Energy dissipation in the inviscid limit of a 2D dipole-wall collision

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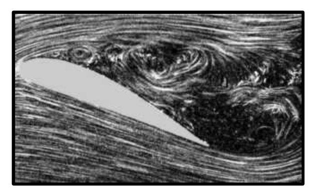
CONTEXT

- The broad context of this work is the **inviscid limit problem for wall-bounded incompressible flows**.
- We focus on the 2D Navier-Stokes equations with **no-slip boundary conditions**:

$$\text{(NS)}\quad \begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u} \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{u}_{|\partial\Omega} = \mathbf{0}, \qquad \mathbf{u}(0,\cdot) = \mathbf{v} \end{cases} \qquad \begin{array}{l} \mathbf{u} := \mathbf{u}(t,\mathbf{x}), \quad \mathbf{x}\in\Omega \\ t\in[0,T], \quad \Omega\subset\mathbb{R}/\mathbb{Z} \end{array}$$



- Existence and uniqueness for smooth initial data and smooth boundary are not an issue.
- We are interested in the behavior of the solution $\mathbf{u}_{\mathrm{Re}}$ as $\mathrm{Re}\to\infty$ (inviscid limit), all other parameters being kept fixed.
- Boundary layer theory[1] assumes that the limit satisfies Euler equation (NS with $\mathrm{Re}=\infty$),
- This assumption seems in **contradiction with experiments**, where **separation** is commonly observed,
- The goal of this study is to observe numerically a **nonvanishing energy dissipation rate** in the inviscid limit.

ROLE OF ENERGY DISSIPATION

From (NS) one can get the following equation for the kinetic energy $e = \frac{1}{2}\mathbf{u}^2$: $\quad \partial_t e + (\mathbf{u}\cdot\nabla)(e+p) = \frac{1}{\mathrm{Re}}\Delta e + \frac{1}{\mathrm{Re}}|\nabla\mathbf{u}|^2$

And by integrating on Ω we get the global energy budget: $E = \int_\Omega e \;\Rightarrow\; \frac{d}{dt}E = -\frac{2}{\mathrm{Re}}Z$ where $Z = \frac{1}{2}\int_\Omega \omega^2$

which can also be integrated on a time interval $[t_a,t_b]$: $\quad \Delta E(t_a,t_b) := E(t_b) - E(t_a) = \frac{2}{\mathrm{Re}}\int_{t_a}^{t_b} Z$ (enstrophy)

Energy dissipation plays an important role in the following theorem:

Theorem 1 (Kato 1984)
For flow in a 2D domain without forcing, the following assertions are equivalent:
i) the Navier-Stokes flow converges to the Euler flow in $C([0,T],L^2(\Omega))$,
ii) the **energy dissipation** of the Navier-Stokes solution in a strip proportional to $\mathrm{Re}^{-1/2}$ around the solid during the time interval [0,T] tends to zero as Re goes to infinity.

Theorem 2 (Sammartino & Caflisch 1998)
For flow in a half-plane with analytic initial data, there exists a critical time τ > 0 (τ = +∞ allowed) such that the Navier-Stokes flow converges to the Euler flow in an analytical norm for t in [0, τ[, and not for t > τ.

SETUP

We perform numerical experiments, taking for the domain Ω an **horizontal channel** of height 0.9.
Time varies in the invertal [0 0.5].
The initial condition is a **vorticity dipole**[4] given by:

$$\begin{aligned} \omega_0 &= \omega_e \sum_{i=1}^{2}\left(1 - \frac{(x-x_i)^2}{r_0^2}\right)\exp\left(-\frac{(x-x_i)^2}{r_0^2}\right) \\ x_1 &= (0.445, 0.5), \quad x_2 = (0.555, 0.5), \\ r_0 &= 0.045, \quad \omega_e = 299.5 \end{aligned}$$

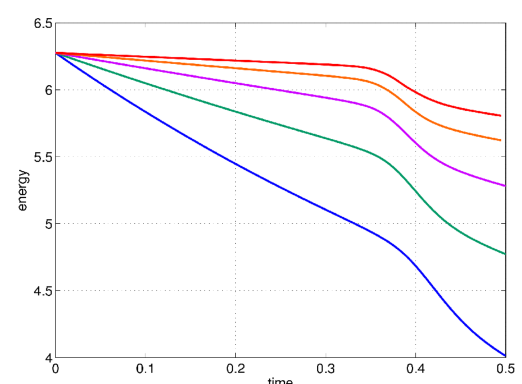
where vorticity is defined by $\omega = \nabla\times\mathbf{u}$

Summary of numerical computations reported here because of Th. 1 we take N ~ Re and not N ~ $\mathrm{Re}^{1/2}$

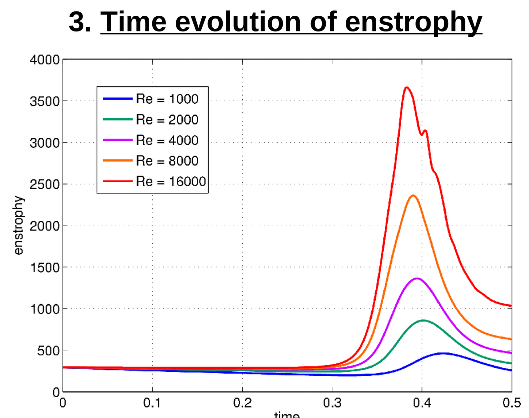
Re	1000	2000	4000	8000	8000	16000	∞
N	4096	8192	8192	8192	16384	16384	4096
η	$2.\ 10^{-5}$	$1.\ 10^{-5}$	$1.\ 10^{-5}$	$2.\ 10^{-5}$	$0.5\ 10^{-5}$	$2.\ 10^{-5}$	$2.\ 10^{-5}$
CPUs	4	8	8	8	1024	1024	4

RESULTS

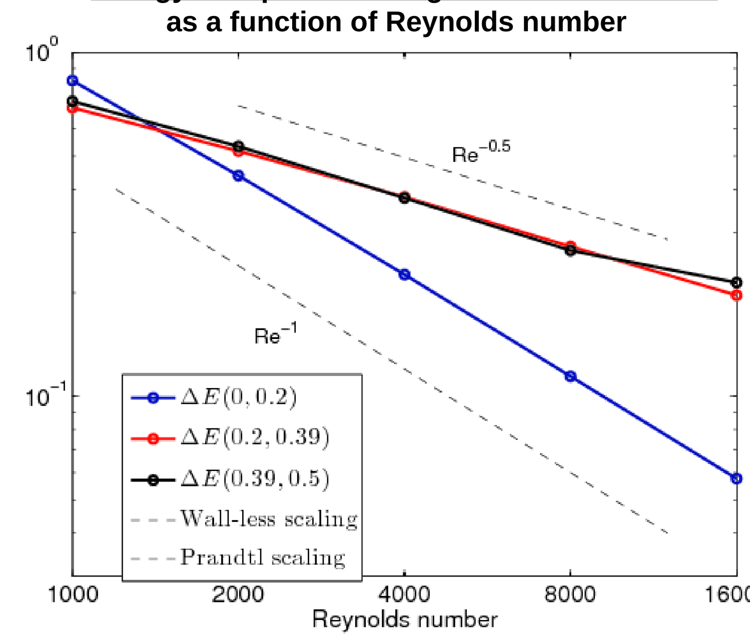
2. Time evolution of energy



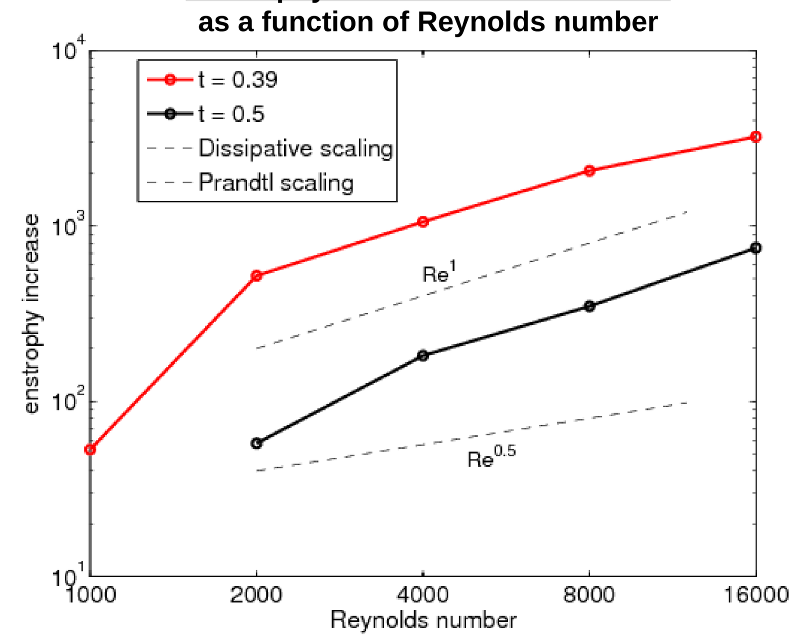
3. Time evolution of enstrophy



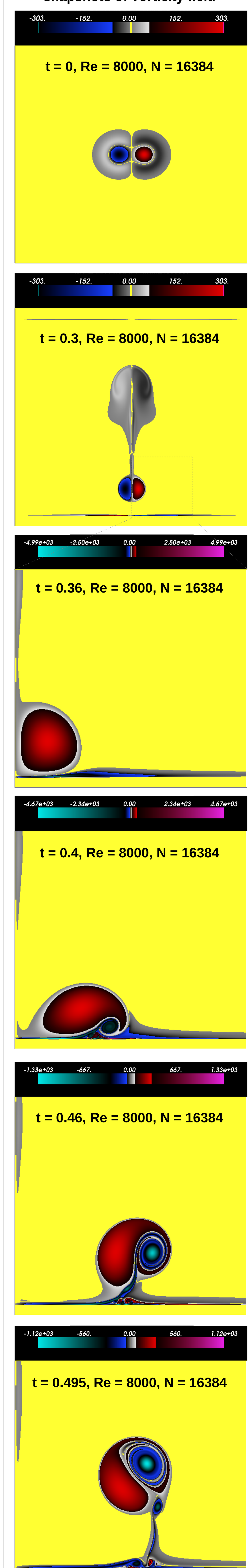
4. Energy dissipation during a fixed time interval as a function of Reynolds number



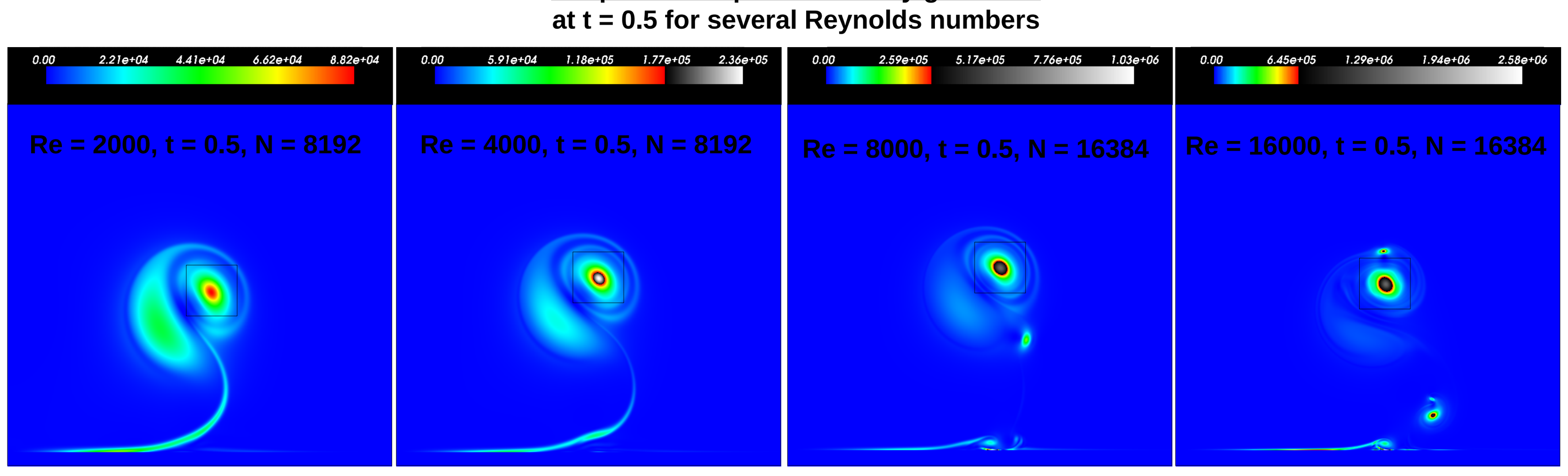
5. Enstrophy increase from initial time as a function of Reynolds number



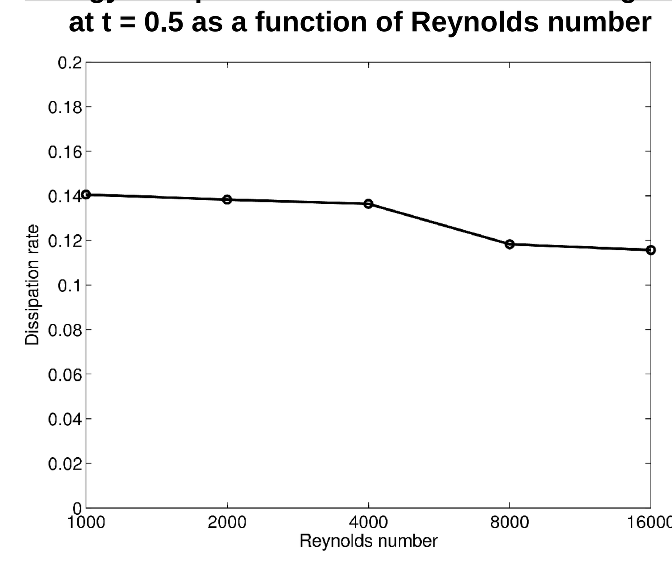
1. Dipole-wall collision at Re=8000 snapshots of vorticity field



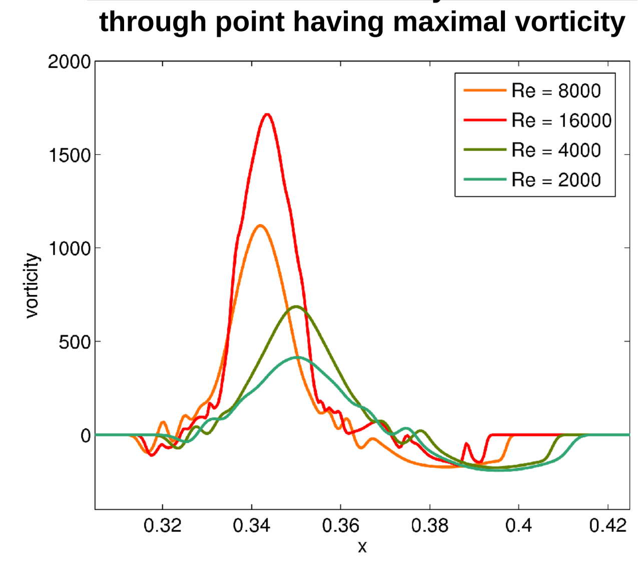
6. Snaphots of squared velocity gradients at t = 0.5 for several Reynolds numbers



7. Energy dissipation rate inside framed subregions at t = 0.5 as a function of Reynolds number



8. Horizontal cuts of vorticity field at t = 0.5 through point having maximal vorticity



INTERPRETATION

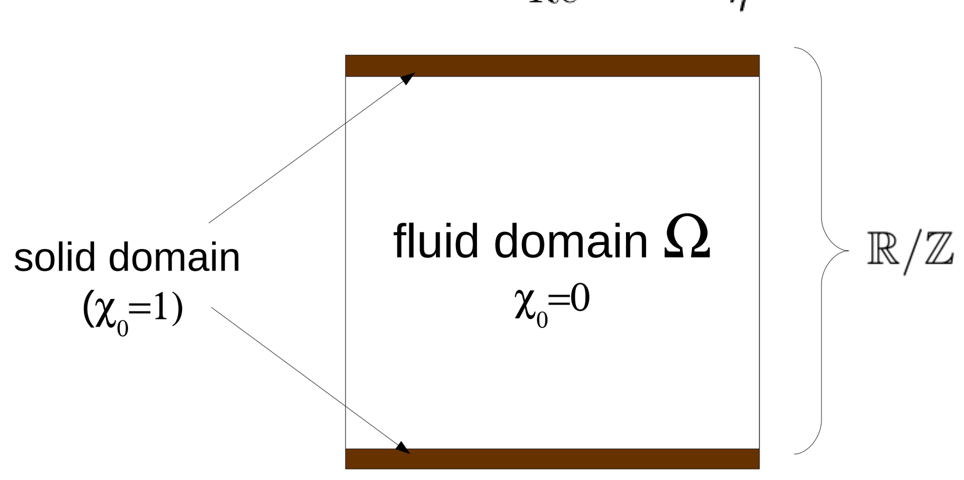
- For the initial condition we have considered, up to Re = 16000, the energy dissipation rate after t ~ 0.35 is **not consistent with Prandtl's boundary layer theory**.
- This contradicts results from earlier computations using Chebichev discretization[4], which may have been underresolved in the wall parallel direction.
- At these Reynolds numbers, the **residual energy dissipation** in the bulk of the channel remains larger than the limiting value, which makes it hard to observe on Figs. 2 and 4,
- however, the scaling is apparent when substracting its initial value to the enstrophy (Fig. 5).
- To make the finite energy dissipation rate stand out even more unambiguously, we have isolated a small domain containing intense vorticity (Fig. 6).
- The energy dissipation in this domain is almost Reynolds independent (Fig. 7).
- We suggest that this region contains a **dissipative structure**, which is advected inside the domain.
- If this scenario is confirmed, it means that **after the critical time τ ~ 0.35** (cf Th. 2) **the limiting flow does not satisfy the Euler equations in the strong sense**,
- The necessity to resolve scales finer than Re^{-1} to approximate the inviscid limit for wall-bounded 2D flows poses a great challenge. **Adaptive schemes are a must**.
- The question remains open as to whether the limit flow is a weak Euler solution inside the domain for t > τ.

NUMERICAL METHOD

Formulation

We use the **volume penalization** method[5], which consists in adding a term to the equations and solving them in a larger domain:

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u} - \frac{1}{\eta}\chi_0\mathbf{u}$$



Spatial discretization is achieved using **K Fourier modes**, and the nonlinear term is evaluated on a N x N collocation grid with N = 3K to avoid aliasing errors
Temporal discretization relies on a Runge-Kutta 3[rd] order explicit scheme.

The parallel C++ numerical code is available online under the GPL licence. Support can be provided on demand.
http://justpmf.com/romain/kicksey_winsey

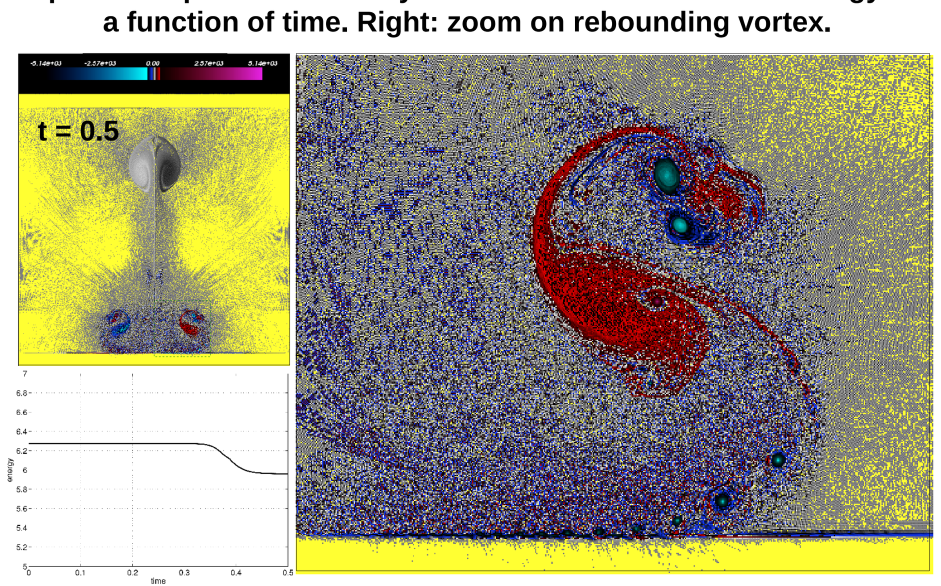
Stability and energy conservation

To impose energy decay for the discretized equations, we replace χ_0 by a mollified function χ,which is positive and band limited.
Thus the discrete solution $\mathbf{u}_{K,\chi,\eta}$ satisfies :

$$\frac{d}{dt}|\mathbf{u}_{K,\chi,\eta}|^2 + \frac{2}{\mathrm{Re}}|\nabla\mathbf{u}_{K,\chi,\eta}|^2 + \frac{1}{\eta}\int_{\mathbb{T}}\chi|\mathbf{u}_{K,\chi,\eta}|^2 = 0$$

To see how the scheme behaves, we have performed an experiment with **Re=∞** (Fig. 8). The solution is very noisy but stable.

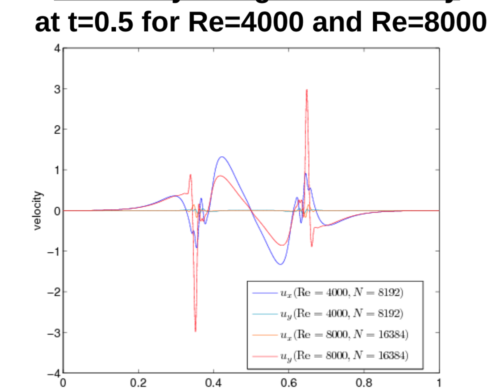
8. Inviscid experiment with N = 4096
Top left: snapshot of vorticity field at t = 0.5. Bottom left: energy as a function of time. Right: zoom on rebounding vortex.



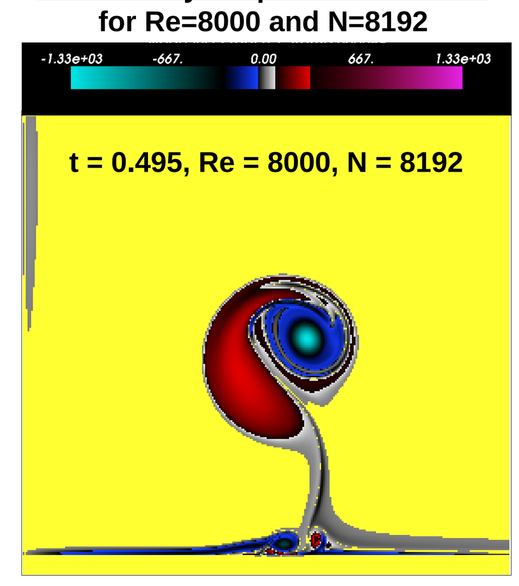
Convergence

- A rigorous convergence proof for the volume penalization method is known[5], but all **estimates blow-up exponentially with Re**.
- Numerically we observe that above Re~4000 it is very difficult to have satisfactory convergence up to t = 0.5.
- Theorem 1 (Kato) indicates that **resolutions N ~ Re**, orders of magnitudes larger than what is traditionally used (namely N ~ $\mathrm{Re}^{1/2}$), are required to observe genuine boundary layer detachment !!
- The tangential velocity at the boundary (Fig. 9) has a sharp peak with maximum value ~3 for Re=8000, to be compared with maximal values ~27 in the bulk,
- We compare (Fig. 10) the flows obtained for N=8192 and N=16384 at Re=8000,
- The result seem good enough to derive **Reynolds number scalings** of energy dissipation, especially since our scheme has **no systematic numerical enstrophy production**,
- Since we were not able to compute a reference solution at a higher resolution, we cannot guarantee that the run at Re=16000 with N=16384 is converged.

9. Velocity along the boundary at t=0.5 for Re=4000 and Re=8000



10. Vorticity snapshot at t = 0.495 for Re=8000 and N=8192



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5. P. Angot, C.-H. Bruneau, P. Fabrie, *Num. Math.* **81** pp. 497-520 (1999)

Acknowledgements :
- This work was supported by the French Federation for Fusion Studies (CNRS-CEA-EFDA) and by Wiko Berlin,
- Computations were carried out in part at the IDRIS-CNRS national computing centre,
- We would like to heartfully thank Claude Bardos and Dmitry Kolomenskiy for many inspiring discussions without which this work would not have been possible, and Ruppert Klein for important remarks.