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Interpretation of Two-Dimensional Turbulence Energy Spectrum in Terms of Quasi-Singularity in Some Vortex Cores.

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Abstract. – We propose a new geometrical interpretation of two-dimensional turbulence energy spectrum, supposing the presence of at least one cusplike axisymmetric coherent structure in the flow. Such a coherent structure is a scaling distribution of vorticity which presents, instead of a characteristic radius, a range of radii corresponding to all scales where the associated energy spectrum has a power law behaviour. We compute a relation between the spectral slope and the exponent of the singularity in the Euler limit (Reynolds number tending to infinity). We predict a sensitivity of the two-dimensional flow dynamics to the regularity of the initial vorticity field: if it is initially regular, *i.e.* smooth, the coherent structures formed will be regular presenting a flat core, and if it is initially nonregular they will have a cusplike shape with only the vortex core regularized by dissipation, which corresponds to quasi-singularities.

1. Introduction.

A generic behaviour of two-dimensional turbulent flows is the emergence of coherent structures which are observed both in laboratory [1, 2] and in numerical experiments [3-7]. The statistical theory of two-dimensional turbulence [8, 9] predicts a k^{-3} energy spectrum due to a supposed cascade of enstrophy in the inertial range, but most numerical experiments observe a steeper slope instead, between k^{-4} and k^{-6}, depending on the initial conditions and on the forcing [4-7]. To explain this discrepancy, some adjustments have been made to the statistical theory of two-dimensional turbulence supposing an intermittency of

interpretation of two-dimensional turbulence based on the local scaling of some coherent structures, which does not refer to any statistical mechanics arguments and for which there is no cascade hypothesis. This scaling seems confirmed by both numerical [7] and laboratory experiments [2].

2. The regularity of initial conditions in numerical experiments.

Let us consider Euler's equations, describing the time evolution of a perfect incompressible fluid

$$\partial_t \omega + J(\psi, \omega) = 0\,, \qquad \omega = \Delta\psi\,, \tag{2.1}$$

where ω is defined as the vorticity field, ψ the stream function field, and J the Jacobian operator. Throughout this paper we assume periodic boundary conditions, that is we are on the two-dimensional torus $\boldsymbol{T}^2$.

Let us consider an initial condition for ω that satisfies the following scaling law:

$$|\widehat{\omega}(k_x, k_y)| = |\widehat{\omega}(k)| \sim k^{-\gamma}\,, \qquad \gamma \in \boldsymbol{R} \text{ and } k \to \infty\,, \tag{2.2}$$

where $k = \sqrt{k_x^2 + k_y^2}$ and $\widehat{\omega}$ is the two-dimensional Fourier transform ([1]) of ω. The phases, however, are chosen randomly. The corresponding energy E also has a power law spectrum for arbitrarily small scales:

$$|E(k)| = \int_0^{2\pi} k\,\mathrm{d}\theta \frac{1}{2}(\widehat{v}_x^2(k,\theta) + \widehat{v}_y^2(k,\theta)) \sim k^{-2\gamma-1}\,, \qquad k \to \infty\,, \tag{2.3}$$

with $\boldsymbol{v}(v_x, v_y)$ the velocity field associated to $\omega = \nabla \times \boldsymbol{v}$. The scaling of $\widehat{\omega}(k)$ (2.2) implies that in physical space the initial vorticity field ω is not smooth. Indeed, it is known [12] that a periodic function f of regularity Λ^γ, namely

$$|f(x) - f(y)| \leq C|x-y|^\gamma \quad \text{with some constant } C \text{ and for } x \to y\,,$$

has Fourier coefficients that decay as

$$|\widehat{f}(k)| \leq C' k^{-\gamma} \quad \text{with some constant } C'. \tag{2.4}$$

Therefore, because of (2.2), the regularity of the initial vorticity field ω cannot be better than γ, a property that we denote by $\omega \notin \Lambda^\gamma$, otherwise its Fourier transform would decay faster at infinity. If we choose $-2\gamma - 1 = -3$, which corresponds to the energy spectrum of the initial conditions commonly used in most numerical experiments [6, 7], it implies that the regularity γ of the vorticity field cannot be better than 1 because

$$\gamma = 1 \quad \Rightarrow \omega \notin \Lambda^1.$$

Now, if we want to compute Euler's equations numerically, we have to do two things: discretize the initial field on a grid of mesh size Δx and add some dissipation in the very small scales. Upon discretizing one actually regularizes the initial field in the sense that, even if there is a finite number of singularities with negative exponents, the random field will be finite with probability one; we will then call the discretized singularities quasi-singularities. Now let us consider as an example the numerical experiment of Benzi *et al.* [7], which

([1]) Note that k takes only integer values because of the periodic boundary conditions. However, we write integrals instead of sums to clarify the notations.

energy power law spectrum scaling as k^{-3} until the cut-off scale $k_{\max}=256$, which corresponds to a discretization on a 512^2 grid, and they add a dissipation term $\nu\,\Delta^2\omega$, ν being the kinematic viscosity of the fluid, to Euler's equations without any forcing. The time evolution of this decaying flow shows the emergence of coherent structures, namely the condensation of the vorticity field into several axisymmetric vortices, more or less deformed if they are interacting with each other.

From what we have previously said, this initial vorticity field necessarily presents some quasi-singularities of exponent $\gamma \leqslant 1$. In the following section we shall propose a new model for two-dimensional turbulence based on axisymmetric coherent structures presenting such a local singularity. In sect. 4 we shall discuss how evidence can be found, from both numerical and laboratory experiments, for the existence and the stability of such quasi-singular vortices, even in the viscous case ($\nu \neq 0$).

3. Local scaling properties in axisymmetric vortices.

Now let us consider an axisymmetric solution $\omega=\omega(r)$, $\psi=\psi(r)$ of Euler's equations, similar to the observed coherent structures in two-dimensional turbulent flows. We assume that in the reference frame, where this vortex solution is at rest, the axisymmetric approximation will always be valid for small values of r, the radial distance to the vortex core. All scaling laws presented herein hold only asymptotically for $r \to 0$, whereas at infinity it is assumed that all functions are rapidly decreasing. We will first develop our theory in the inviscid case and from there infer a new way to interpret numerical and experimental results obtained in the viscous case.

We suppose that in the vicinity of the vortex core the vorticity field has a power law behaviour

$$\omega \sim r^{\alpha}. \tag{3.1}$$

Equation (2.1) implies that

$$\psi \sim r^{\alpha+2}+\psi(0). \tag{3.2}$$

By definition of the stream function

$$v \sim r^{\alpha+1}+v(0). \tag{3.3}$$

Combining (3.1) and (3.2) leads to the function $\omega(\psi)$, that we will call the coherent structure function

$$\omega(\psi) \sim (\psi-\psi(0))^{\alpha/(\alpha+2)}. \tag{3.4}$$

In order to have a finite circulation

$$\Gamma(r)=2\pi\int_0^r \omega(r')\,r'\,\mathrm{d}r' \sim r^{\alpha+2}, \tag{3.5}$$

a finite total enstrophy

$$\Omega(r)=2\pi\int_0^r \omega(r')^2\,r'\,\mathrm{d}r' \sim r^{2\alpha+2}, \tag{3.6}$$

and a finite total energy

$$E(r)=2\pi\int_0^r v(r')^2\,r'\,\mathrm{d}r' \sim r^{2\alpha+4}, \tag{3.7}$$

the exponent α should be restricted to values larger than or equal to -1. Notice that we only need to consider the convergence of the integrals around the origin, since all functions are supposed to have a fast decay at infinity.

Constraints (3.5), (3.6), (3.7) and relation (3.2) imply in particular that ψ is regular.

The fourier transform of the velocity (3.3) leads to the equation

$$v(k) = k^{-2} \int_0^r v(r'/k), \quad J_0(r')\, r'\, \mathrm{d}r' \sim k^{-\alpha-3}, \tag{3.8}$$

where J_0 is the zeroth-order Bessel function. Taking the energy density as

$$v^2(k) \sim k^{-2\alpha-6}$$

and integrating over shells of constant k leads to

$$E(k) \sim k^{-2\alpha-5}. \tag{3.9}$$

4. Experimental identification of exponent α.

Coherent structures can be characterized by a pointwise correlation (2) between vorticity ω and stream function ψ, that we have called the coherent structure function $\omega(\psi)$. Results from both laboratory experiments [2] and numerical simulations [7] suggest that, at least for some (3) vortices, we have $\omega \sim \psi^\beta$, with $\beta < 0$, which implies a cusplike shape for the coherent structure function of these vortices (fig. 4 in Benzi *et al.* [7] and fig. 9 in Nguyen Duc and Sommeria [2]).

Benzi *et al.* [7] have shown, in their numerical experiments of two-dimensional decaying turbulence, that coherent structures can be described by an axisymmetric function $\omega \sim f(r/r_0)$, with r_0 a rescaling constant. Benzi *et al.* [7]'s fig. 6 shows that $\Omega(\Gamma^2)$ scales as r^{-2}, which corresponds to the scaling obtained with our model by combining (3.5) and (3.6), a scaling in fact independent of α. In Benzi *et al.* [7]'s fig. 7, ω^2 approximately follows a r^{-1} law, at least in a certain range, before dissipation regularizes the core. Therefore we evaluate that $\alpha \simeq -1/2$, which satisfies the finite energy (3.7), circulation (3.5) and enstrophy (3.6) constraints and leads us to the coherent structure function $\omega(\psi)$ scaling with an exponent $\beta \simeq -1/3$. This result is confirmed by looking at Benzi *et al.* [7]'s fig. 4, which shows a cusplike behaviour of the coherent structure function near the vortex cores.

Another argument supporting the singular behaviour of coherent structures can be found in Nguyen Duc and Sommeria [2]'s fig. 9, which shows the same type of nonlinear coherent structure function obtained from laboratory experiments performed on a two-dimensional decaying turbulent flow in a layer of mercury submitted to a transverse magnetic field. In addition to Nguyen Duc and Sommeria [2]'s comment concerning the fact that the first derivative of the coherent structure function $\partial\omega/\partial\psi$ reaches its maximum in the vortex cores, we note that the finite enstrophy constraint $\alpha \geqslant -1$ implies $-1 < \beta = \alpha/(\alpha+2) < 1$. In consequence, as ψ tends to $\psi(0)$, *i.e.* in the vortex cores, $\partial\omega/\partial\psi$ is either asymptotically

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dissipation is active. In the asymptotic limit of arbitrary small dissipation, this explains the stability of coherent structures.

(3) Recall that we do not claim all vortices to be singular. We only want to give evidence for the existence of some quasi-singular ones.

constant for $\alpha \to \infty$, which corresponds to regular coherent structures with a flat core, or diverges for $\alpha < 0$, which corresponds to quasi-similar coherent structures. This, of course, is true only in the asympototic regions near the vortex cores where the axisymmetric vortex hypothesis remains valid, and within the limit of an asymptotically small viscosity $\nu \to 0$.

5. Prediction of the presence of some quasi-singular vortices.

The scaling exponent $\alpha \simeq -1/2$ that we have identified from Benzi *et al.* [7]'s numerical experiments leads us to the prediction that some of the coherent structures they observe present a cusplike shape until the dissipative scales. This result seems confirmed by the wavelet analysis of coherent structures we have performed for a two-dimensional geostrophic flow, which shows that the smallest scales of the vorticity field are confined inside some vortex cores [6, 12, 13]. Benzi *et al.* [7] have proposed a different interpretation of their numerical experiments in terms of a self-similar hierarchy of vortices of different radii. We think that their interpretation is due to the threshold they use to define what they will call (or not) a coherent structure and does not reflect the fact that, according to the wavelet analysis we have performed for similar flows [6, 13, 14], some coherent structures are scaling distributions of vorticity which present, instead of a characteristic radius, a range of radii corresponding to all scales of the inertial range.

The prediction of a singularity in some vortex cores does not contradict the existing theorems [15, 16] which state that in a bounded domain the two-dimensional Euler flow preserves regularity C^∞ and boundness L^∞ for arbitrary finite times, assuming a certain regularity (at least $\Lambda^{1+\varepsilon}$) of the initial conditions. Therefore if the initial vorticity field is regular the flow will not develop singular coherent structures. But, as we have shown in sect. 2, initial conditions of the type one usually considers in numerical experiments do not—at least in the asymptotic limit of an infinitely small mesh size $\Delta x \to 0$—correspond to regular vorticity fields. So, we claim that in these initial conditions there are at least some singularities, namely some points where $\omega \to \infty$ for $\Delta x \to 0$. Now, due to the Lagrangian conservation of vorticity for the two-dimensional Euler's equations (for $\nu = 0$), it follows that, if N such singular points are present in the initial vorticity field, they will remain singular for all times, being only advected by the flow dynamics.

When there is some dissipation (for $\nu > 0$), *i.e.* for the two-dimensional Navier-Stokes dynamics, we conjecture that those singular points will be germs for the condensation of the vorticity field into a set of N coherent structures, a number which will then be reduced due to the merging process permitted by viscosity. The vortices thus formed will present a cusplike shape for all scales corresponding to the inertial range, but their centre will no longer be singular due to the smoothing effect of dissipation in the smallest scales, and they are therefore called quasi-singularities. To support this scenario we conjecture that the effect of dissipation in the limit of large Reynolds numbers, *i.e.* $\nu \to 0$, is only very local in both wave number and physical spaces; therefore it will not change the overall shape of the coherent structures until the very small dissipative scales where actual singularities (for $\nu = 0$) will be smoothed out and become quasi-singularities (for $\nu > 0$).

6. Energy spectrum of quasi-singular vortices.

Identifying $\alpha \simeq -1/2$ from Benzi *et al.*'s [7] numerical experiments and using relation (3.9) we see that one single quasi-singular vortex $r^{-1/2}$ alone will have an energy power law spectrum $E(k) \sim k^{-4}$. This result is obtained without any statistical argument but via a geometrical approach supposing the presence in the flow of at least one cusplike vortex scaling as $r^{-1/2}$—among a set of more regular structures, vortices and vorticity filaments—, because the spectral slope is dominated by the scaling of the strongest singularity, supposing all singularities are isolated. In the case of an accumulation of singularities [17, 18], the spectral slope may be altered due to the fractal properties of the support of singularities.

Such a geometrical interpretation in terms of singularities had previously been devised for two-dimensional turbulence by Saffman [19], who also predicts a k^{-4} energy spectrum, but unlike the coherent structures observed in laboratory and numerical experiments the singularities he proposes are not axisymmetric. They correspond instead to a distribution of vorticity along fronts, which are only observed numerically during the very early time evolution of the flow and before the emergence of coherent structures. Gilbert [18] has proposed a different scenario for the formation of coherent structures, according to which singular vorticity sheets, such as those proposed by Saffman, will form spiral structures with an accumulation of singularities in their centres.

Finally we have numerically checked [20] that the quasi-singularity we propose to model some coherent structures encountered in two-dimensional turbulent flows, namely a cusplike vortex $r^{-1/2}$ discretized on a finite grid, remains stable under the Navier-Stokes dynamics, even if it is perturbed by some noise. We have also observed [20] that such a cusplike vortex organizes the random field in its vicinity, which supports our hypothesis of the vorticity field condensation around the quasi-singular points of the initial flow as a scenario for the formation of coherent structures. During this process the initial k^{-1} energy spectrum in the small scales, which corresponds to the superimposed random vorticity distribution, becomes k^{-4} due to the accretion of noise within the quasi-singularity [20]. We will now perform extensive numerical experiments with initial conditions presenting different levels of regularity, and superpositions of quasi-singularities having different scaling exponents, in order to check if the flow dynamics will (or not) preferably select certain exponents. We will also use the wavelet transform to measure the scaling behaviour of a given coherent structure all along the flow evolution and see how its exponent varies (or not) in time. We will also follow the time evolution of the coherent structure function $\omega(\psi)$ corresponding to this coherent structure. We will then extend this study to a set of coherent structures and compute the histogram of both the scaling exponent α of the vorticity field and the scaling exponent β of the coherent structure function, this for all coherent structures. We will also test the merging of two same-sign cusplike coherent structures and verify that the distance below which they merge is much smaller than for flat core coherent structures. In fact, we predict this is the case, and that in consequence quasi-singular (cusplike) coherent structures will have less chance of merging than regular (flat core) coherent structures, which may explain the fact that the former survive on very long time scales.

7. Conclusion.

To conclude our analysis of Nguyen Duc and Sommeria [2]'s experiment: we predict that the vortices with a linear coherent structure function $\omega(\psi)$ correspond to smooth initial distributions of the vorticity field, while the vortices with a nonlinear coherent structure function they also observe have been created by a condensation of the vorticity field around [illegible]

coherent structures have a cusplike shape with only the vortex cores regularized by dissipative effects.

This conclusion also applies to the interpretation of numerical experiments of two-dimensional turbulent flows: if the initial energy is confined in a limited-band spectrum [21] with no energy in the smallest scales, then the initial vorticity field is smooth and the structures which may appear in this field during the time evolution would be very regular with flat vortex cores, while if the energy is initially distributed among all the resolved scales, namely until the cut-off scale, as is the case for most numerical experiments of two-dimensional turbulence [5-7], then it approximates—due to the limited resolution—an initial distribution of vorticity which contains some quasi-singularities; these local extrema will act as germs for the condensation of the vorticity field into cusplike coherent structures which are observed after a long-time evolution of the flow. This sensitivity of the Navier-Stokes dynamics to the initial nonregularity of the flow, if it is confirmed, may impair the quest for a universal behaviour of fully developed turbulence in two dimensions.

In conclusion, we propose a new geometrical «picture» of two-dimensional turbulence, in which the power law energy spectrum is interpreted in geometrical terms without any statistical argument, supposing the presence in the flow of some (at least one) quasi-singular coherent structures scaling as $r^{-1/2}$. These coherent structures are scaling distributions of vorticity which present, instead of a characteristic length, a range of lengths, corresponding to all scales where the energy spectrum has a power law behaviour. We conjecture that the flow dynamics selects and condensates the vorticity field around some quasi-singular points already present in the initial conditions, which gives rise to cusplike vortices with a core smoothed by dissipation.

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