Wavelets in Burgers land

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1. Introduction

The initial value problem for the Burgers equation (1) with $\nu > 0$ has a unique solution [1]. When $\nu \to 0$, this solution converges to a weak solution of (2), known as 'entropy solution' because it is characterized among all the weak solutions of (2) by an entropy condition. Although the viscosity is vanishing, the energy still decays due to the presence of negative jumps in the velocity, known as shocks.

A classical way of approximating this entropy solution numerically is to solve (1) with a 'small' viscosity. We compare this approach with an attempt to solve (2) directly, applying a nonlinear wavelet filter at every timestep. We check if the solutions thus obtained also converge to the entropy solution when numerical resolution increases.

To solve equations (1) and (2) numerically, we use a classical Fourier pseudo-spectral method, x being discretized on N equidistant gridpoints. One important point is that the nonlinear term is discretized in skew-symmetric form in order to ensure conservation of energy. We advance in time from t = 0 to t = 5, using the Runge-Kutta fourth order scheme. The timestep is 1/16 N, which has been chosen small enough so that there is virtually no numerical dissipation.

Viscous Burgers equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \qquad \textbf{(1)}$$

Inviscid Burgers equation

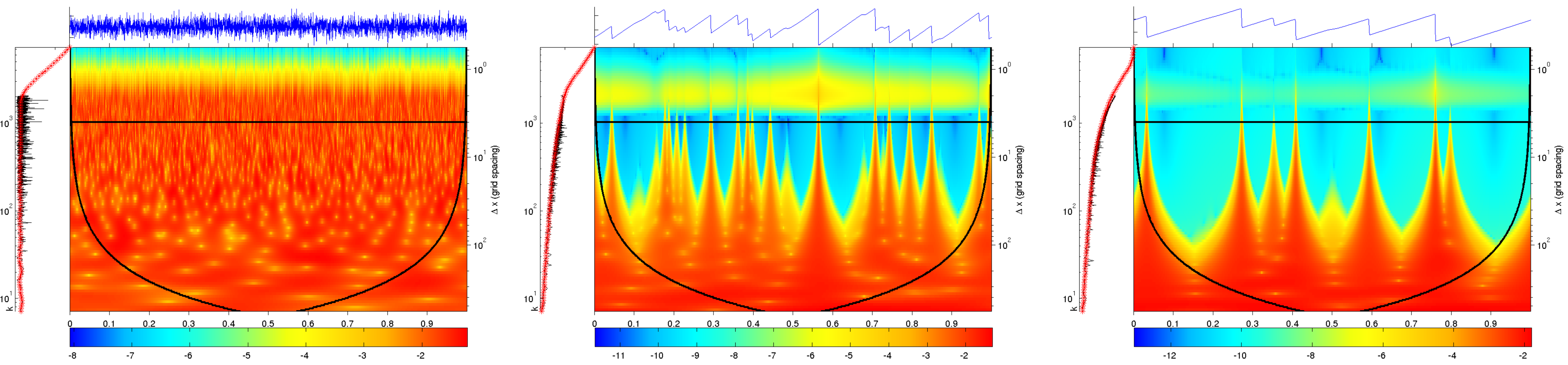
$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0 \qquad \textbf{(2)}$$

- u(x,0) is one reqlization of a standard Gaussian white noise
- boundary conditions are periodic on [-1,1]

2. Wavelet analysis of a viscous solution

We analyze the behaviour of the reference viscous solution in space and scale by computing its **continuous wavelet transform** using a complex Morlet wavelet. As shocks form, the velocity becomes very sparse in wavelet space, which reveals its intermittency.

On each of these figures, the scale varies logarithmically from coarse to fine when going upwards. The region outside the black curve is affected by boundary conditions or aliasing. To every scale corresponds a Fourier wavenumber, which allows us to represent the spectrum as a the tilted black plot on the left. The red curve is the wavelet scalogram, which is a smoothed version of the Fourier spectrum. The horizontal axis corresponds to the space coordinate, and the velocity is plotted in blue at the top of the figure.



t = 0 t = 0.5 t = 2

The initial Gaussian white noise (left) has its energy homogeneously distributed throughout the space-scale plane. At t=0.5, one can already see a red cone corresponding to each shock. The decay exponent of the wavelet coefficients from coarse to fine scale is the same for all shocks, because they share the same order of singularity.

Between t = 0.5 and t = 2, shocks merge together and loose energy at fine scales, but their overall shape is preserved. Dissipation occurs because of the negative jumps in the velocity.

3. Description of the filter

The Coherent Vortex Simulation (CVS) filter consists in retaining only the largest coefficients of a wavelet decomposition of the velocity. It has been designed to extract coherent structures out of turbulent flows [3]. In order to preserve the translation invariance of the Burgers equation, we use a quasi-orthogonal complex valued wavelet transform [5] instead of orthogonal wavelets.

At each timestep, we decompose the velocity in the following way :

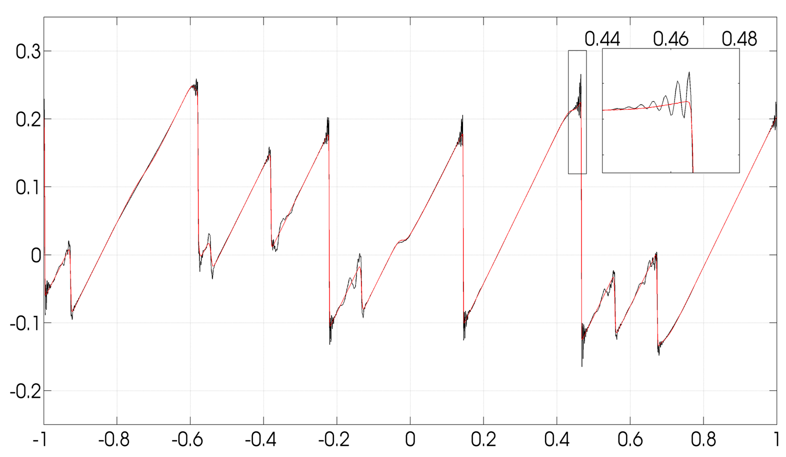
$$u(x,t) = \Re\left(\sum_{i=1}^{2^L} \langle u|\phi_{Li}\rangle(t)\,\phi_{Li}(x) + \sum_{j=L}^{J-1}\sum_{i=1}^{2^j} \langle u|\psi_{ji}(t)\rangle\,\psi_{ji}(x)\right)$$

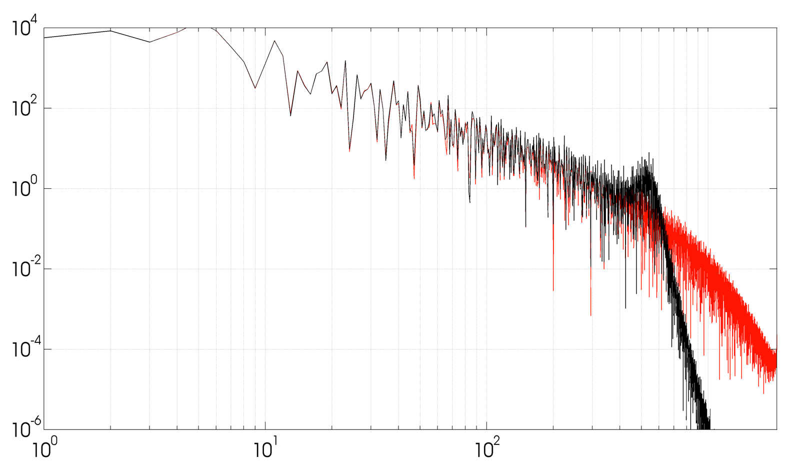
where j is the scale, i is the position, the ϕ_{Li} are the scaling functions and the ψ_{Li} are the wavelets. The coarsest scale of wavelet decomposition was chosen as L = 3.

The **CVS filter** then consists in retaining only the wavelet coefficients above a threshold. The threshold T has to be estimated at each timestep from the velocity itself, We impose the condition that T equals 5 times the standard deviation of the wavelet coefficients below it. The threshold satisfying this condition is found using a fixed point iterative procedure [4]. Additionally, we discard the finest scale j=J-1. This step is necessary to stabilize the method.

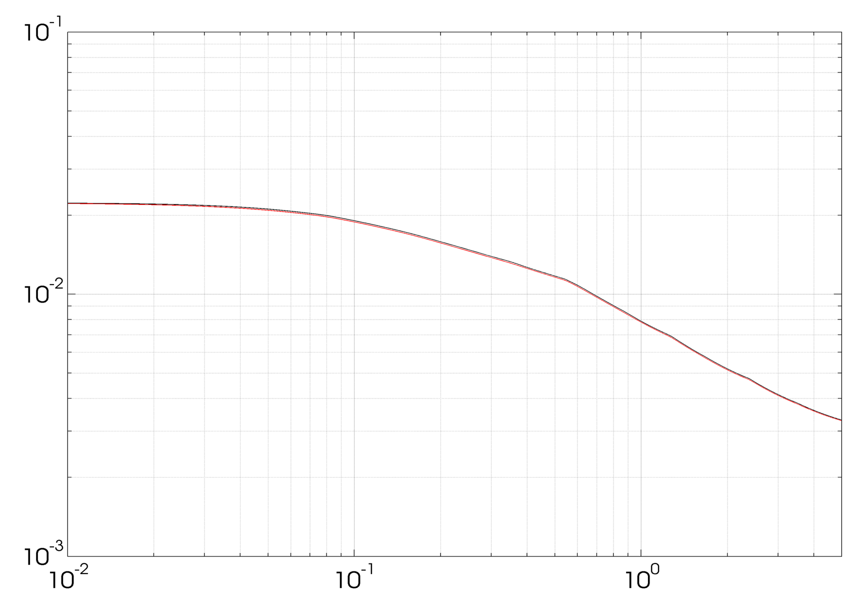
In order to avoid discarding the whole set of coefficients before shocks can form, we only apply the filter for t >= 0.3.

4. Properties of the filtered solution





Velocity (top) and its Fourier spectrum (bottom) for N = 4096 at t = 0.9

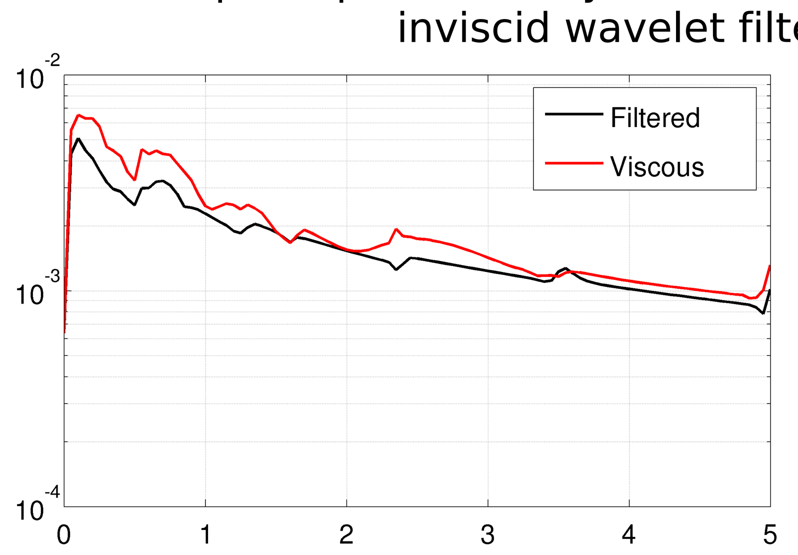


Energy as a function of time

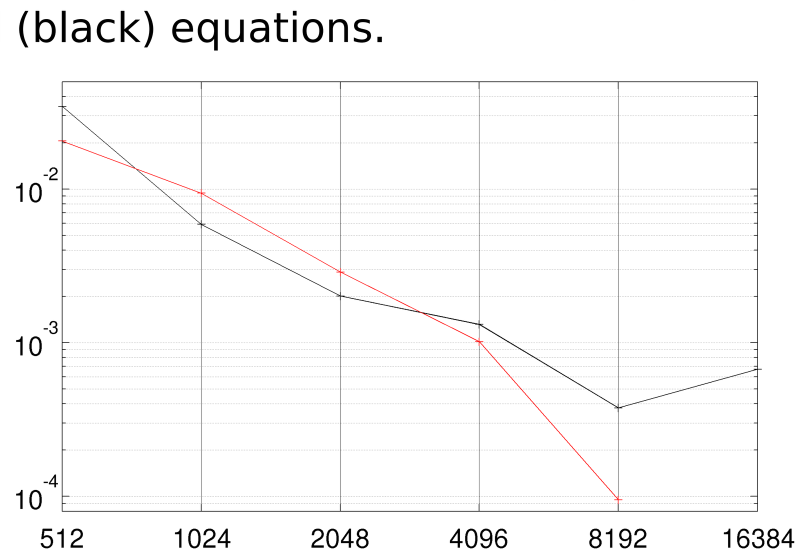
For a given resolution, the solution of the inviscid wavelet- filtered (black) and of the viscous (red) equation fit well together (left, top). The filtered ones has spurious oscillations, but these remain confined to the vicinity of shocks. They correspond to the bump at high wavenumber on the Fourier spectrum (left, bottom). The energy decay behaviors (right) also match.

5. Convergence to the entropy solutions

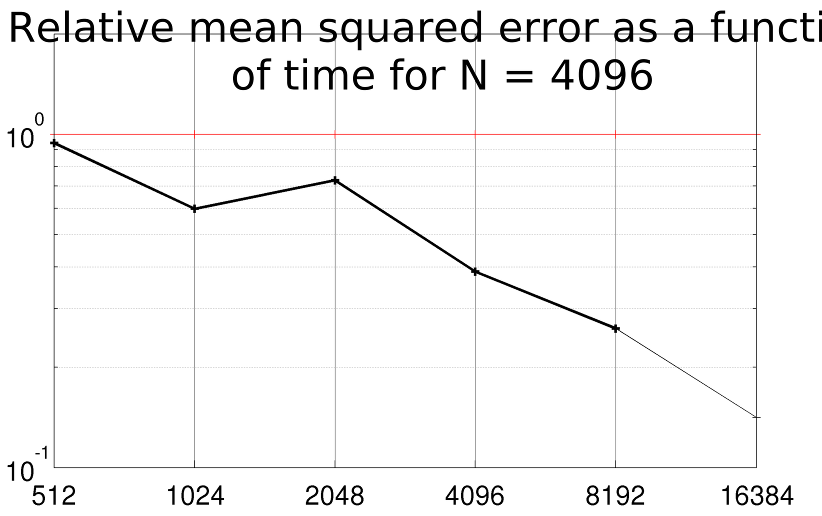
We compare quantitatively the numerical solutions of the viscous (red) and inviscid wavelet filtered (black) equations.



Relative mean squared error as a function of time for N = 4096



Relative mean squared error at t = 5 as a function of resolution



Fraction of retained coefficients at t = 5 as a function of resolution

6. Conclusion and perspectives

We have shown that the inviscid Burgers equation with CVS filtering at every timestep is equivalent to the viscous Burgers equation with a small viscosity. Both methods yield good approximations of the entropy solution. The CVS approximation is slightly better for long integration times.

The translation invariant complex wavelet transform plays a key role in the success of our method. We have not been able to reproduce these results using a real-valued orthogonal wavelet transform. Further investigation will be needed to fully understand this point.

In the future, we plan to apply this approach to other differential equations with singular non-dissipative limits, like the Navier-Stokes/Euler system. The CVS filter has already been applied in this context [2].

References :

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