dissipation is contained in the two terms.

The nonlinear vortex stretching term turns out to be more complicated as the rate-of-strain tensor contains both coherent and incoherent contributions. This can be seen by noting that we can calculate the coherent and incoherent velocity, $\boldsymbol{u}^c(\boldsymbol{x},t)$ and $\boldsymbol{u}^i(\boldsymbol{x},t)$, from the decomposed vorticity field via Biot-Savart's law and subsequently the coherent and incoherent rate-of-strain tensor $\mathrm{S}^c(\boldsymbol{x},t)$ and $\mathrm{S}^i(\boldsymbol{x},t)$. Hence, the vortex stretching term may be split up into four terms containing all possible combinations of coherent and incoherent parts of the vorticity and the rate-of-strain tensor,

$$\left\langle \mathrm{S}\boldsymbol{\omega} \middle| \boldsymbol{\Omega} \right\rangle = \left\langle \mathrm{S}^c\boldsymbol{\omega}^c \middle| \boldsymbol{\Omega} \right\rangle + \left\langle \mathrm{S}^i\boldsymbol{\omega}^i \middle| \boldsymbol{\Omega} \right\rangle + \left\langle \mathrm{S}^i\boldsymbol{\omega}^c \middle| \boldsymbol{\Omega} \right\rangle + \left\langle \mathrm{S}^c\boldsymbol{\omega}^i \middle| \boldsymbol{\Omega} \right\rangle \tag{14}$$

$$= \left[s^{cc}(\Omega) + s^{ii}(\Omega) + s^{ic}(\Omega) + s^{ci}(\Omega) \right] \widehat{\boldsymbol{\Omega}}. \tag{15}$$

The interesting fact now is that the different functions characterize the interaction of coherent and incoherent contributions of the rate-of-strain field with coherent and incoherent contributions of the vorticity field.

3. Results

The presented flow is generated with a standard, dealiased Fourier pseudospectral code [18, 19] for the vorticity equation. The integration domain is a triply periodic box of side-length 2π

at a spatial resolution of $N = 512^3$ grid points. The time stepping scheme is a third-order Runge-Kutta scheme [20]. The Reynolds number based on the Taylor micro-scale is 112. For the statistically stationary simulations a large-scale forcing is applied to the flow, for which we chose one which conserves the kinetic energy of the flow by amplifying the magnitude of Fourier modes in a wavenumber band and letting their phases evolve freely. This forcing has been found to deliver satisfactory results concerning the statistical symmetries.

For the numerical evaluation we average over twenty statistically independent realizations of the vorticity field to ensure a good statistical quality. Furthermore, the presented data stems from a particularly well-resolved simulation ($k_{\max}\eta \approx 2$), which is necessary as, e.g., second derivatives of the vorticity field will be considered.

In figure 1 volume visualizations of the total, coherent and incoherent contributions of the vorticity are shown. It can be seen that the coherent contribution represents the global structure of the total vorticity field to a good extent, differences are only visible in the details. The incoherent contribution appears very noisy and is small in amplitude compared to the total field. The color scales for the visualization of the incoherent field has been adjusted to account for that issue.

This observation can be made more quantitative by investigating the PDF of the magnitude of the vorticity which is presented in figure 2. This figure shows that the PDF of the coherent part of the vorticity leads to a PDF almost indistinguishable from the total PDF. The incoherent part has a largely reduced variance and displays a nearly exponential decay consistent with previous findings [16, 4, 5].

In the same figure the conditional balance of vortex stretching and vorticity diffusion is shown. It can be seen that the vortex stretching term is positively correlated with the vorticity, whereas the diffusive term is negatively correlated. This is physically quite intuitive as it mirrors the fact that the vortex stretching term tends to amplify the vorticity, while the dissipative term depletes vorticity. The fact that the sum of both averages nearly identically vanishes represents an *a posteriori* justification for the approximation leading to the relation (8). In the same figure the functions expected for the case where the rate-of-strain tensor is statistically independent of the vorticity and the corresponding diffusive term balances this term are shown for comparison. The slope of these linear functions is obtained such that these function yield the correct ordinary enstrophy budget (9). The difference compared to the functions obtained from the DNS proves, as expected, that pronounced correlations between the field of the rate-of-strain tensor, the Laplacian of the vorticity, and the vorticity, respectively, exist.

To now quantify the contributions of the coherent structures, we investigate the diffusive term, which is presented in figure 3. It is observed that this term is almost fully represented by the coherent part of the field and it has been calculated that it contributes with about 91% to the dissipative term of the ordinary enstrophy budget. The incoherent contribution appears significantly smaller, contributing with the remaining 9%. As the Laplacian of a field enhances its small-scale features, this demonstrates that the CVE captures these features especially well. Similar observations can be made for the terms related to the conditional vortex stretching term also shown in figure 3. For this term the coherent contribution matches almost perfectly the total contribution, about 98% of the enstrophy production is contained within this term. The remaining terms are strongly reduced in amplitude, still their investigation is interesting. It becomes apparent that the interaction of the coherent part of the rate-of-strain field with the incoherent part of the vorticity field is positively correlated with the vorticity, i.e., a positive contribution to the average enstrophy budget originates from this term. An interesting interpretation of this observation is that the rate-of-strain field produced by the coherent vortex structures is able to produce additional coherent vortex structures; in a sense coherent structures breed coherent structures. In contrast to that, the interaction of the rate-of-strain field induced by the incoherent vorticity depletes the coherent vorticity, such that it can be concluded that

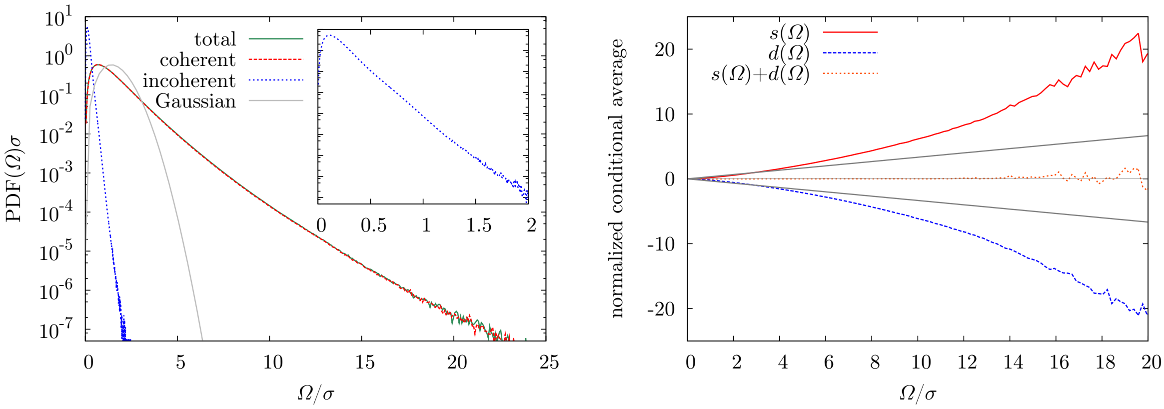

Figure 2. Left: PDFs of the magnitude of the total vorticity, the coherent part and the incoherent part for wavelet-decomposed fields. Right: Conditional balance of the conditional averages related to vortex stretching and diffusion of vorticity. The gray lines indicate the functional form of the conditional averages expected when assuming statistical independence of the rate-of-strain tensor and the vorticity magnitude and a corresponding balance of the diffusive term. All conditional averages are normalized with the factor $\langle \varepsilon_\omega \rangle / \sigma$, i.e., the fraction of the enstrophy dissipation and the standard deviation of the vorticity field.

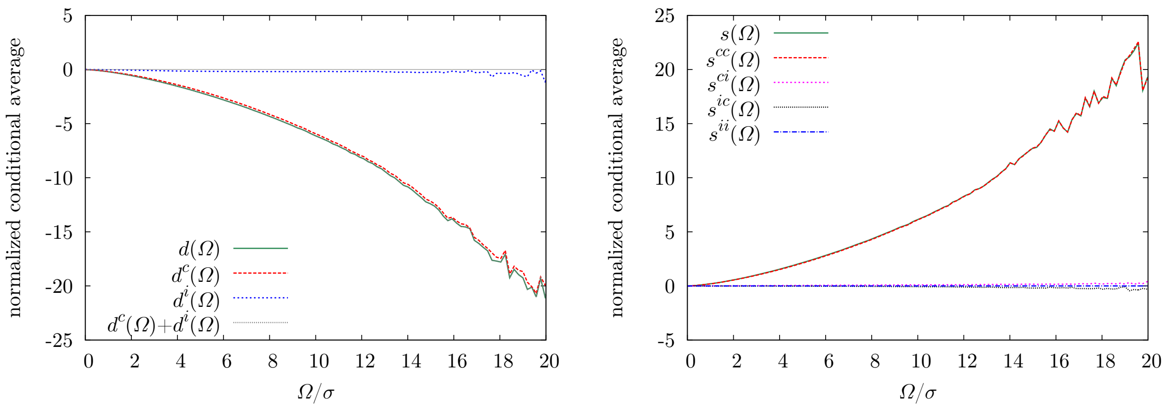

Figure 3. Left: Coherent and incoherent parts of the diffusive term compared to the total one. As a benchmark, the sum of both contributions is shown to add up to the total diffusive term. Right: Coherent and incoherent contributions to the conditional average related to the vortex stretching term.

the incoherent rate-of-strain field destroys coherent vortex structures and hence has a dissipative character. The contribution of the incoherent rate-of-strain field times the incoherent vorticity field is negligible in view of its even smaller amplitude.

4. Conclusion

To summarize, we presented an analysis of the conditional vorticity budget in terms of coherent vorticity. For that purpose we made use of the CVE technique to separate the noisy incoherent contributions of the vorticity field from the coherent ones. It was shown, in accordance with previous results, that CVE yields an excellent representation of the total flow using a reduced number of degrees of freedom. This is particularly interesting as the conditional budget of vortex stretching and vorticity diffusion represents a dynamical rather than a purely kinematic relation. It has been shown that most of the enstrophy production can be accounted to the coherent vorticity and the correspondingly induced rate-of-strain field. We have found that the incoherent rate-of-strain field tends to deplete vorticity, i.e., it tends to destroy coherent structures, while the rate-of-strain field induced by the coherent vorticity contributes positively. In this sense the coherent structures are able to maintain or even amplify themselves, whereas the incoherent contributions tend to have a dissipative effect.

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