Production of dissipative vortices by solid bodies in incompressible fluid flows: comparison between Prandtl, Navier-Stokes and Euler solutions

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How can the resistance of fluids be explained ?

Famously put forward by Leonard Euler as challenge to the mathematicians of his time, this question still resists the most recent investigations.

Here is a non-exhaustive collection of quotes related to the problem, which serves as introduction to the work presented below.

« Propose a theory, based on fully new principles and using the simplest deduction, to explain the resistance that a body in a moving fluid is subject to, in function of the velocity, shape and mass of the body, and of the density and compressibility of the fluid. »

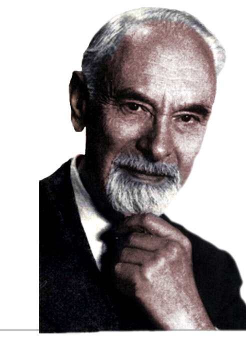
Leonhard Euler, statement of the problem for the Prize of Mathematics for 1750, Prussian Academy of Sciences, Berlin, 16 May 1748

«I do not see, I admit, how my theory can explain in a satisfactory manner the resistance of fluids. It seems to me, on the contrary, that this theory, dealt with and deepened in all possible rigor, yields, in several cases at least, a strictly vanishing resistance ; a singular paradox that I leave for future Geometers to elucidate. »

Jean le Rond d'Alembert, Opuscules Mathématiques, vol. 5, chap. 34 (1768)

« When the mass of water contained in a vessel […] is stirred round, and then left to itself, it presently comes to rest. This, no doubt, is owing to the friction against the sides of the vessel. But if the fluidity of water were perfect, it does not appear how the retardation due to this friction could be transmitted to the mass. It would appear that in that case a thin film of fluid close to the walls of the vessel would remain at rest, the remaining part of the fluid being unaffected by it. »

George G. Stokes, « On some cases of fluid motion », *Trans. Cambridge Phil. Soc.* **8**, p.105 (1845)



« The viscosity is supposed to be so small that it can be disregarded wherever there are no great velocity differences. […] The most important aspect of the problem is the behavior of the fluid on the surface of the solid body. […] In the thin transition layer, the great velocity differences will […] produce noticeable effects in spite of the small viscosity constants. »

Ludwig Prandtl, « Über Flüssigkeitsbewegung bei sehr kleiner Reibung », *Verh. III. Intern. Math. Kongr., Heidelberg*, p.484 (1904)

« It can be checked […] that the total kinetic energy of the liquid remains bounded, but it doesn't seem possible to me to deduce from this fact that the motion itself remains regular. I have even indicated a reason which makes me believe in the existence of motions which become irregular after a finite time. I have unfortunately not succeeded in constructing an example of such a singularity. »

Jean Leray, « Sur le mouvement d'un liquide visqueux emplissant l'espace », *Acta Mathematica* **63** p. 193 (1934)

For flow in a 2D domain with smooth initial data and without forcing, the following assertions are equivalent:
i) the Navier-Stokes flow converges to the Euler flow in C([0,T],L²(Ω)),
*ii) the **energy dissipation** associated to the Navier-Stokes flow in a strip proportional to Re^{-1} around the solid during the time interval [0,T] tends to zero as Re goes to infinity.*

Tosio Kato, « Remarks on Zero Viscosity Limit for Nonstationary Navier- Stokes Flows with Boundary », *Sem. Nonlinear Partial Diff. Eq.* **2**, p. 85 (1984)

MOTIVATION

According to a well known experimental result, **the rate of kinetic energy dissipation in incompressible flows at high Reynolds number does not tend to vanish**, despite the fact that the coupling constant responsible for dissipation – viscosity, denoted ν – is, in this regime, very small compared to macroscopic scales. This phenomenon, still often inappropriately called « anomalous dissipation », has been found to occur in many other dissipative systems. Since the dissipation rate is a product of viscosity by a certain norm depending on velocity gradients, efficient dissipation at small viscosity requires that **the velocity gradients blow up at vanishing viscosity**, if all other parameters are kept constant.

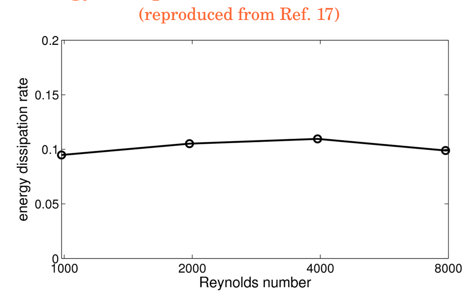
Fundamental questions in fluid mechanics are to understand **which flow structures can support such large gradients** and are thus responsible for the observed dissipation, and **how these structures can be produced** starting from smooth flows. For the simple model of a 1D pressure-less compressible fluid, as described by the Burgers equation, such structures are proven to exist and are called « shocks ». In incompressible flows, shocks cannot occur, but there are two main mechanisms which can amplify the velocity gradients: vorticity stretching (possible only in 3D), and **interaction of the flow with solid boundaries**. In this work, we will focus only on the second mechanism, and therefore we choose to work with 2D flows, which also present the advantage to be less demanding in computational ressources.

The production of gradients at solid boundaries is related to d'Alembert's paradox concerning the resistance exerted by fluids on immersed solid bodies. Working with the inviscid potential flow equations, later generalized as the **Euler** equations, **d'Alembert** realized in 1749, in his answer to the problem for the Mathematics Prize of Prussian Academy, that the flow would exert no drag force onto solid obstacles. The **Navier-Stokes** equations (NSE) were then derived during the 19th century by including molecular friction effects. It was gradually realized that the paradox came from the singular nature of the vanishing viscosity limit, mostly due to the no-slip boundary condition imposed along the solid boundary. In 1904, **Prandtl** [1] resolved the paradox in the special case of flows in which the effects of viscosity are **confined to a boundary layer of thickness proportional to $\nu^{1/2}$** attachted to the wall. He was able to compute a drag coefficient, and hence also an energy dissipation rate, which are both proportional to $\nu^{1/2}$ in the vanishing viscosity limit, and therefore **do not describe the experimentally observed dissipation**. This shortcoming is related to the fact that Prandtl's theory does not apply when the boundary layer detaches from the wall, because the Euler equations can then no more be used to describe the flow, even far from the solid boundary.

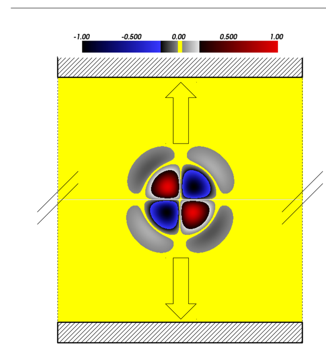
The most important remaining question is now to understand **how the Prandtl asymptotic regime breaks down**, what are the new scalings coming into play during this break-down, and what structures emerge of it. A fundamental constraint on the break-down, unfortunately largely unknown to the fluid mechanics community, was provided in 1984 by **Kato** [4]. He proved that, in the vanishing viscosity limit, **the energy dissipation rate tends to zero if and only if the solution of the NSE converges to the solution of the Euler equations with the same initial data**. He also proved in the same paper that, for dissipation to occur anywhere in the flow at any time, at least some dissipation had to occur within a very thin boundary layer of thickness proportional to **ν** in the neighborhood of the wall. Increasingly convincing evidence [2,3,5,6,14] suggests that the break down is due to the occurrence of a finite-time singularity in Prandtl's boundary layer equations characterized by a blow-up of the parallel vorticity gradient.

In a previous work [17], we have shown that the collision of a 2D vorticity dipole into a wall leads to the formation of dissipative structures. Indeed, by integrating the local energy dissipation rate in a small box centered on the structure, and comparing four cases at increasing Reynolds numbers, we have shown that the dependency on Reynolds number was weak (see figure to the right). This work had the inconvenient of relying on a volume penalization scheme to approximate the no-slip conditions, and thus did not account fery precisely for the behavior at the wall. In this work, we focus closely on the detachment process that precedes the formation of the structure, thanks to a new scheme implemeting both divergence free and no-slip conditions exactly.

Energy dissipation in detached structure
(reproduced from Ref. 17)



MODELS



We consider a two-dimensional channel flow, with no-slip boundary conditions at the channel walls (y=0 and y=1), and periodic boundary conditions in the x direction. As initial condition, we choose a quadrupole of vorticity localized in the center of the channel, defined by the following stream function, which is designed to trigger a dipole-wall collision, as previously studied in the literature [9,12,13,17] :

$$\psi_i(x,y) = Axy\exp\left(-\frac{(x-x_0)^2+(y-y_0)^2}{2s^2}\right) \qquad s = 0.06366 \qquad A = 0.6258$$

Since our goal is to understand the break-down of the boundary layer leading to the production of a dissipative structure, we consider on the one hand the Navier-Stokes equations with large Reynolds number (between 10000 and 85000), and on the other hand the association between the Euler and the Prandtl equations. The link between ν and Re is given by : Re = 2.19 10^{-4} ν^{-1}.

As a first step, we have reformulated both problems in terms of the **vorticity** field ω. Since the no-slip boundary condition applies to the velocity field which depends on ω via an integral formula, it translates into a set of integral constraints on ω, which are conveniently formulated by introducing the Fourier coefficients of ω in the x direction.

The Prandtl asymptotic regime can then be formally derived by assuming the following Ansatz for ω :

$$\omega(x,y,t) = \omega_0(x,y,t) + \mathrm{Re}^{1/2}\omega_1(x,\mathrm{Re}^{1/2}y,t) + R(x,y,t)$$

and assuming that parallel gradients remain sufficiently small. This leads to the equations shown on the right, where y_1 denotes $\mathrm{Re}^{1/2}y$.

Since our essential question concerns the kinetic energy, we also write down its evolution equation, which follows from the Navier-Stokes equations :

$$e = \frac{1}{2}\mathbf{u}^2 \qquad \partial_t e + (\mathbf{u}\cdot\boldsymbol{\nabla})(e+p) = \frac{1}{\mathrm{Re}}\Delta e + \frac{1}{\mathrm{Re}}|\boldsymbol{\nabla}\mathbf{u}|^2$$

And, by integrating in space, we obtain a global energy budget:

$$E = \int_\Omega e \qquad \frac{d}{dt}E = -\frac{2}{\mathrm{Re}}Z \quad \text{where} \quad Z = \frac{1}{2}\int_\Omega \omega^2 \quad \text{(enstrophy).}$$

Navier-Stokes

$$\partial_t\omega + \boldsymbol{\nabla}\cdot(\mathbf{u}\omega) = \nu\Delta\omega$$
$$\Delta\psi = \omega \quad u_x = -\partial_y\psi \quad u_y = \partial_x\psi$$
$$\partial_x\psi(x,0,t) = \partial_x\psi(x,1,t) = 0$$
$$\omega(x,y,0) = \omega_i(x,y)$$
$$\forall k\neq 0 \begin{cases} \int_0^1 \widehat{\omega}_k(y')e^{-2\pi|k|y'}\mathrm{d}y' = 0 \\ \int_0^1 \widehat{\omega}_k(y')e^{-2\pi|k|(1-y')}\mathrm{d}y' = 0 \end{cases}$$
$$\int_0^1 \widehat{\omega}_0(y')y\mathrm{d}y' = 0$$
$$\int_0^1 \widehat{\omega}_0(y')(1-y)\mathrm{d}y' = 0$$

Euler / Prandtl

$$\partial_t\omega_0 + \boldsymbol{\nabla}\cdot(\mathbf{u}_0\omega_0) = 0$$
$$\omega_0 = \boldsymbol{\nabla}\times\mathbf{u}_0, \quad \boldsymbol{\nabla}\cdot\mathbf{u}_0 = 0$$
$$u_{0,y}(x,0,t) = u_{0,y}(x,1,t) = 0$$
$$\omega_0(x,y,0) = \omega_i(x,y)$$
$$\partial_t\omega_1 + \mathbf{u}_1\cdot\boldsymbol{\nabla}\omega_1 = \partial^2_{y_1}\omega_1$$
$$u_{1,x}(x,y_1,t) = -\int_0^{y_1}\mathrm{d}y_1'\,\omega_1(x,y_1',t)$$
$$u_{1,y}(x,y_1,t) = \int_0^{y_1}\mathrm{d}y_1'\int_0^{y_1'}\mathrm{d}y_1''\,\partial_x\omega_1(x,y_1'',t)$$
$$\partial_{y_1}\omega_1(x,0,t) = -\partial_x p_0(x,0,t)$$
$$\omega_1(x,y,0) = 0$$

THEOREMS

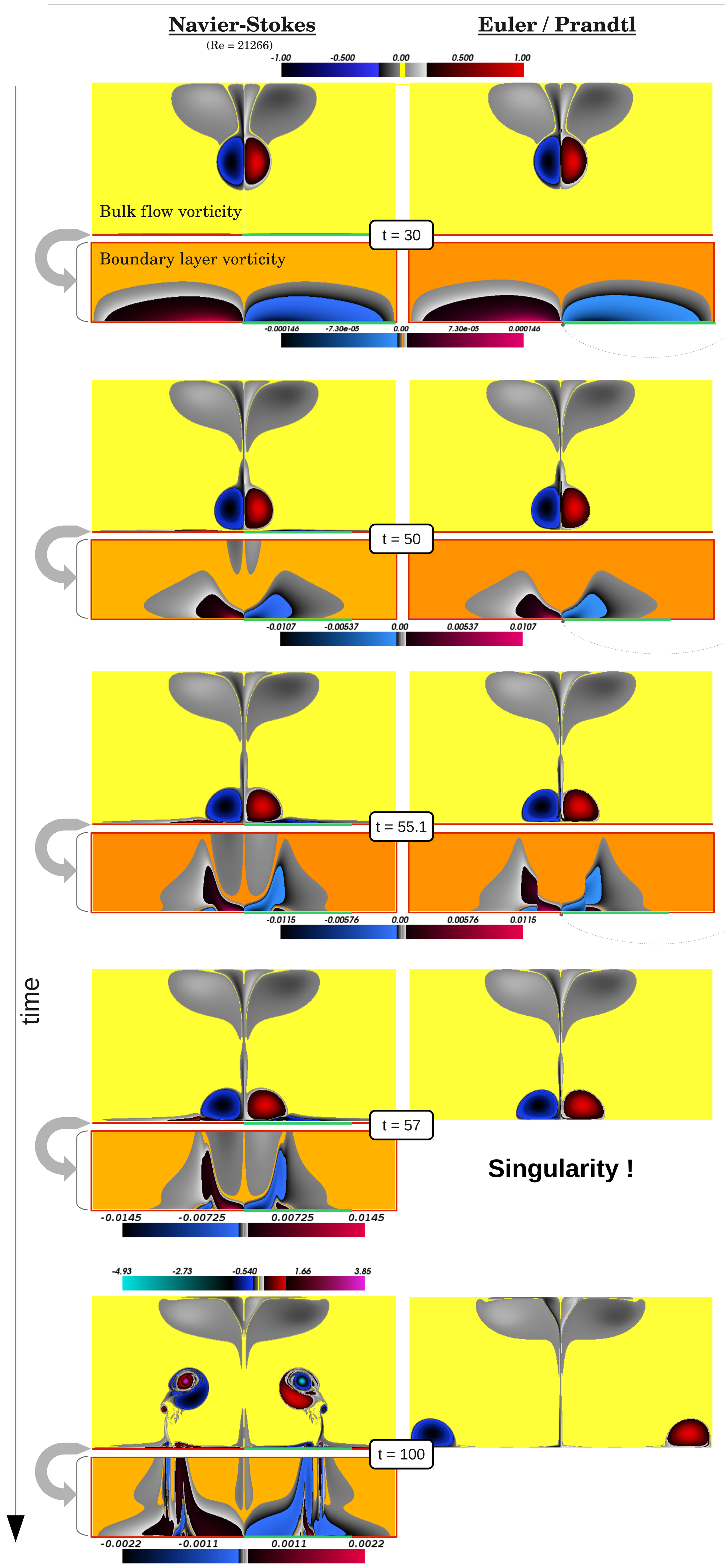
The three following theorems are most relevant to our problem :

Theorem 1 (Lichtenstein, Yudovich, Leray, Ladyzhenskaya...)
In a plane domain and for smooth initial data, both the Navier-Stokes and Euler initial-boundary value problems as written above admit unique smooth solutions for all time. **Our setting excludes the formation of finite time singularities in the Navier-Stokes and Euler solutions.**

Theorem 2 (Kato 1984) [4]
For flow in a 2D domain with smooth initial data and without forcing, the following assertions are equivalent:
i) the Navier-Stokes flow converges to the Euler flow in C([0,T],L²(Ω)) (i.e., *uniformly in time* in the energy norm!),
ii) the **energy dissipation** (as defined above) associated to the Navier-Stokes flow, integrated over a strip proportional to Re^{-1} around the solid during the time interval [0,T], tends to zero as Re goes to infinity.
Note the essential message of this theorem : **the flow has to develop dissipative activity at a scale at least as fine as Re^{-1} for detachment to be possible**.

Theorem 3 (Sammartino & Caflisch 1998) [7]
For flow in a half-plane with analytic initial data, there exists a critical time τ > 0 (τ = +∞ allowed) such that the Navier-Stokes flow converges to the Euler flow in an analytical norm at least for t in [0, τ[. Moreover, the Prandtl asymptotic regime is valid on the same time interval.
Although our geometry is a channel and not a half-plane, there are good reasons to believe that the same conclusion holds. Since our initial data is analytic, we should therefore expect our flow to asymptotically satisfy the Prandtl-Euler model at least for short times.

RESULTS

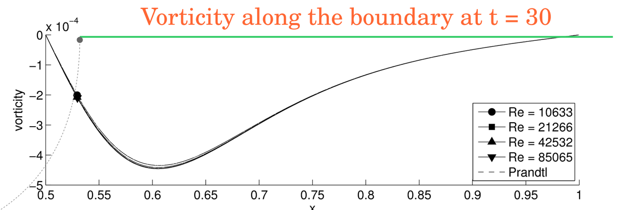


The dipole first shoots towards the lower channel wall. The Navier-Stokes vorticity field in the bulk (top, left) remains very close to the Euler vorticity field (top, right).

Plotting the Navier-Stokes flow in the Prandtl boundary layer units (bottom, left) reveals that it is smooth, and **very well approximated by the solution of the Prandtl equations** (bottom, right).

The vorticity along the boundary converges to the Prandtl values as the Reynolds number increases.

Vorticity along the boundary at t = 30

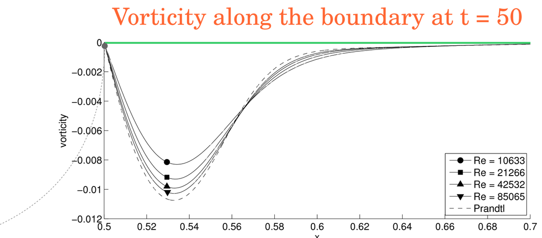


As the dipole approaches the wall, the pressure gradient along the wall becomes both more intense and steeper, which causes a strong inward diffusion of vorticity at the wall as well as increased vorticity gradients within the boundary layer.

At this time, we still observe convergence of the Navier-Stokes solution at high Reynolds number towards the Euler flow in the bulk and the Prandtl flow in the boundary layer.

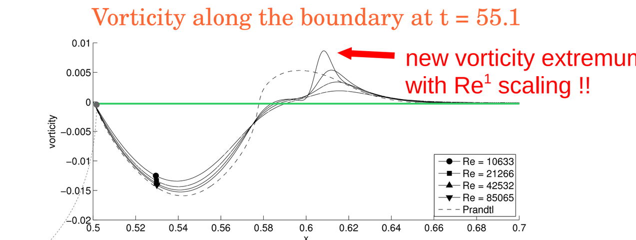
However, looking at the boundary vorticity reveals a larger difference between Prandtl and Navier-Stokes flows than at t=30.

Vorticity along the boundary at t = 50



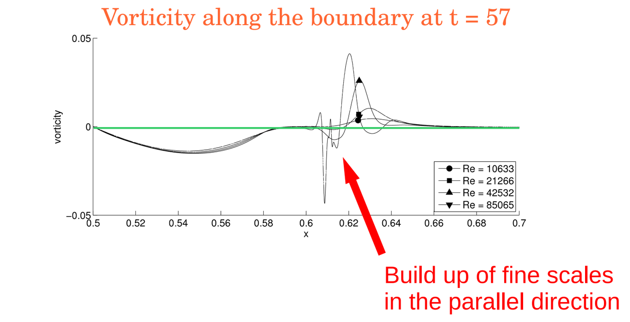
As the Prandtl solution approaches its singularity time t* ~ 55.6, parallel vorticity gradients increase rapidly, and soon the cut-off parallel wavenumber of the numerical scheme becomes insufficient to resolve it.
The convergence of the Navier-Stokes boundary vorticity is lost over a wide interval in x, and vorticity around x=0.61 adopts a stronger scaling with Reynolds.

Vorticity along the boundary at t = 55.1



Following the new vorticity extremum which has appeared at the boundary, a cascade of extrema with opposite signs appear (for sufficiently high Reynolds number), exciting increasingly fine parallel scales.

Vorticity along the boundary at t = 57



At much later times, the Euler and Navier-Stokes solutions have become completely different:
- in the Euler case, the vortices glide along the wall, having paired with their mirror image, and no new vorticity has been generated. The energy is conserved.
- in the Navier-Stokes case, the detachment process has lead to the formation of **two new vortices**, shown in cyan and magenta in the figure, of much larger amplitude than those of the incoming dipole. The activity in the boundary layer remains intense, leading to the ejection of smaller structures.

NUMERICAL IMPLEMENTATION

The way to implement correct boundary conditions for solving the vorticity transport equation in the viscous case has been a subject of heated debate for decades. Fortunately, the simple geometry to which we have restricted ourselves allows a very accurate implementation without too much trouble. First, the equations are discretized in the wall-parallel direction by the Fourier-Galerkin method: the solution is expanded into a Fourier series, and only a finite number of modes are kept.

In the Fourier domain, the Poisson equation relating stream function to vorticity becomes a set of second order linear ordinary differential equations in the variable y, with the parallel wavenumber k as a parameter. These equations are solved using a direct solver based on LU factorization, after discretization using a 5th order compact finite scheme of the type :

$$\alpha_i f'_{i-1} + f'_i + \beta_i f'_{i+1} = A_i f_{i-1} + B_i f_i + C_i f_{i+1}$$
$$\gamma_i f''_{i-1} + f''_i + \delta_i f''_{i+1} = D_i f_{i-1} + E_i f_i + F_i f_{i+1}$$

The derivatives with respect to y in the vorticity transport equation are also discretized using compact finite differences. For the y-advection term, a Neumann boundary condition applies, and we use it to replace the first and last lines of the compact finite differences matrix. For the diffusion term, we use the integral constraints on vorticity instead. Overall, the scheme has 5th order regularity in space, and satisfies all the boundary conditions up to round-off precision. The divergence free condition is automatically satisfied thanks to the vorticity formulation.

Taking advantage of the antisymmetry of the vorticity field with respect to the center point, which is preserved by the time evolution, the Fourier series can in practice be replaced by sine and cosine series, and only half of the domain needs to be considered in the y direction. The parallel cut-off wavenumber is varied proportionally to Re¹ between 512 and 16384. In the wall normal direction, a varying grid resolution with a total of 449 points is used in order to ensure that all active scales of motion remain accurately resolved.

For the Prandtl equations, the same discretization is used, with the addition of an artificial zero vorticity flux condition at y_1 = 64 to have a finite domain.
The Euler equations are solved by a classical pseudo-spectral scheme, relying on the mirror image principle for imposing the non penetration condition at the walls, and including a small amount of hyperdissipation for regularization.

INTERPRETATION

There are several features of the numerical solutions which have not been observed in previous work.

The most striking one is the appearance of the scaling Re¹ for the vorticity maximum, which takes precedence, at the singularity time, over the weaker Prandtl scaling $Re^{1/2}$. As seen on the graphs of the wall vorticity, this new extremum extends over parallel scales which seem to be of order 1, and do not become smaller and smaller as the Reynolds increases. Even more strikingly, this new extremum does not even appear at the location of the Prandtl singularity.

This result contradicts sharply the picture of boundary layer detachment as it was described in earlier work, as essentially a localized process coinciding with the singularity in the Prandtl equations. Thanks to the vorticity formulation we have favoured, the origin of the non-locality can be traced back to the integral constraints on the vorticity field of the type :

$$\int_0^1 \widehat{\omega}_k(y')e^{-2\pi|k|y'}\mathrm{d}y' = 0 \qquad (1)$$

which are themselves consequences of the no-slip boundary conditions. If higher and higher k modes are excited, as occurs in particular due to the Prandtl singularity formation, the reaction of the flow dictated by (1) has no reason to be localized in the x direction.

A more classical, but equivalent way of considering the same phenomenon is to write down (following Lighthill) a Neumann boundary condition for ω:

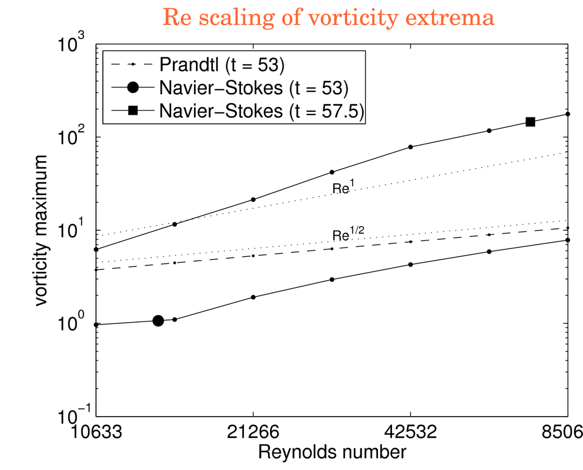
$$\nu\partial_n\omega = -\partial_\tau p$$

where p depends on ω via :

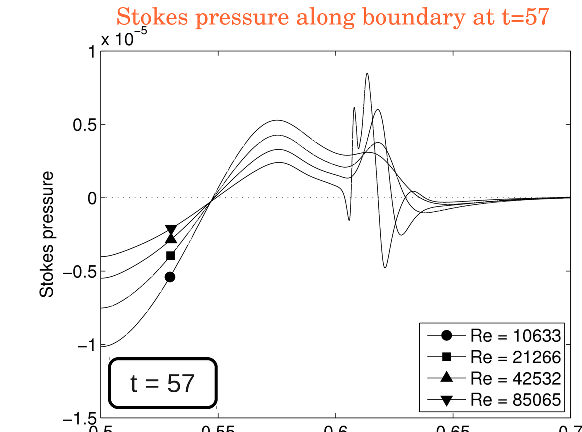
$$\Delta p = -\boldsymbol{\nabla}((\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}) \quad \text{inside } \Omega$$
$$\partial_n p_{|\partial\Omega} = \nu\partial_\tau\omega_{|\partial\Omega} \quad (\textit{Stokes pressure})$$

Therefore, according to our results, it is the **Stokes pressure** (which is the part of the pressure obtained by keeping only the Neumann condition in red and discarding the source term in the Poisson equation) which plays the essential dynamical role in the detachment process. By plotting it for different Reynolds numbers, it appears indeed to have fast oscillations in the detachment region and to increase like $Re^{1/2}$.

Re scaling of vorticity extrema



Stokes pressure along boundary at t=57



PERSPECTIVES

We have studied boundary layer detachment as a process occurring in the vanishing viscosity limit of incompressible fluid flows, at times when singularities form in the corresponding Prandtl equations. We have shown that the Navier-Stokes detachment dynamics are very different from the dynamics of the singularity in Prandtl, in particular because of the two following key elements :

- **non locality in the parallel direction,**
- **formation of smaller scales, at least as fine as Re^{-1}, in both directions.**

These numerical results suggest that a new asymptotic description of the flow beyond the break down of the Prandtl regime is possible, and deriving it would shed much more light on the observed scalings. Another open question concerns the Reynolds-independent description of the flow after the detachment has occurred: does it still converge to a solution of the Euler equations, but a weak singular solution instead of the smooth one? How can this weak solution be defined, and maybe approximated numerically ?

The problem of describing the high Reynolds number dissipative regime mathematically has resisted all attempts up to now. But following the seminal ideas of **Leray**, the most favored research direction today is to develop a notion of **weak dissipative solutions of the Euler equations**, which, associated with appropriate selection conditions, could provide a **Reynolds independent description of dissipative structures**, analog to the inviscid Burgers equation for shocks.

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