Remarks on boundary layers and energy dissipation in high Reynolds number 2D flows

Romain Nguyen van yen[1], Marie Farge[2], Kai Schneider[3], Rupert Klein[1]

[1] FB Mathematik, Freie Universität Berlin and Humboldt Foundation, Berlin, Germany
[2] LMD-CNRS-IPSL, Ecole Normale Superieure, Paris, France
[3] M2P2-CNRS and CMI, Universite de Provence, Marseille, France

Setting of the problem

2D flow in bounded domain with no-slip boundary conditions and Re >> 1

$$\mathbf{u} : \Omega \times]0,T[\to \mathbb{R}^2 \qquad \Omega \subset \mathbb{R}^2$$

$$\text{NSE :} \begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla} p = \nu \Delta \mathbf{u} \\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \\ \mathbf{u}_{|\partial\Omega} = 0, \quad \mathbf{u}(\cdot,0) = \mathbf{u}_0 \end{cases}$$

Vanishing viscosity limit (VVL) problem : study the solution of NSE $\mathbf{u}_\nu(\mathbf{x},t)$ when $\nu \to 0$, all other parameters being fixed.

- What are tne properties of the solution in the VVL ?
- Is there a set of Reynolds-independent equations approximating the behavior in the VVL
- Is there finite energy dissipation in the VVL ?
- What is the scaling of maximum vorticity with Re ?
- How does the finest scale of the flow scale with Re ?

The Prandtl Ansatz [1]

$$\text{PA :} \begin{cases} u_x(x,y,t) \simeq u_x^0(x,y,t) + u_x^1(x,\nu^{-1/2}y,t) \\ u_y(x,y,t) \simeq u_y^0(x,y,t) + \nu^{1/2} u_y^1(x,\nu^{-1/2}y,t) \end{cases}$$

Euler flow — Boundary layer corrector

Euler Eq.

$$\begin{cases} \partial_t \mathbf{u}^0 + (\mathbf{u}^0\cdot\boldsymbol{\nabla})\mathbf{u}^0 + \boldsymbol{\nabla} p^0 = 0 \\ \boldsymbol{\nabla}\cdot\mathbf{u}^0 = 0 \\ \mathbf{u}^0_{|\partial\Omega}\cdot\mathbf{n} = 0, \quad \mathbf{u}^0(\cdot,0) = \mathbf{u}_0 \end{cases} \qquad \begin{cases} x^1 = x \\ y^1 = \nu^{-1/2} y \\ t^1 = t \end{cases}$$

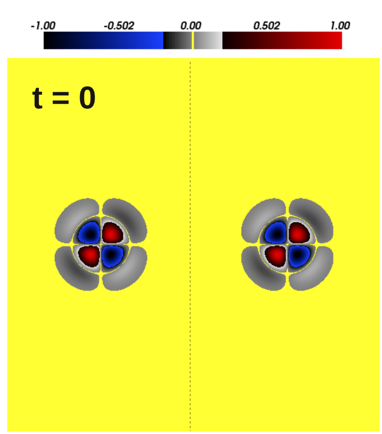
Prandtl Eq.

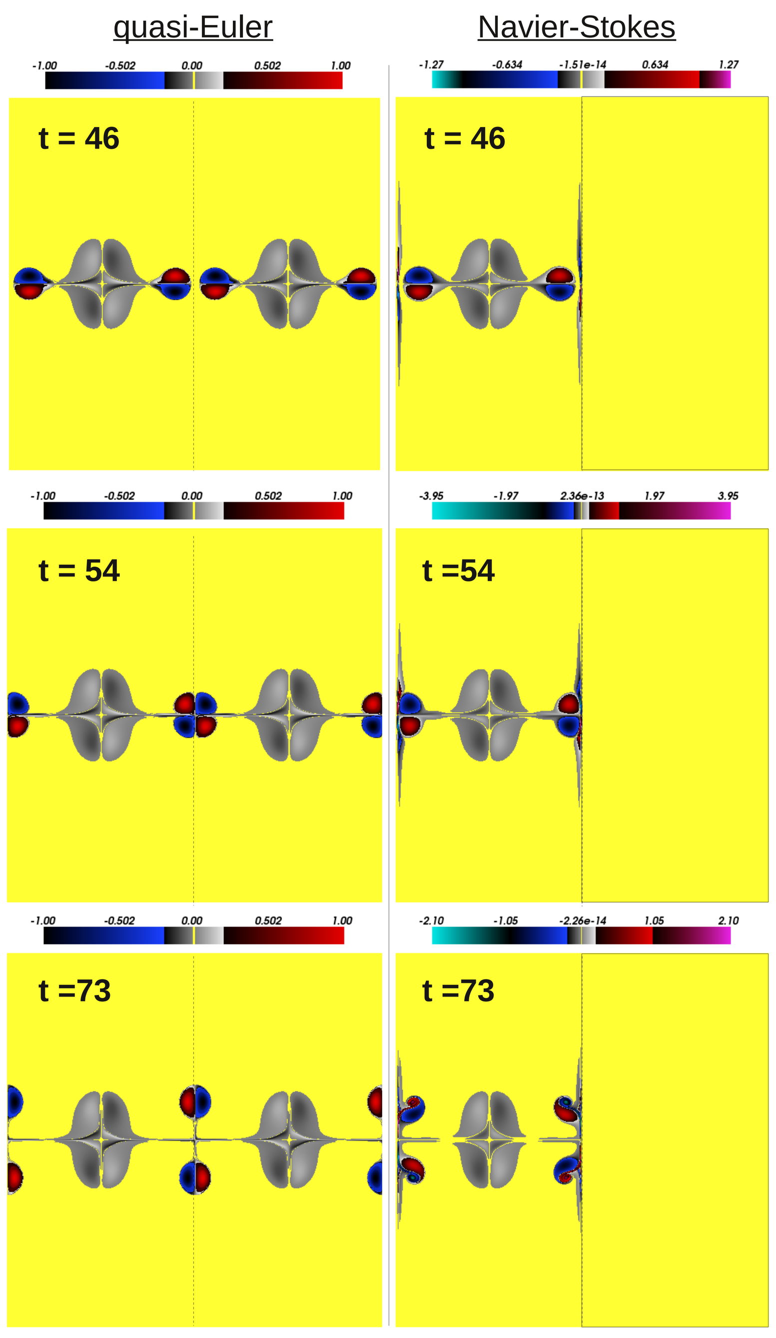
$$\begin{cases} \partial_t u_x^1 + u_x^1 \partial_x u_x^0 + (u_x^0 + u_x^1)\partial_x u_x^1 + (y^1 \partial_y u_y^0 + u_y^1)\partial_{y^1} u_x^1 - (\partial_{y^1})^2 u_x^1 = -\partial_x p^0 \\ \partial_x u_x^1 + \partial_{y^1} u_y^1 = 0 \\ \mathbf{u}^1(x,0,t) + \mathbf{u}^0(x,0,t) = 0, \quad u_x^1(x,y^1,t) \xrightarrow[y^1\to\infty]{} 0, \quad \mathbf{u}^1(\cdot,0) = 0 \end{cases}$$

Problem : no regularization parallel to the wall ! Ill-posed [4]. Also no wall-normal pressure gradient.

Numerical experiments

Fig 1. Illustration of the discrepancy between the Euler and Navier-Stokes solutions. The Euler solution is approximated by the NSE with hyper-dissipation, using mirror symmetry to mimic the effect of the boundary. Top to bottom : time evolution of vorticity.





To understand how the Prandtl Ansatz breaks down, we study the solution numerically as a function of Reynolds number [5]. The solution is approximated using the volume penalization method (see other poster by the same authors for more info).

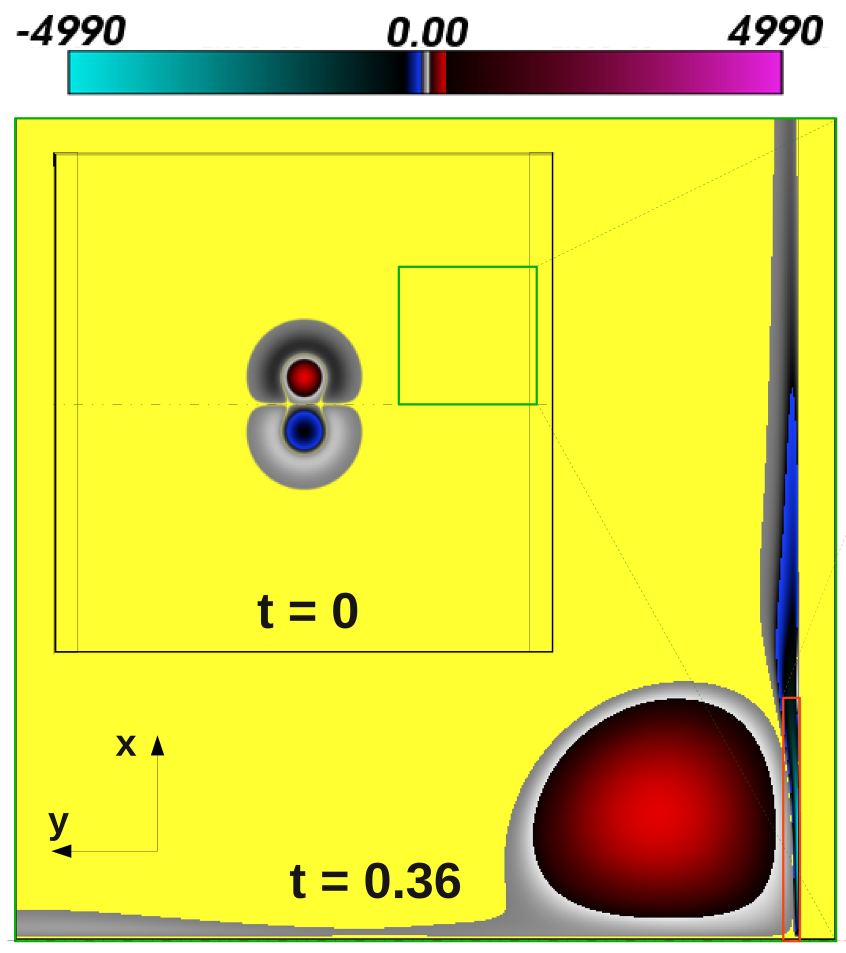


Fig 2. Vorticity field during a dipole-wall collision at Re = 7880, computed using pseudo-spectral scheme with volume penalization, and grid size 16384^2 [5].The inset shows the initial condition.

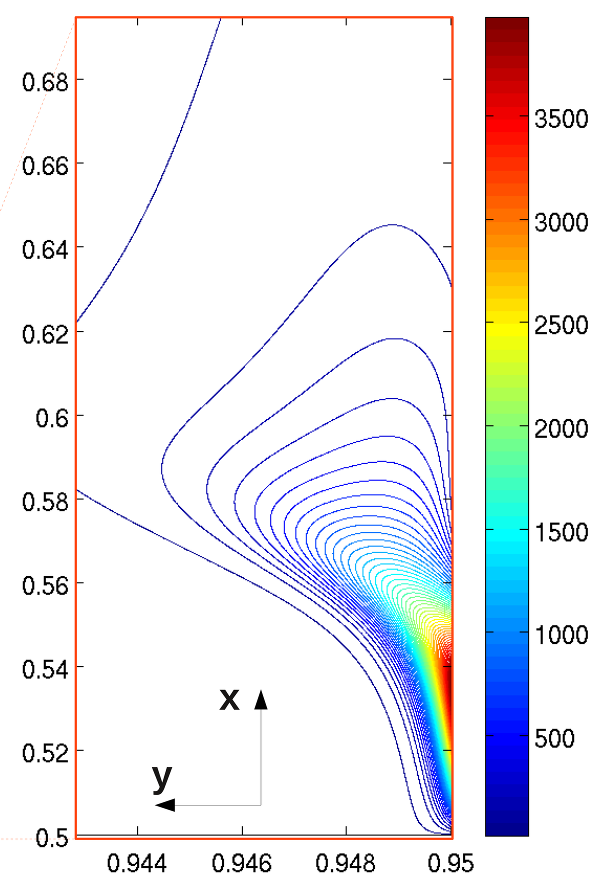


Fig. 3 Zoom on contour lines of vorticity near the boundary, showing the squeezing of the Prandtl boundary layer and concentration of vorticity within a much thiner region near the impact point.

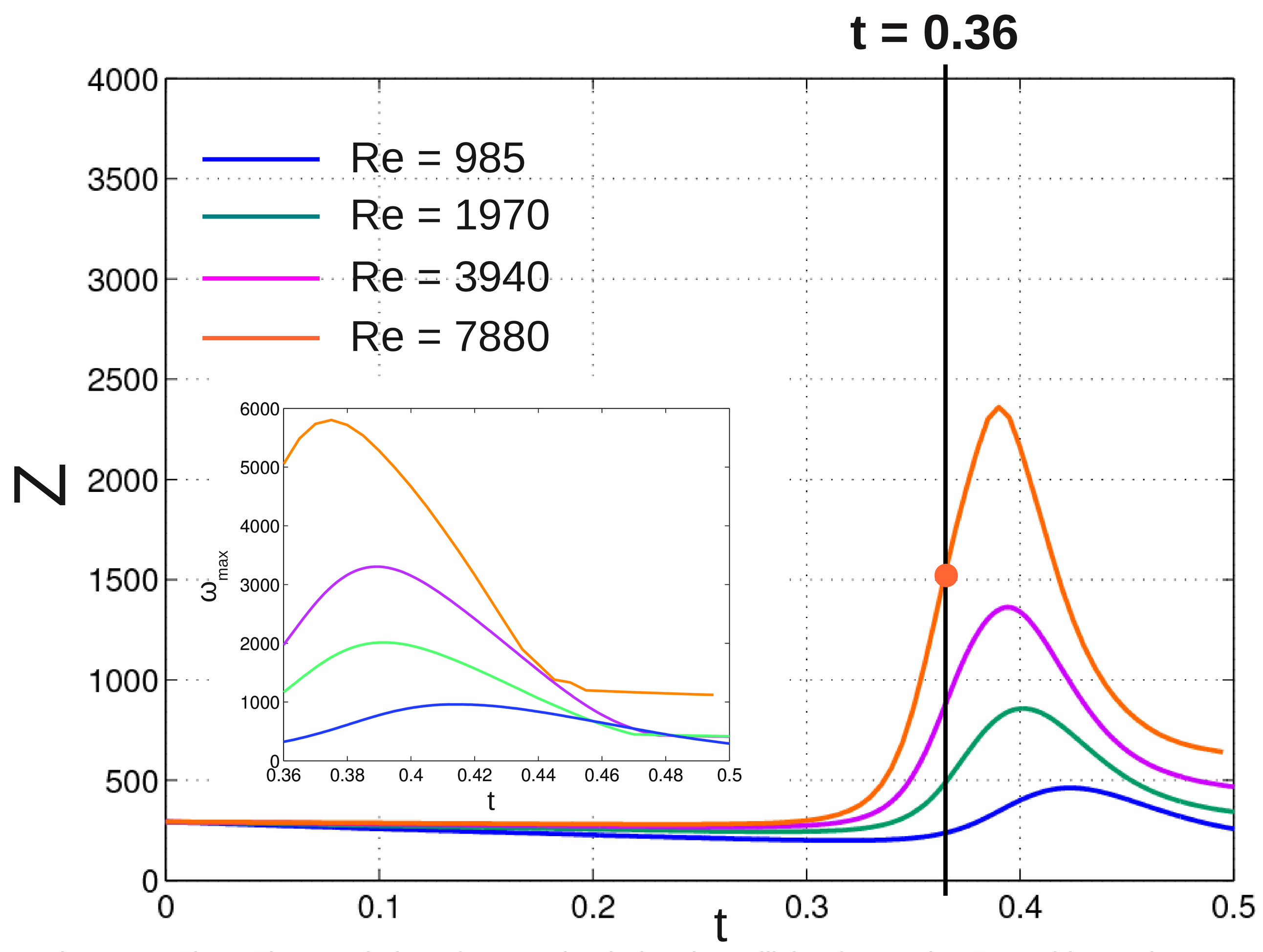


Fig. 4 Time evolution of enstrophy during the collision for varying Reynolds numbers, showing the appearance of a peak where Z-Z(0) scales like Re. The time evolution of the vorticity maximum is shown in the inset.

Can we find vorticity scalings that are consistent with these results ?
It is very important to keep in mind that all the vorticity production occurs by diffusion through the boundary.
Z blows up like Re and dZ/dt as well, which means that the vorticity production term probably scales like Re.

The Kato theorem [6]...

...relates L^2 convergence to the Euler solution inside the domain (which is necessary for PA to hold) to dissipation of energy in a thin layer of thickness proportional to Re^{-1} along the wall.

Theorem 5.1 *Let $u(x,t) \in W^{1,\infty}((0,T)\times\Omega)$ be a solution of the Euler dynamics, and let u_ν be a sequence of Leray–Hopf solutions of the Navier–Stokes dynamics with no-slip boundary condition.*

$$\partial_t u_\nu - \nu\Delta u_\nu + \nabla\cdot(u_\nu \otimes u_\nu) + \nabla p_\nu = 0\,,\; u_\nu(x,t) = 0 \text{ on } \partial\Omega\,,$$

with initial data $u_\nu(x,0) = u(x,0)$. Then, the following facts are equivalent.

(i) $\lim_{\nu\to 0} \nu \int_0^T \int_{\partial\Omega} (\nabla \wedge u_\nu)\cdot(\vec{n}\wedge u)\, d\sigma dt = 0$

(ii) $u_\nu(t) \to u(t)$ in $L^2(\Omega)$ uniformly in $t \in [0,T]$

(iii) $u_\nu(t) \to u(t)$ weakly in $L^2(\Omega)$ for each $t \in [0,T]$

(iv) $\lim_{\nu\to 0} \nu \int_0^T \int_\Omega |\nabla u_\nu(x,t)|^2 dx dt = 0$

(v) $\lim_{\nu\to 0} \nu \int_0^T \int_{\Omega\cap\{d(x,\partial\Omega)<\nu\}} |\nabla u_\nu(x,t)|^2 dx dt = 0\,.$

Scalings of energy dissipation in the vanishing viscosity limit

The requirement of energy dissipation imposes some constraints on the scaling of the solution in the VVL, which we summarize here.

$$E = \frac{1}{2}\int_\Omega \mathbf{u}^2 \qquad Z = \frac{1}{2}\int_\Omega \omega^2$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -2\nu Z$$

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = -2\nu P + \frac{\nu}{2}\oint_{\partial\Omega} \boldsymbol{\nabla}(\omega^2)\cdot\mathbf{n}$$

$$P = \frac{1}{2}\int_\Omega |\boldsymbol{\nabla}\omega|^2$$

$$\omega = \partial_x u_y - \partial_y u_x$$

	Euler	Wall-less	Prandtl	Dissipative
$-\mathrm{d}E/\mathrm{d}t$	0	ν	$\nu^{1/2}$	$\to C > 0$
Z	= constant	~ constant	$\nu^{-1/2}$	ν^{-1}
$\mathrm{d}Z/\mathrm{d}t$	0	$-1/\log(\nu)^\alpha$	$\nu^{-1/2}$	ν^{-1}
$\omega_{\max}$	= constant	~ cst	$\nu^{-1/2}$	$\nu^{-5/4}$?
δx	0	$\nu^{1/2}$	$\nu^{1/2}$	$< \nu$

Scenario of attached breakdown

One possibility of breakdown of the Prandtl scaling in the unstationnary case is that it leads to fluid particles crashing into the wall without rescue by viscous effects. To see this, write the NSE in Lagrangian form:

$$\frac{Du_x}{Dt} = -\partial_x p + \nu\Delta u_x$$

O(1) tangential friction will prevent particles from escaping out of the zone where pressure gradient coming from the outer flow is pushing them towards the wall.

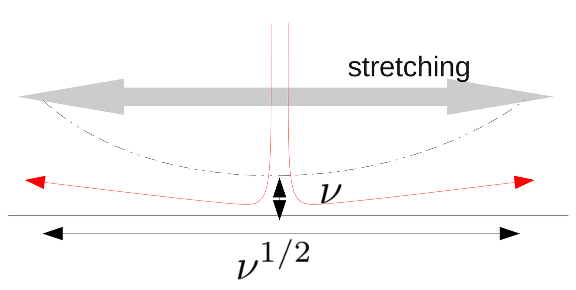
$$\frac{Du_y}{Dt} = -\partial_y p + \nu\Delta u_y$$

$$\Delta u_y = \partial_x \omega$$

Normal friction will be very much reduced because of wall-parallel stretching of isovorticity lines :

$$\frac{D\boldsymbol{\nabla}\omega}{Dt} = \nu\Delta\boldsymbol{\nabla}\omega - S\boldsymbol{\nabla}\omega$$

The stretching can be compensated by viscosity only at scales down to Re^{-1} !!



Since vorticity can be injected only at the boundary, it may be more suitable to look for a boundary layer Ansatz in the velocity-vorticity vormulation. This allows for more freedom, because a localized vorticity can already perturb the velocity field at finite distances from the wall.
The only possibility for a localized boundary layer to break-down the Euler convergence is through the nonlocal effect of pressure.

Outlook and open questions

- Is there a matched asymptotic expansion similar to (PA) which can describe the dissipative solution ?
- If yes, when does the new boundary layer detach ? What determines the amount of vorticity injected into the flow ?
- Should we expect compressible effects due to the diverging Lagrangian acceleration ? Can this be measured ?
- Once detachment has occured, is the flow still a weak dissipative solution of the Euler equations inside the domain ?
- How is this modified in the case of the quasi-geostrophic equations ?

References :
[1] L. Prandtl, *Proc. 3Rd Int. Math. Cong. Heidelberg* (1904)
[2] J.M. Burgers, *Proc. KNAW* **26** (1923)
[3] T. Kato, in *Sem. Nonlin. PDEs* (1984)
[4] D. Gerard-Varet & E. Dormy, *JAMS* **23** (2010)
[5] RNVY, MF & KS, *PRL* **108** (2011)
[6] C. Bardos & E. Titi, *Russ. Math. Surv.* **62** (2007)

Acknowledgements : ANR, PEPS-CNRS, IDRIS-CNRS, Humboldt Foundation, Federation Fusion, D. Gerard-Varet