The Continuous Wavelet Transform of Two-Dimensional Turbulent Flows*

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'In the last decade we have experienced a conceptual shift in our view of turbulence. For flows with strong velocity shear ... or other organizing characteristics, many now feel that the *spectral description has inhibited fundamental progress*. The next "El Dorado" lies in the *mathematical understanding of coherent structures in weakly dissipative fluids*: the formation, evolution and interaction of metastable vortex-like solutions of nonlinear partial differential equations ... .' Norman Zabusky (1984)

INTRODUCTION

I have chosen to present here a personal point of view concerning the current state of our understanding of fully-developed turbulence. By this I mean the study of dissipative flows in the limit of large Reynolds numbers

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(the Reynolds number being a dimensionless number characterizing the ratio of the nonlinear advection to the linear dissipation), that is, the limit where the dissipation becomes negligible so that the dynamics of the flow is essentially dominated by the nonlinear interactions.

After more than a century of turbulence study (Reynolds, 1883), no convincing theoretical explanation has given rise to a consensus among physicists (for a historical review of the various theories of turbulence, see (Von Neumann, 1949), (Monin and Yaglom, 1975), (Farge, 1990). In fact, there exist a large number of *ad hoc* models, called 'phenomenological', that are widely used by fluid mechanicians to interpret experiments and to compute many industrial applications where turbulence plays a role. However, it is still not known whether fully-developed turbulence actually has the universal behavior (independent of initial conditions and boundary conditions) assumed for it in the limit of infinitely large Reynolds numbers and infinitely small scales. Already in 1979, in an unpublished article (Farge, 1979), I expressed reservations about our understanding of turbulence and thought that we did not yet know which are the 'good questions' to ask. Ten years of work on the subject have persuaded me that we have not yet identified the 'good objects,' by which I mean the structures and elementary interactions from which it will be possible to construct a satisfying statistical theory of fully-developed turbulence.

In my opinion, and as Zabusky expresses in the quotation (Zabusky, 1984) I have used as an epigraph to this article, ignorance of the elementary physical mechanisms at work in turbulent flows arises in part from the fact that we reason in Fourier modes (wave vectors), constructed from functions that are not well-localized; this viewpoint ignores the presence of the coherent structures that can be observed in physical space and whose dynamic role seems essential to us. In fact, these coherent structures are observed both in experiments carried out in the laboratory (Jimenez, 1981), (Van Dyke, 1982), (Couder and Basdevant, 1986) and in numerical experiments based on the fundamental equations of fluid mechanics (Kim and Moin, 1979), (Basdevant, Legras, Sadourny and Béland, 1981), (McWilliams, 1984), (Farge and Sadourny, 1989), but the current statistical theory (Monin and Yaglom, 1975) does not take them into account. Thus, the visualization of the evolution of two-dimensional turbulent fields numerically computed (Figure 1) leads us to conjecture that the dynamics of a two-dimensional turbulent flow is essentially dominated by the interactions between the coherent structures that advect the residual flow situated between them; the latter itself seems to play no dynamic role. We think that this point of view can be generalized to three dimensions as well, because the existence of coherent structures has also been observed in the context of three-dimensional flows (Kim and Moin, 1979), (Jimenez, 1981), (Hussain, 1986), but their topology is more complex (Moffatt, 1990). Consequently, the wavelet transform, which decomposes the fields on a set of functions with compact (or quasicompact) support and thus permits an analysis in both space *and* scale, seems to be a good tool, not only for analyzing and interpreting the experimental results obtained in two-dimensional turbulence, but also in the long term for attempting to construct a more satisfactory statistical theory of fully-developed turbulence (see color plate for Figure 1).

1. FULLY-DEVELOPED TURBULENCE

1.1. THE EQUATIONS

The fundamental equation of the dynamics of an incompressible (constant density throughout time) and Newtonian (stress proportional to the velocity gradients) fluid is the Navier-Stokes equation:

$$\left\{\begin{array}{l}\partial_t\vec{\mathrm{V}} + (\vec{\mathrm{V}}\cdot\nabla)\vec{\mathrm{V}} + \frac{1}{\rho}\nabla\mathrm{P} = \nu\nabla^2\vec{\mathrm{V}} + \vec{\mathrm{F}},\\ \nabla\cdot\vec{\mathrm{V}} = 0,\\ \text{initial conditions},\\ \text{boundary conditions},\end{array}\right. \tag{1}$$

where $\vec{\mathrm{V}}$ is the velocity, $\vec{\mathrm{F}}$ is the resultant of the external forces per unit of mass, and ν is the kinematic viscosity.

We remark here that the mathematical intracability of the Navier-Stokes equation arises from the fact that the small parameter ν, which tends to zero in the limit of large Reynolds numbers, i.e. for very turbulent flows, appears in the term containing the highest-order derivative, namely the dissipation term $\nu\nabla^2\vec{\mathrm{V}}$. Thus the character of the equation, which is given by the term containing the highest-order derivative, changes as ν tends to zero, since in this limit it is the advection term $(\vec{\mathrm{V}}\cdot\nabla)\vec{\mathrm{V}}$ that dominates. When $\nu = 0$, or $\mathrm{Re} = \infty$, the Navier-Stokes equation is called Euler's equation and the nonlinear advection term is no longer controlled by the linear dissipation term. Moreover, Euler's equation conserves energy whereas the Navier-Stokes equation dissipates it; thus the former is reversible in time whereas the latter is irreversible.

If one takes the curl of equation (1), one can eliminate the pressure term; this gives the equation of the curl of the velocity, also called the vorticity

$\vec{\omega} = \nabla \times \vec{\mathrm{V}}$, as:

$$\begin{cases} \partial_t \vec{\omega} + (\vec{\mathrm{V}} \cdot \nabla)\vec{\omega} = (\vec{\omega} \cdot \nabla)\vec{\mathrm{V}} + \nu \nabla^2 \vec{\omega} + \nabla \times \vec{\mathrm{F}} \\ \vec{\omega} = \nabla^2 \psi \,. \end{cases} \tag{2}$$

If one considers a regime state, i.e. a state of the flow such that the energy contribution from the external forces is dissipated by the viscous friction, then:

$$\frac{d\vec{\omega}}{dt} = (\vec{\omega} \cdot \nabla)\vec{\mathrm{V}} \,. \tag{3}$$

Thus, in three dimensions, the Lagrangian variation of vorticity is equal to the product of the vorticity by the velocity gradients, which leads to stretching of the vorticity tubes by the velocity gradients, a mechanism that may explain the transfer of energy towards the smallest scales of the flow in three dimensions (cf. Section 1.2).

In two dimensions, vorticity becomes a pseudo-scalar, for $\vec{\omega} = (0, 0, \omega)$ is then perpendicular to $\nabla \vec{\mathrm{V}}$. Therefore in this case, the vorticity stretching by the velocity gradients is no longer possible. Indeed in two dimensions, the vorticity is a Lagrangian invariant of the motion because, in the absence of dissipation, it is conserved throughout time along a fluid trajectory:

$$\frac{d\vec{\omega}}{dt} = 0 \,. \tag{4}$$

If we now consider the vorticity gradients, we have:

$$\partial_t \overrightarrow{\nabla \omega} + (\vec{\mathrm{V}} \cdot \nabla)\overrightarrow{\nabla \omega} = -(\overrightarrow{\nabla \omega} \cdot \nabla)\vec{\mathrm{V}} \,. \tag{5}$$

Thus, in two dimensions, the Lagrangian variation of the vorticity gradients is equal to the product of the vorticity and velocity gradients, a mechanism that may explain the transfer of enstrophy towards the smallest scales of the flow in two dimensions (cf. Section 1.2).

1.2. THE INVARIANTS

In the absence of external forces ($\vec{\mathrm{F}} = 0$) and of dissipation ($\nu = 0$), Euler's equation (i.e. the Navier-Stokes equation for $\nu = 0$) conserves energy in two and three dimensions:

$$\mathcal{E}(t) = \frac{1}{2} \int_{-\infty}^{+\infty} \mathrm{V}^2(\vec{x}, t) d^n \vec{x} = \text{constant} \,, \tag{6}$$

where n is the dimension of the space. Using Plancherel's identity, we have:

$$\mathcal{E}(t) = \frac{1}{2} \int_{-\infty}^{+\infty} \hat{\mathrm{V}}^2(\vec{k}, t) d^n \vec{k} = \int_0^{\infty} \mathrm{E}(|\vec{k}|, t) d^n |\vec{k}| = \text{constant} \,, \tag{7}$$

where

$$\hat{\mathrm{V}}(\vec{k}) = \int_{-\infty}^{+\infty} \mathrm{V}(\vec{x}) e^{i\vec{k}\cdot\vec{x}} d^n\vec{x}$$

and $\mathrm{E}(k)$ is the energy integrated in spectral space over crowns of constant radius $|\vec{k}|$. $\mathrm{E}(k)$ characterizes the distribution of energy among the various scales (in the sense of wave numbers) of the motion, a modal distribution predicted by the statistical theory (cf. Section 1.4).

In the special case of two dimensions, which is particularly interesting for studying the large-scale dynamics of geophysical flows for which the two-dimensional approximation is valid, Euler's equation also preserves enstrophy:

$$\Omega(t) = \frac{1}{2} \int_{-\infty}^{+\infty} \omega^2(\vec{x}, t) d^2\vec{x} = \text{constant} \tag{8}$$

or using Plancherel's identity:

$$\Omega(t) = \frac{1}{2} \int_{-\infty}^{+\infty} \hat{\omega}^2(\vec{k}, t) d^2\vec{k} = \int_{0}^{\infty} \mathrm{Z}(|\vec{k}|, t) d^2|\vec{k}| = \text{constant} \tag{9}$$

where

$$\hat{\omega}(\vec{k}) = \int_{-\infty}^{+\infty} \omega(\vec{x}) e^{i\vec{k}\cdot\vec{x}} d^2\vec{x}$$

and $\mathrm{Z}(k)$ is the enstrophy integrated in spectral space over crowns of constant radius $|\vec{k}|$. $\mathrm{Z}(k)$ characterizes the distribution of enstrophy among the various scales (wave numbers) of the motion and, on imposing hypotheses of homogeneity and of isotropy in the statistical sense (cf. Section 1.4), one can relate it to the modal energy and obtain:

$$\mathrm{Z}(k) = k^2 \mathrm{E}(k) \tag{10}$$

where $k = |\vec{k}|$.

1.3. UNIQUENESS, REGULARITY AND ANALYTICITY OF THE SOLUTIONS

We shall attempt to summarize briefly the existing theorems on uniqueness, regularity and analyticity of the solutions of the Euler and Navier-Stokes equations in two and three dimensions. In general, these various theorems consider regular initial conditions, i.e. they use C^∞ functions with bounded support in $\mathbf{R}^n$. In this case, in two dimensions the Lagrangian conservation of vorticity (4) implies that:

$$\|\omega\|_{\mathrm{L}^\infty} = \sup |\omega(x, y)| = \text{constant}\,, \tag{11}$$

which leads to the prediction that Euler's solutions remain regular for all times in a bounded domain (Lichtenstein, 1925), (Wolibner, 1933), (Hölder, 1933), (Schaeffer, 1937). Kato (Kato, 1972) has shown that this remains true even if the initial velocity is $C^{1+\epsilon}$ ($\epsilon > 0$). However, we still do not have a regularity theorem for the Euler solutions in an unbounded domain, unless the solutions are constrained to decrease at infinity. An interesting special case, worth noting here, is that of the Kelvin-Helmholtz instability that develops at the interface between two flows of different velocities; if the initial flow presents a discontinuity of the velocity at the interface, it was conjectured by Birkhoff (Birkhoff, 1962), and then proved (Babenko and Petrovich, 1979), (Sulem et al, 1981), that if the interface is initially an analytic curve, then it remains so for a finite time. However, an asymptotic expansion of Moore (Moore, 1979) and numerical results (Meiron and Baker, 1982) suggest that this curve will ultimately always develop a singularity. In two dimensions, the global regularity of the Navier-Stokes equation in an unbounded domain — and this for any viscosity — is a consequence of the regularity of Euler's equation for a bounded domain. Ladyzhenskaya (Ladyzhenskaya, 1963) and Lions (Lions, 1969) have proven the global regularity of the Navier-Stokes equation in two dimensions, provided that the viscosity is sufficiently high. However, for the limit of ν tending to zero, the problem remains open because one does not know how to take into account the boundary layers that develop at the walls.

In three dimensions one shows that, assuming regular initial conditions, one has uniqueness, regularity and analyticity in the following cases:

- for all times, provided that the viscosity is high enough (Reynolds < 1 initially) (Leray, 1933);
- for arbitrary viscosity and arbitrary boundaries, provided that time is sufficiently long (Leray, 1933), or for short times in the absence of boundaries (Kato, 1972);
- for all times, but for dissipation of the form $-\nu'(-\nabla^2)^\alpha$, where $\nu' > 0$ and $\alpha \geq 5/4$ (Ladyzhenskaya, 1963), (Lions, 1969).

For more detailed expositions on this subject, see (Rose and Sulem, 1978), (Frisch, 1983) and (Temam, 1984).

We note here that the Lagrangian conservation of vorticity (4) for Euler's equation in two dimensions implies that, if the initial field of vorticity has N singularities, these will be conserved for all time, and thus there will not be any regularization of the flow in the absence of dissipation (Farge and Holschneider, 1990); we will return to this point in the last part of the article.

1.4. STATISTICAL THEORY AND SPECTRAL SLOPES

It is first of all necessary to note that the statistical theory of fully-developed turbulence, which is attributed to Kolmogorov, was discovered quasi-simultaneously by him (Kolmogorov, 1941a,b,c), and others (Obukhov, 1941), (Onsager, 1945), (Heisenberg, 1948), and (Von Weizsäcker, 1948), each using different methods that we shall not present here. (see (Battimelli and Vulpiani,1982) for a review of the history of this subject.) Kolmogorov studied the way in which the Navier-Stokes equation in three dimensions distributes the energy among the various degrees of freedom of the flow. This type of approach is very classical in statistical mechanics, but the difficulty here arises from the fact that turbulent flow is an open thermodynamical system, that is, not isolated from the exterior, due to the forces acting on the flow either at large scale (external forces) or at small scale (viscous frictional forces). It is therefore necessary to limit oneself to a range of intermediate scales, called the inertial range, where one supposes that energy is transferred between the various degrees of freedom, namely the scales of the flow, and this in a conservative manner. We thus have:

$$\lambda \ll \ell \ll \mathrm{L}\,, \tag{12}$$

where

- λ denotes the dissipative scales where kinetic energy is transformed into thermal energy under the effect of viscous friction,
- ℓ denotes the scales of turbulent motion dominated by nonlinear advection that transfer kinetic energy between them in a conservative manner, and
- L denotes the integral scales where energy is injected by external forces.

Kolmogorov assumes that for this range of scales ℓ, the flow is statistically homogeneous, that is, invariant under translation, and isotropic, that is, invariant under rotation. ('Statistical' here means in the sense of Gibbs ensemble averages, i.e. in averaging from a set of realizations of the same flow.) He also assumes that the energy is transferred, from the large to the small scales, at a constant rate ε which is independent of scale and equal to the quantity of energy dissipated by the scales smaller than λ. He adds to this the hypothesis that the skewness S , namely the departure from Gaussian behavior of the velocity probability distribution, is constant.

From this he deduces the following scale law for the two-point correlation function of the velocity:

$$\langle |v(r+\ell) - v(r)|^2 \rangle \sim C\varepsilon^{2/3}\ell^{2/3}, \tag{13}$$

where $C = -4\ell/(5S)$ is Kolmogorov's constant. Upon Fourier transforming to the space of wave vectors $\vec{k}$, this yields:

$$\mathcal{E} = \frac{1}{2}\int_{-\infty}^{+\infty} \hat{V}^2(\vec{k})d^3\vec{k} \sim C\varepsilon^{2/3}k^{-2/3}. \tag{14}$$

Considering now the energy integrated over crowns of constant radius $k = |\vec{k}|$, we obtain the (Kolmogorov) spectrum (Figure 2a):

$$E(k) \sim C\varepsilon^{2/3}k^{-5/3}. \tag{15}$$

Notice here that Kolmogorov never expressed his law in Fourier space, whereas the same result is obtained from the theory of Heisenberg (Heisenberg, 1948) or Von Weizsäcker (Von Weizsäcker, 1948) working directly in spectral space.

Following a remark of Landau (Landau and Lifchitz, 1971) concerning the random character of energy transfers in the inertial zone, Kolmogorov (Kolmogorov, 1961) added to his law (13) a lognormal correction in $(\ln(L/\ell))^\beta$, β being the dispersion constant of the logarithm of ε. Various experiments carried out in wind-tunnels (Batchelor and Townsend, 1949), (Anselmet, Gagne, Hopfinger and Antonia, 1984) have shown that the energy associated with the small scales of a turbulent flow is not densely distributed in space. This observation of a spatial intermittency of the support of the energy transfers has led several authors to conjecture that this support is fractal (Mandelbrot, 1975 and 1976), (Frisch, Sulem and Nelkin, 1978) or multifractal (Parisi and Frisch, 1985), which also gives rise to a correction of the Kolmogorov spectrum (15), of the form $(kL)^{-(3-D/3)}$, D being the Hausdorff dimension of the dissipative structures.

In two dimensions, the conservation of enstrophy, related to energy by the relation (10), leads to a modification of the statistical theory, conjectured by Von Neumann (Von Neumann, 1949), since it prevents the energy from cascading from large to small scales in the limit k tending to infinity. The energy is, on the contrary, transferred to the large scales according to a spectral law similar to that of Kolmogorov (15); this is the inverse energy cascade of two dimensional turbulence (Figure 2b). Kraichnan (Kraichnan, 1967) and Batchelor (Batchelor, 1969) have shown that there is then another cascade, but of enstrophy (9) from large to small scales (Figure 2b), and, assuming the rate of enstrophy transfer η to be

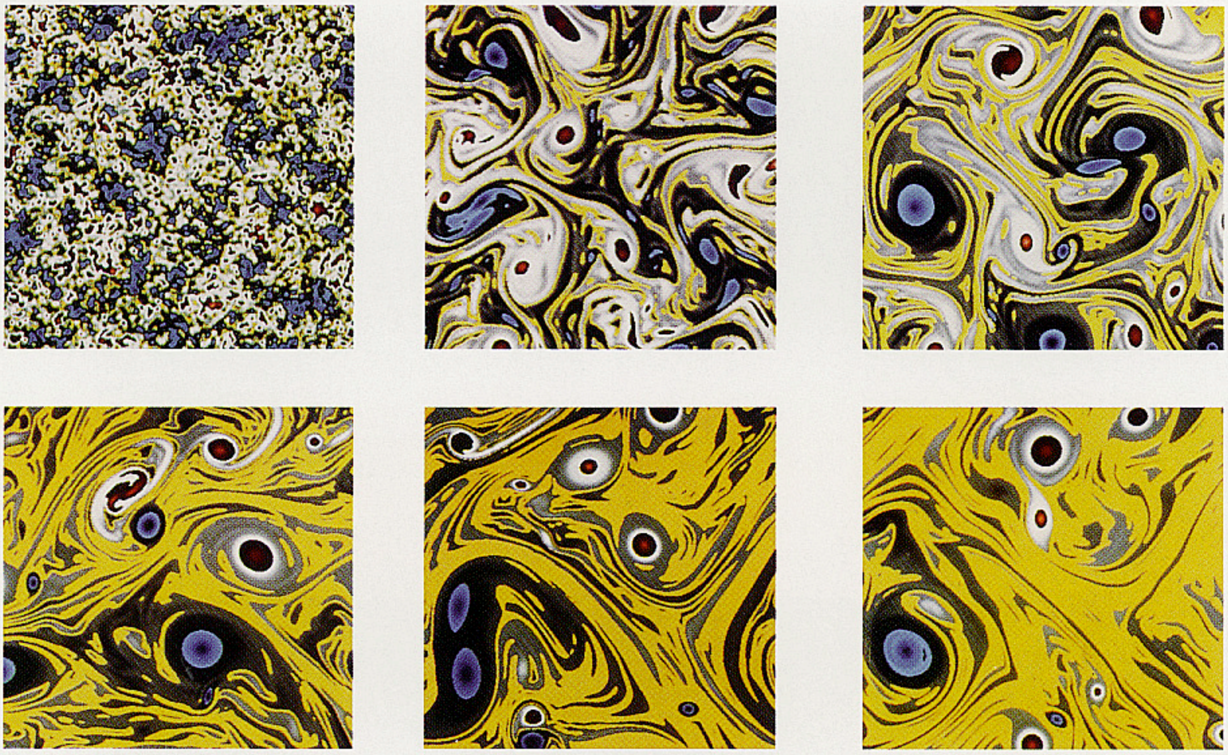


FIGURE 1. Direct numerical simulation of a decaying two-dimensional turbulent flow: time evolution during 10^5 time steps computed from a random initial vorticity field (Farge 1988, Farge and Sadourny 1989).

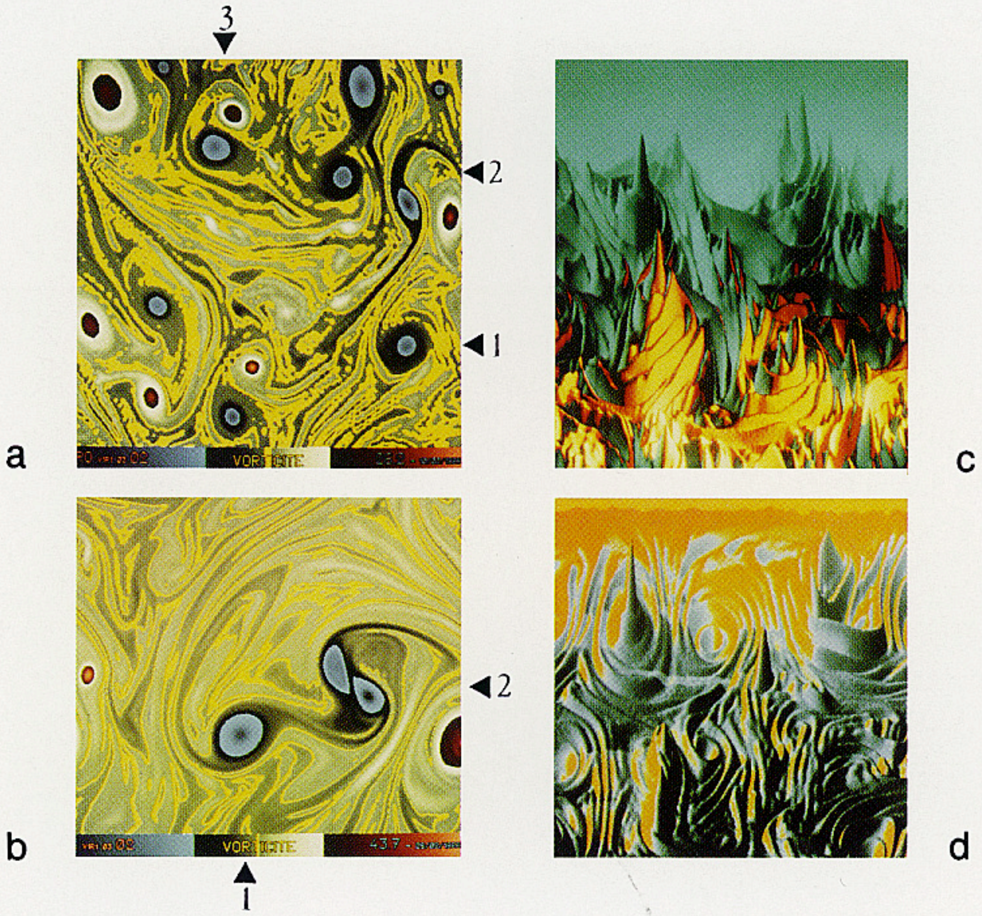


FIGURE 4. Dynamics and topology of coherent structures in two-dimensional turbulence.

a. and b. Elementary interactions: a1) Axisymmetrization, a2) Filamentation, a3) Binding, b1) Deformation, b2) Merging.

c. and d. Cusp-like shape of the most excited coherent structures (Farge 1988, Farge and Sadourny 1989).

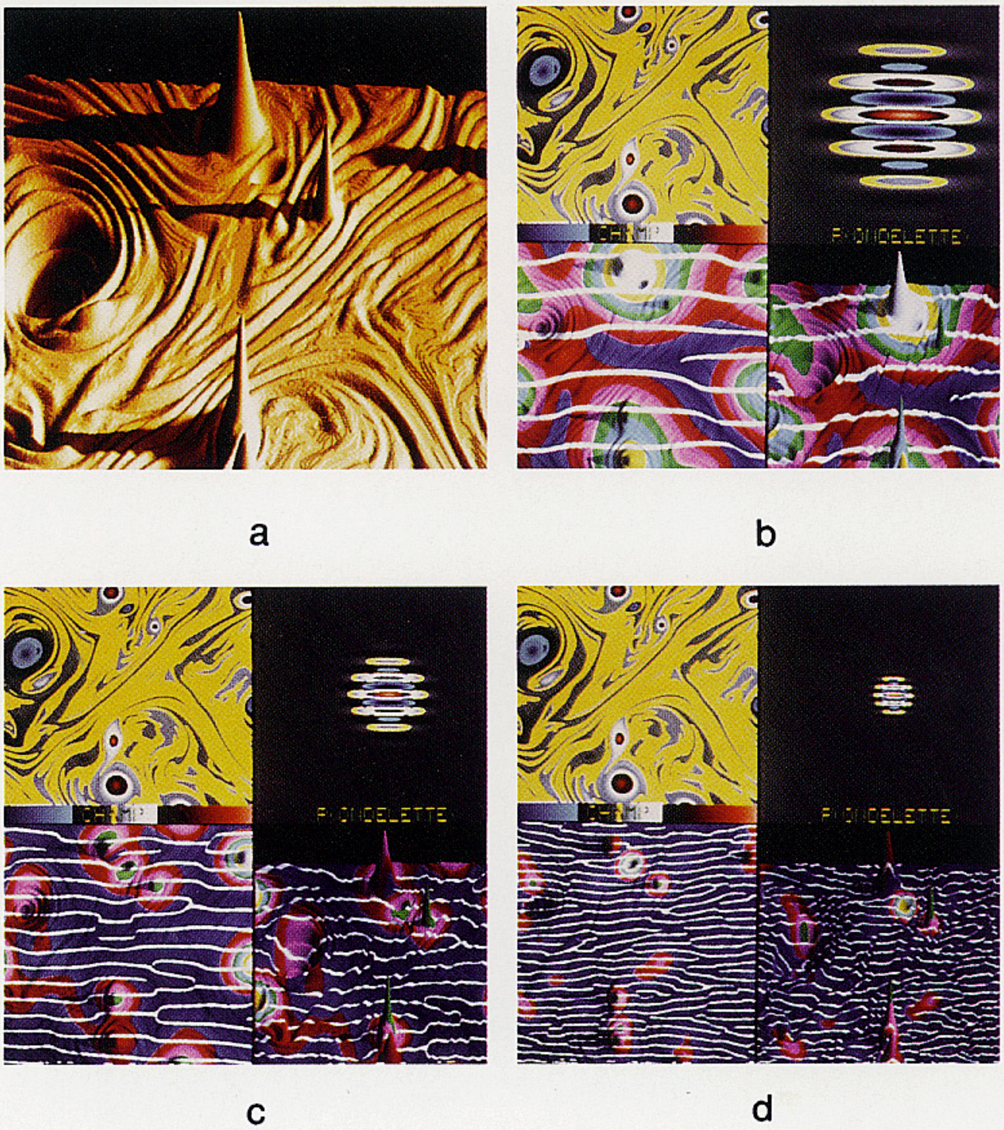


FIGURE 7. Two-dimensional wavelet analysis computed in L1-norm, using Morlet wavelet with $k\Psi = 5$ and $\theta = 0$, of the same two-dimensional turbulent flow as Figure 6 (Farge, Holschneider, and Colonna 1990).

On the display we have superposed the vorticity field, in perspective representation, the wavelet coefficients module, color coded in order of increasing luminance (blue, red, magenta, green, cyan, yellow, white), and the zeroes of the wavelet coefficients phase, represented by white isolines.

a. Vorticity field to be analyzed (sampled on 512^2 points).
b. Wavelet coefficients at scale $k = 8$.
c. Wavelet coefficients at scale $k = 16$.
d. Wavelet coefficients at scale $k = 32$.

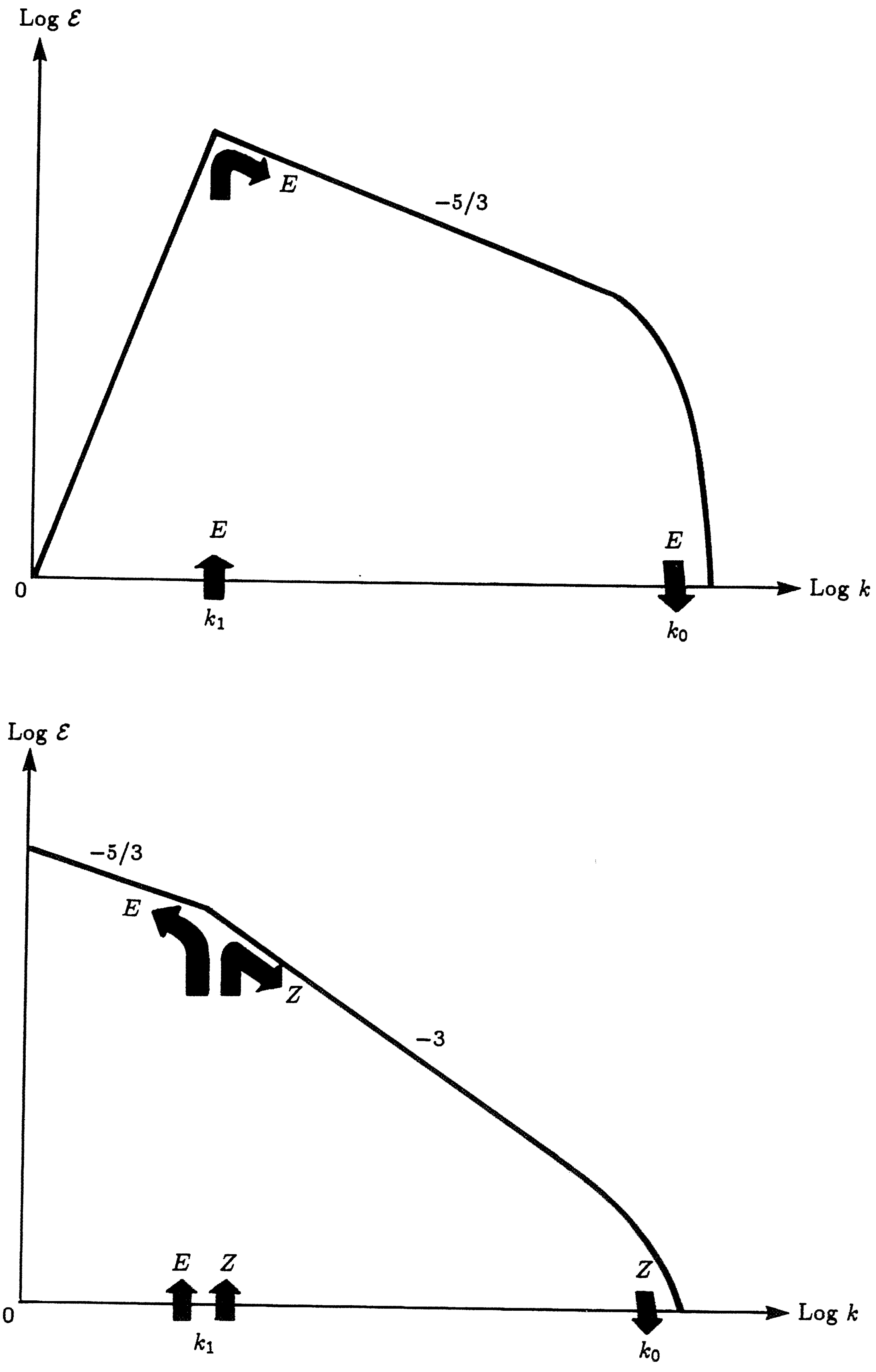


FIGURE 2. Fourier energy spectra predicted by the statistical theory of turbulence: a. Kolmogorov spectrum for three-dimensional turbulence (Kolmogorov 1941a, b, c), b. Kraichnan-Batchelor spectrum for two-dimensional turbulence (Kraichnan 1967, Batchelor 1969).

constant in the inertial range, they predicted the following energy spectrum for two dimensional turbulence:

$$E(k) \sim \eta^{2/3} k^{-3} . \tag{16}$$

Kraichnan (Kraichnan, 1971) added a correction term in $(\ln(kL))^{-1/3}$ to account for the fact that in two dimensional turbulence the transfers are not local in spectral space.

1.5. NUMERICAL SIMULATIONS AND COHERENT STRUCTURES

We shall limit ourselves here to the case of incompressible two-dimensional turbulence. It is very difficult to carry out laboratory experiments under rigorously two dimensional conditions, whereas numerical simulations can attain Reynolds numbers much higher in two than in three dimensions because, for in two dimensions the number of mesh points varies directly with $(Re)^1$, while in three dimensions it varies as $(Re)^{9/4}$. In general, the numerical experiments are carried out with periodic boundary conditions and are initialized with random fields whose energy is distributed over a large spectral band, usually up to the cutoff scale of the mesh (Figure 3a). We then follow the flow evolution both in physical space, by visualizing the vorticity field (Figure 1) which is the most significant quantity since it is a Lagrangian invariant of the flow for $\nu = 0$, as well as in spectral space, by plotting the energy spectrum integrated over the crowns $|\vec{k}| =$ constant (Figure 3b).

In the numerical experiments of two dimensional turbulence, the observed energy spectra usually follow a power law in k^{-4} (Figure 2b), and not in k^{-3} as is predicted by the theory of Kraichnan (Kraichnan, 1967) (Figure 2b). It is thought that this disparity is due to the intermittency of the flow, for which we shall propose a geometric interpretation (cf. Section 2.2) based on the presence of coherent structures we observe in numerical experiments, but that are not addressed by the statistical theory.

What is a coherent structure? We do not presently have any theory to describe them, therefore we must content ourselves with a qualitative description, more similar to the approach of a zoologist than to that of a fluid mechanician. But this taxonomic and descriptive stage is a necessary preliminary to any further theory. Basing ourselves on visualizations of numerically computed two-dimensional turbulent fields (Figure 1), we may characterize coherent structures in the following way:

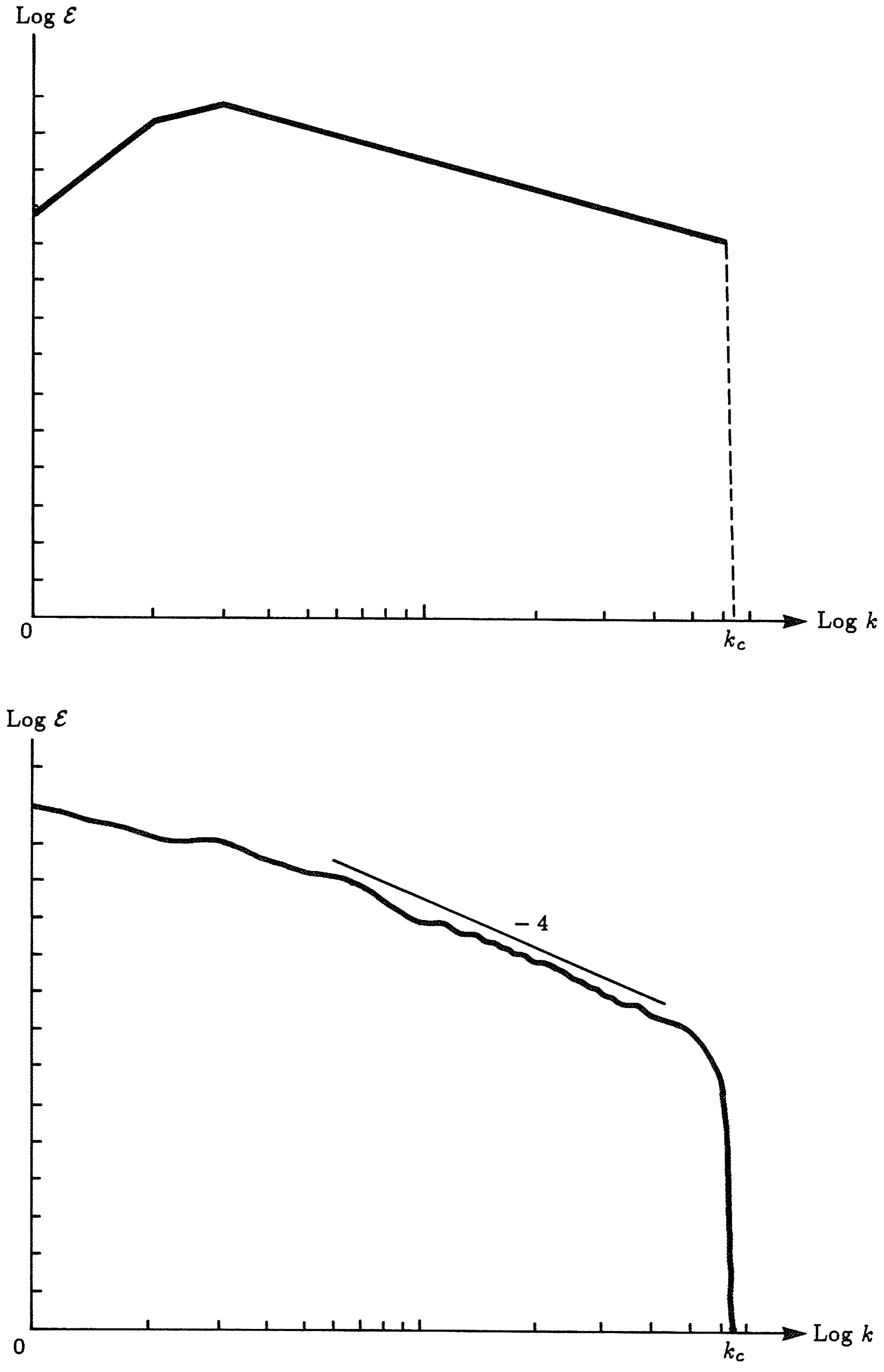


FIGURE 3. Time evolution of the Fourier energy spectrum of a decaying two-dimensional turbulent flow numerically computed (Farge 1988, Farge and Sadourny 1989): a. Initial energy spectrum, b. Energy spectrum after 10^5 time steps.

- they are vortical structures, that is, regions of the flow where vorticity prevails over deformation,
- they contain most of the energy and enstrophy of the flow,
- they form spontaneously by a condensation of the vorticity field, for which we do not have any theory at the present time,
- they are encountered over a large range of scales; in fact throughout the whole inertial range if there is no forcing,
- they survive on time scales much larger than the eddy turnover time $\tau = Z^{1/2}$ selected by the statistical theory as being the characteristic time of the enstrophy transfers.

When one studies the dynamics of coherent structures, one can distinguish the following states and elementary interactions:

1. *relaxed state*, characterized by
   - *aximetrization* (see arrow 1 on Figure 4a) in the absence of interaction with nearby coherent structures,
2. *weakly excited states*, with
   - either *deformation* (see arrow 1 on Figure 4b) under the influence of a nearby coherent structure having the same intensity,
   - or *filamentation* (see arrow 2 on Figure 4a) when the deformation becomes too strong under the influence of a more intense nearby coherent structure,
3. *strongly excited states*, with
   - either *binding* (see arrow 3 on Figure 4a) of two very close coherent structures of opposite sign and comparable intensity,
   - or *merging* (see arrow 2 on Figure 4b) of two or more very close coherent structures having the same sign.

(See color plate for figure 4.)

Although we have defined coherent structures in a purely qualitative manner using observations obtained from numerical simulations, it is equally possible to characterize them in a more quantitative manner by studying the velocity gradient tensor, called the stress tensor:

$$\nabla V = (\nabla V)_{ij} = \partial_j V_i = V_{i,j} \,. \tag{17}$$

Its symmetric part characterizes the strain undergone by the fluid element, and is written:

$$\frac{1}{2}(\nabla V+\nabla V^t) = \begin{vmatrix} \partial_1 v_1 & \frac{1}{2}(\partial_2 v_1 + \partial_1 v_2) \\ \frac{1}{2}(\partial_1 v_2 + \partial_2 v_1) & \partial_2 v_2 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} s_1 & s_2 \\ s_2 & -s_1 \end{vmatrix}, \quad (18)$$

where ∇V^t is the transposed matrix of ∇V, and $s_1 = 2\partial_1 v_1 = -2\partial_2 v_2$ since the fluid is incompressible.

Its antisymmetric part corresponds to the rotation of the fluid element, and is written:

$$\frac{1}{2}(\nabla V - \nabla V^t) = \begin{vmatrix} 0 & \frac{1}{2}(\partial_2 v_1 - \partial_1 v_2) \\ \frac{1}{2}(\partial_1 v_2 - \partial_2 v_1) & 0 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 0 & \omega \\ -\omega & 0 \end{vmatrix}. \quad (19)$$

Adding (18) and (19), the equation (17) becomes:

$$\nabla V = \frac{1}{2}\begin{vmatrix} s_1 & s_2 + \omega \\ s_2 - \omega & -s_1 \end{vmatrix}. \quad (20)$$

Calculating the curl of the vorticity equation (2), with neither forcing ($\vec{F} = 0$) nor dissipation ($\nu = 0$), one obtains the equation of the curl of the vorticity, also called the divorticity $\xi = \nabla \times \omega = (\partial_2\omega - \partial_1\omega, 0)$:

$$\left\{ \begin{array}{c} \partial_1\xi + (\vec{V}\cdot\nabla)\xi = (\nabla V)\xi \\ \nabla\cdot\xi = 0 \end{array} \right\}. \quad (21)$$

Assuming that the spatio-temporal variations of the strain tensor ∇V are slow compared to those of the curl of the vorticity ξ (the hypothesis of (Weiss, 1981)), equation (21) becomes linear in Lagrangian coordinates:

$$\left\{ \begin{array}{c} \omega(x_0, y_0, t) = \omega_0 \\ \frac{d\xi}{dt} = (\nabla V)\xi \end{array} \right\}. \quad (22)$$

The eigenvalues of the stress tensor are

$$\alpha = \pm\frac{1}{2}\sqrt{-\det(\nabla V)} = \pm\frac{1}{2}[s_1^2 + s_2^2 - \omega^2]^{1/2}. \quad (23)$$

One can then separate the flow into two types of regions where the Lagrangian dynamic is different:

a) **elliptic regions** (Figure 5a) corresponding to the purely imaginary eigenvalues, where rotation dominates strain and for which two initially close fluid particles remain nearby for all time, their distance oscillating only slightly throughout time. These regions are thus associated with the geometrically stable coherent structures,

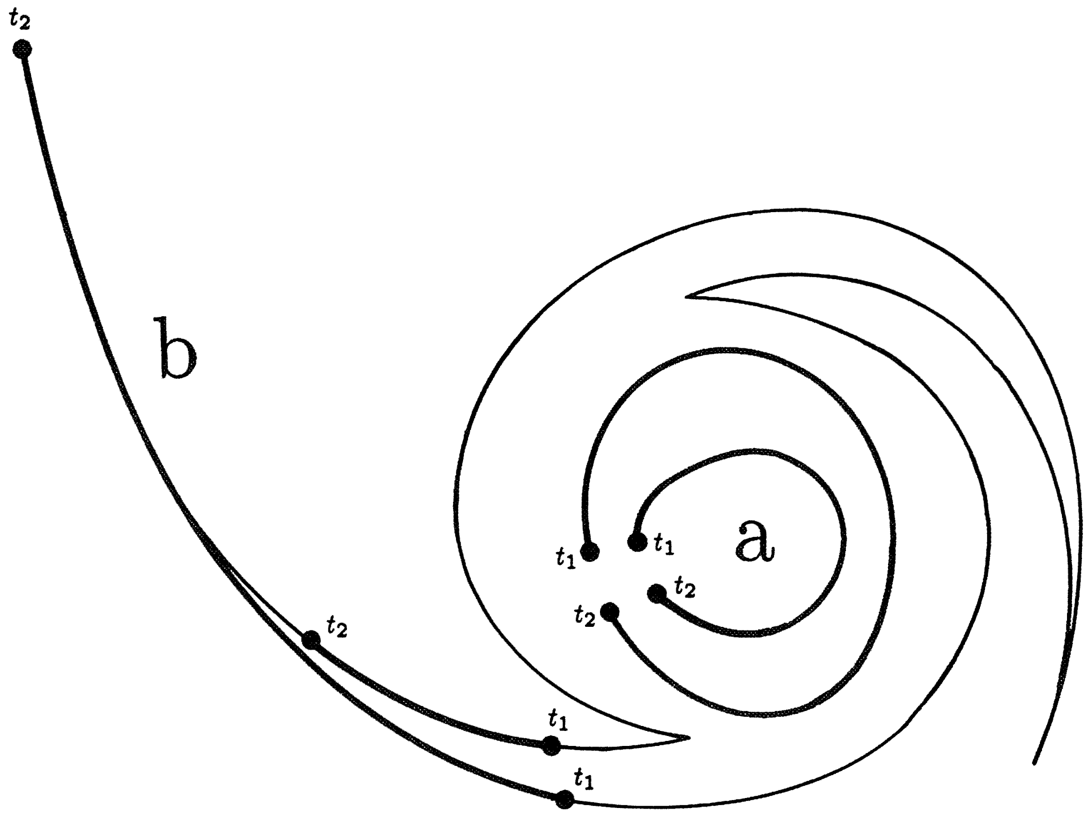


FIGURE 5. Dynamical characterization of a coherent structure in two-dimensional turbulence. a. Elliptic region corresponding to the coherent structure: two nearby particles at t_1 will remain close at t_2. b. Hyperbolic region corresponding to the vorticity filaments emitted by the coherent structure during a deformation: two nearby particles at t_1 will separate exponentially after t_2.

b) **hyperbolic regions** (Figure 5b) corresponding to the real eigenvalues of opposite sign, where strain dominates rotation and for which two initially close fluid particles separate exponentially as time goes, their distance being contracted in one direction and dilated in the other. These regions are thus associated with the vorticity filaments stretched by the velocity gradients.

In conclusion, in view of the numerical simulations we have carried out, we conjecture that the dynamics of two-dimensional turbulent flows is essentially dominated by the interactions between the coherent structures that advect the residual flow. This background flow is formed by the vorticity filaments emitted during the vortex interactions, but it only plays a passive role, contrarily to the coherent structures which are dynamically active. If this conjecture is verified, it then becomes important to find a method capable of separating the coherent structures from the background flow, not by a thresholding technique such as the one we have just presented, but rather by using a filtering technique that respects the local regularity of the flow (cf. Section 2.3).

2. WAVELET ANALYSIS OF TWO-DIMENSIONAL TURBULENT FLOWS

2.1. RESULTS

It is important to realize that the wavelet transform is not being used to study turbulence simply because it is currently fashionable; but rather because we have been searching for a long time for a technique capable of decomposing turbulent flows in both space *and* scale simultaneously. If, under the influence of the statistical theory of turbulence, we had lost in the past the habit of considering the flow evolution in physical space, we have now recovered it thanks to the advent of supercomputers and their associated means of visualization. They have revealed to us a menagerie of turbulent flow patterns, namely, the existence of coherent structures and their elementary interactions (cf. Section 1.5) for which the present statistical theory is not adequate.

During the 1980's, I was very involved with displays of turbulent fields in physical space, and I have proposed a normalization for their representations in order to compare results obtained by numerical simulations and by laboratory experiments which was essentially done in morphological terms (Farge, 1987), (Farge, 1990a). Just before Alex Grossman first spoke to me about wavelets in 1984, I had envisioned making bandpass filters in two-dimensional Fourier space and then reconstructing the filtered field in physical space, spectral band by spectral band, so as to match up certain interactions observed in physical space with the turbulence cascades predicted by the statistical theory in Fourier space. This method, however, ran into problems because of the Gibbs phenomenon, which occurs in physical space when the frequency filtering is too abrupt. The wavelet transform now allows us to analyze two-dimensional turbulence in a much more satisfactory way, and it provides us with new mathematical tools for analyzing the local regularity of a function (Holschneider and Tchamitchian, 1988), (Jaffard, 1989). In an earlier paper (Farge and Rabreau, 1988), we carried out one-dimensional wavelet transforms, using Morlet's wavelet, of sections of several vorticity fields numerically computed (Farge, 1988). This showed how, during the flow evolution, starting from an initial random distribution of vorticity, the smallest scales of the flow become more and more localized and concentrated in the centers of coherent structures (Figure 6). We have subsequently confirmed this result using a two-dimensional Morlet wavelet (Farge, Holschneider and Colonna, 1990) (Figure 7). The smallest scales of the flow are localized in the coherent structures cores and are excited when the latter are deformed by interactions with other nearby structures; this led us to conjecture that, contrary to generally accepted ideas, dissipation also acts in the center of coherent structures. This is confirmed when one visualizes the vorticity Laplacian field which corresponds to the dissipation term; the maxima of the dissipation are well localized in the cores of the coherent structures (Farge, Holschneider and Colonna, 1990). Moreover, some enstrophy dissipation is necessary inside two same-sign coherent structures which are strongly interacting in order that they ultimately merge, which corroborates our hypothesis. (See color plate for Figure 7)

2.2. INTERPRETATION

Now we will present a new model (Farge, 1990c), (Farge and Holschneider, 1990), (Farge, Holschneider and Philipovitch, 1991) in which we relate the energy spectrum of two-dimensional turbulent flows to the presence of cusp-like, or quasi-singular, coherent structures. This model was suggested by our previous wavelet analysis of two-dimensional turbulent fields.

Consider the immediate neighborhood of a coherent structure which, by definition, is a quasi-stationary solution of Euler's equation

$$\left\{ \begin{matrix} \partial\omega/\partial t + J(\psi, \omega) = 0 \\ \omega = \nabla^2\psi \end{matrix} \right\} \tag{24}$$

where ω denotes the vorticity and ψ the stream function, and suppose that locally, in a domain as small as one wishes, this solution is axially symmetric, i.e.

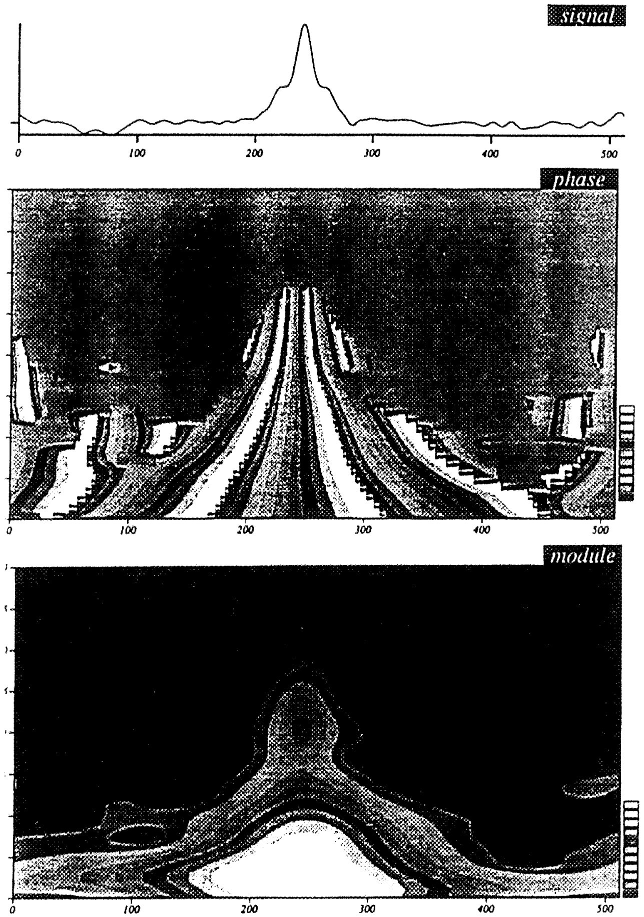
$$\omega \sim r^{\alpha} \qquad \text{with} \qquad \alpha \in \mathbf{R} \tag{25}$$

where r denotes the distance to the center of the coherent structure. Then one infers the following scaling laws:
for the circulation

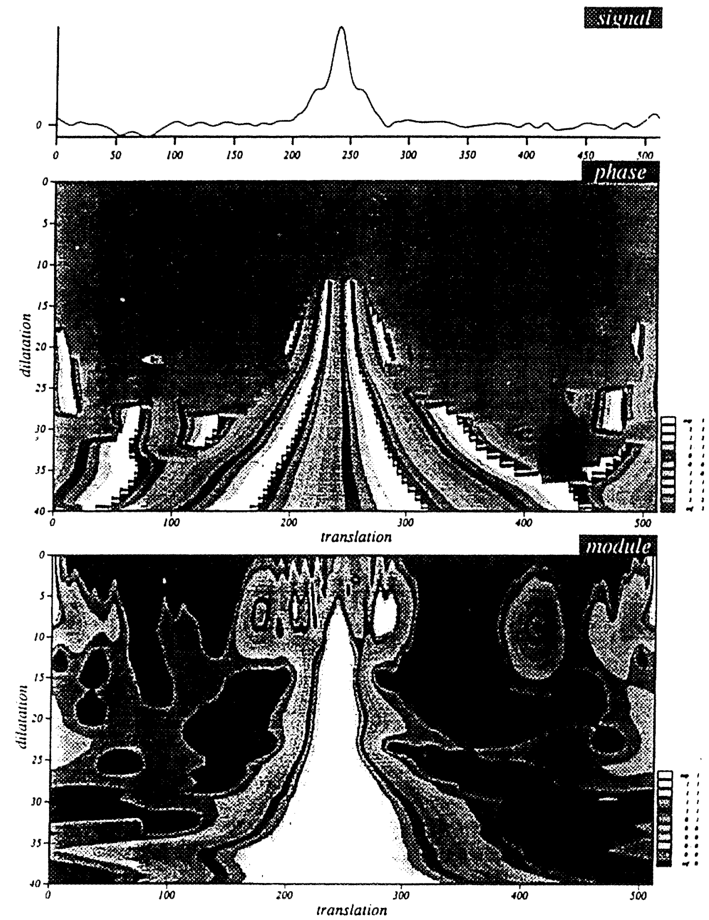
$$\Gamma(r) = 2\pi \int_0^r \omega(r')\, r'\, dr' \sim r^{\alpha+2}, \tag{26}$$

for the energy

$$\varepsilon(r) = 2\pi \int_0^r V(r')^2 r'\, dr' \sim r^{2\alpha+4}, \tag{27}$$



a. Normalisation $a^{-1/2}$



b. Normalisation en a-1

I GURE 6. One-dimensional wavelet analysis, using Morlet wavelet with $k\psi = 5$, of a decaying two-dimensional turbulent flow (ut of the vorticity field sampled on 512 points) corresponding to the last time step of Figure 1 (Farge and Rabreau 1988): a Complex-valued wavelet coefficients (their phase and their modul) in L^2-norm, b. Complex-valued wavelet coefficients (their p ase and their modul) in L^1-norm.

and for the enstrophy

$$\Omega(r) = 2\pi \int_0^r \omega(r')^2 r'\, dr' \sim r^{2\alpha+2}. \tag{28}$$

In order that these three invariants remain finite, we must have

$$\alpha \geq -1\,. \tag{29}$$

A coherent structure is characterized by a pointwise relation between the vorticity and the stream function, called the coherent structure function, for which we predict:

$$\omega \sim (\psi - \psi_0)^{\frac{\alpha}{\alpha+2}} \tag{30}$$

where ψ_0 is the value of the stream function at the core of the coherent structure. Finally, calculating the energy in Fourier space and using Plancherel's identity, we obtain the spectral distribution of energy integrated over the crowns $|\vec{k}|$ = constant, which should scale according to a power law of the form

$$E(k) \sim k^{-2\alpha-5}. \tag{31}$$

Relying on the results of numerical experiments (Benzi et al, 1987), we identify the coefficient $\alpha = -1/2$, value which guarantees that the circulation (26), energy (27), and enstrophy (28) all remain finite. In the neighborhood of the coherent structures, we then have the following scaling laws

$$\omega(r) \sim r^{-1/2}, \tag{32}$$

$$\omega(\psi) \sim (\psi - \psi_0)^{-1/3}, \tag{33}$$

$$E(k) \sim k^{-4}. \tag{34}$$

Equation (32) corresponds to a singular axially symmetric vortex distribution in the center of the coherent structures. We have proven (Farge, Holschneider and Philipovitch, 1991) that such cusp-like structures are stable under the flow dynamics, even if they are perturbed by a strong noise. A singular distribution of vorticity in the core of the coherent structures is not a contradiction to the existing theorems (Wolibner, 1933), (Hölder, 1933), (Cottet, 1987) for Euler equation in two dimensions, which predict that in a bounded domain this equation conserves regularity (at least $C^{1+\epsilon}$) and the boundedness (L^∞) of the initial conditions. In fact, we can consider another case where initially one has functions with a finite energy ($L^{4-\epsilon}$) but presenting singularities. Then, due to the Lagrangian conservation of vorticity (4), if one has initial singular points ($\omega(x_0, y_0) = \infty$ at $t = 0$), then these initial singularities are advected by the flow, but are conserved for all times ($\omega(x_0, y_0) = \infty$ as $t \to \infty$).

Equation (33) seems confirmed by laboratory experiments of turbulent flow in mercury, where the dynamics is constrained to two dimensions by the presence of a magnetic field (Nguyen Duc and Sommeria, 1988). In such a flow one observes, either a linear coherent structure function, or a cusp-like one very similar in form to (33); this is achieved without modifying the experimental parameters, quite apart from the initial conditions which are never completely reproducible. Our theory permits us to propose the following interpretation of these results: the nature of the coherent structures, characterized by their coherent structure function (30), depends on the initial conditions. If these are *regular,* then the coherence structure function is linear and the distribution of the vorticity is regular with a constant vorticity in the center such that

$$\omega \sim \psi^{\frac{\alpha}{\alpha+2}} \sim \psi \qquad \text{for} \qquad \alpha = \infty, \tag{35}$$

but if the initial distribution has *singularities,* the flow organizes itself around them and one obtains

$$\omega \sim \psi^{-1/3}, \tag{36}$$

corresponding to cusp-like coherent structures of the form

$$\omega \sim r^{-1/2}, \tag{37}$$

with a smoothing of the vorticity in the cores of the vortices caused by dissipation due to viscosity. Contrary to the remark we made previously in Section 1.1, which implied that the nature of the solution to the Navier–Stokes equation may change in the limit $\nu \to 0$ because this singular limit affects the highest order terms of the equation, we are led to the following conjecture: at large scales the singular solutions to Euler's equation dominate, whereas at very small scales, on the order of V/ν, they are locally smoothed by dissipation, and become what we call "quasi-singularities" (Farge and Holschneider, 1990).

Such cusp-like coherent structures are observed by visualizing the vorticity field of two-dimensional turbulent flows, where we see very spiky vortex cores (cf. Figures 4c and 4d). Recently Benzi and Vergassola, by performing a wavelet analysis of numerically computed two-dimensional homogeneous turbulent flows (Benzi and Vergassola, 1990), have confirmed the existance of coherent structures having negative scaling exponents between −0.4 and −0.6, close to the value of −0.5 that we have predicted.

Our theory also suggests a new interpretation of the energy spectrum of two-dimensional turbulent flows in purely geometrical terms, reminiscent of Saffman's interpretation (Saffman, 1971), and quite different from the statistical arguments currently applied to the problem. Thus, according to equation (34), a flow with many coherent structures will have a spectral energy distribution varying as k^{-4} if it has at least one isolated coherent structure with a quasi-singularity in $r^{-1/2}$, up to the dissipative scales where viscosity will smooth the vortex core. Indeed, the spectral slope is determined by the strongest isolated singularity present in the flow. This slope in k^{-4} is steeper than the slope in k^{-3} predicted by the statistical theory of two-dimensional turbulence (Kraichnan, 1967), (Batchelor, 1969), but agrees well with the slopes obtained from most of the numerical experiments of two-dimensional turbulent flows (Basdevant et al, 1981), (McWilliams, 1984), (Farge and Sadourny, 1989). In fact, for these numerical experiments one chooses initial energy spectrum presenting a power-law behavior up to the cut-off scale (Figure 3a), which corresponds to quasi-singular vorticity distributions in physical space (Farge and Holschneider, 1990). The only numerical simulations which obtain k^{-3} energy spectrum (Brachet et al, 1988) are actually initialized with a band-limited spectrum with no energy in the small scales, which corresponds to a smooth vorticity field; in these experiments one does not observe the emergence of isolated coherent structures. If this sensitivity to initial conditions of Euler and Navier-Stokes equations in two dimensions is confirmed, we will have to reconsider the hypothesis of a universal behaviour of turbulence (i.e. independent of the flow initial conditions). Perhaps universality isn't where we think it is; perhaps we should, instead, search for it in the shape of the coherent structures and in their elementary interactions.

2.3. PERSPECTIVES

Applications of the wavelet transform to the theory of turbulence presently follows three directions: analysis, filtering, and numerical integration of turbulent flows. Concerning analysis, the wavelet transform offers the possibility of observing the flow from both sides of the Fourier transform at once (up to the limit of the uncertainty principle); this gives us a method for relating the dynamics of coherent structures in physical space to the redistribution of energy among the various Fourier modes. The wavelet transform is a particularly ideal tool for studying intermittency, one of the major unsolved problems in the theory of turbulence today (Frisch and Orszag, 1990). Indeed, the statistical theory of Kolmogorov (Kolmogorov, 1941a,b,c) assumes that the transfer of energy is distributed densely in physical space, as would be true, for example, for a Gaussian distribution of velocities. But several laboratory results (Batchelor and Townsend, 1949), (Anselmet, Gagne, Hopfinger and Antonia, 1984) seem to violate this hypothesis. Likewise, in two dimensions, we interpret the fact that the numerical results give steeper spectral slopes (of the order of k^{-4}) than the predictions of the statistical theory (k^{-3}) in terms of an intermittency of the enstrophy transfers: the spatial support of enstrophy transfers would not be dense, but would diminish with scale until reaching the dissipative scales, where dissipation would only act on a very small sub-domain of physical space. Many theoretical models, all based on *ad hoc* probabilistic considerations, have been proposed to explain intermittency (Kolmogorov, 1961), (Mandelbrot, 1976), (Frisch, Sulem and Nelkin 1978), (Parisi and Frisch, 1985). The theory we have proposed to interpret our results (cf. Section 2.2), deduced from our wavelet analysis of coherent structures (cf. Section 2.1), allows a purely geometrical interpretation of intermittency: it may be related to the cusp-like shape of some very excited axisymmetric coherent structures whose spatial support would decrease with scale following a power-law behavior until the dissipative scales are reached; this would only represent a very small subdomain in physical space, corresponding to the cores of the coherent structures where vorticity would be locally smoothed by dissipation. Recently, by applying the wavelet transform to analyze three-dimensional flows, we have also found some very strong intermittency that we have related to the presence of coherent structures; this led us to a similar geometrical interpretation of intermittency in three dimensions (Farge, Guezennec, Ho and Méneveau, 1990).

The second direction in which the wavelet transform could play an important role in turbulence is the possibility of extracting coherent sructures from the rest of the flow by filtering the wavelet coefficients. In effect, this would allow us to test our conjecture (cf. Section 1.5) that the dynamics of a turbulent two-dimensional flow may be essentially dominated by nonlinear interactions among the coherent structures, the rest of the flow being only passively advected by them. Thanks to Bruno Torrésani, Pierre Jean Ponenti (Centre de Physique Théorique du CNRS-Luminy) and Richard Kronland-Martinet (Laboratoire de Mécanique et Acoustique du CNRS-Marseille), we have tried a technique (described in the chapter of this book by Tchamitchian and Torrésani) to extract the skeleton of the wavelet coefficients of the vorticity field. This method, which has given interesting results for phase stationary signals (Escudié and Torrésani, 1989), doesn't let us easily separate coherent structures from the rest of the flow. However, the skeleton allows us to delimit the influence cones associated with the coherent structures. Then, after cancelling the wavelet coefficients outside of these influence cones, we may be able to use an inverse wavelet transform to reconstruct new vorticity fields, whose background flow would have been filtered out so that only the coherent structures remain. Another, and quite similar, approach, we are presently trying with Victor Wickerhauser from Yale University, is to perform wavelet packet decomposition of the vorticity field and only retain the wavelet packet coefficients which are attached to coherent structures before reconstructing the filtered vorticity field. We then plan to perform comparative numerical simulations of both filtered and unfiltered flows. If the dynamics of the original and filtered vorticity fields are similar, then our conjecture will have been verified.

Finally, from the point of view of numerical analysis, if our conjecture is verified, one could then hope to reduce the number of degrees of freedom required to numerically integrate the evolution of turbulent flows. Thus, if one defines the degrees of freedom using Fourier modes, their number varies as $(Re)^{9/4}$ in three dimensions or $(Re)^1$ in two dimensions, where Re is the Reynolds number of the flow being calculated. If it could be proved that coherent structures are the dynamically active part of the flow, it would then be sufficient to consider only the degrees of freedom associated with them, that we could express, for example, using orthogonal wavelets or wavelet packets. This would allow us to design new algorithms, variants of the Large Eddy Simulation technique (Ferziger, 1981), based not on the traditional separation between resolved small Fourier wave numbers and parametrized large Fourier wave numbers, but rather on a more physical separation between the wavelet coefficients attached to coherent structures, which would be computed explicitly, and the remaining wavelet coefficients which would be only parametrized globally to take into account their energy and enstrophy but not their phases.

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