Energy dissipating structures in the vanishing viscosity limit of 2D incompressible flows with solid boundaries

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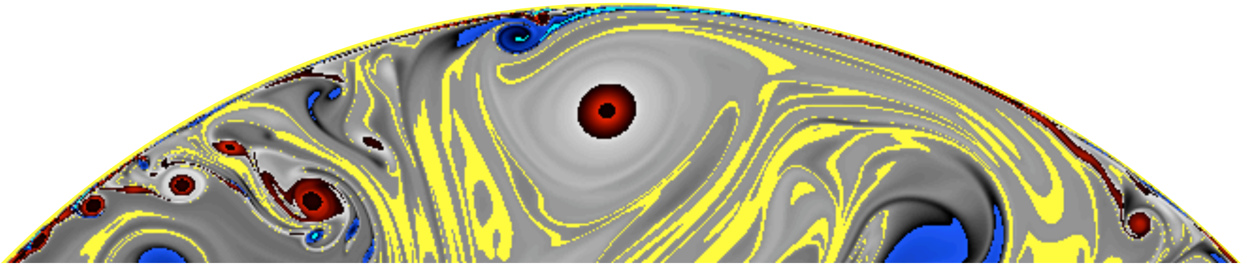
Romain Nguyen van yen[1], Marie Farge[1], Kai Schneider[2]

[1] Laboratoire de météorologie dynamique-CNRS, ENS Paris, France
[2] Laboratoire de mécanique, modélisation et procédés propres-CNRS, CMI-Université d'Aix-Marseille, France



Spirit

Review of mathematical litterature

$\Downarrow$

Numerical experiments

$\Downarrow$

Physical understanding (?)

- **No rigorous results.**

Outline

1. **Vanishing viscosity limit ?**
2. **Design of numerical experiments.**
3. **Dissipation of energy ?**
4. **Dissipative structures ?**

What do we mean by inviscid limit ?

Mathematical formulation

- We consider a single **incompressible** fluid with constant density contained in a 2D torus (we only briefly mention the 3D case below),
- the torus may either
  - be filled with fluid only => **wall-less case**,
  - contain solid obstacle(s) => **wall-bounded case**.
- The difference between these two situations is the main subject of this talk.

Mathematical formulation

Navier-Stokes equations with
no-slip boundary conditions:

$$
\text{(NS)}\begin{cases}\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \frac{1}{\mathrm{Re}}\boldsymbol{\nabla}^2\mathbf{u}\\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0\\ \mathbf{u}_{|\partial\Omega} = \mathbf{0}, \qquad \mathbf{u}(0,\cdot) = \mathbf{v}\end{cases} \longrightarrow \mathbf{u}_{\mathrm{Re}}(t,\mathbf{x}) \xrightarrow[\mathrm{Re}\to\infty]{} \; ?
$$

solution

the Reynolds number Re = ULν^{-1} appears when non dimensional quantities are introduced.

Incompressible Euler equations:

$$
\text{(E)}\begin{cases}\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p\\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0\\ \mathbf{u}_{|\partial\Omega}\cdot\mathbf{n} = \mathbf{0}, \qquad \mathbf{u}(0,\cdot) = \mathbf{v}\end{cases} \longrightarrow \mathbf{u}(t,\mathbf{x})
$$

solution

?

Well posedness

- In 2D,
  - for smooth initial data,

    both problems are well posed

    (long time existence and uniqueness),
  - the Navier-Stokes problem is well posed in L^2
  - the Euler problem is well posed **for bounded vorticity** (Yudovich 1963),
  - many open questions for cases with unbounded vorticity (cf later).

Ladyzhenskaya 1963, Foias, Manley, Rosa & Temam 2001, Bardos & Titi 2007

Known convergence results

- <u>Without walls</u>, for smooth initial data, we have the strong convergence result (Golovkin 1966, Swann 1971, Kato 1972) :

$$\left\| u_{\mathrm{Re}} - u \right\| \underset{\mathrm{Re}\to\infty}{=} \mathrm{O}(\mathrm{Re}^{-1})$$

in all Sobolev spaces for all time in 2D

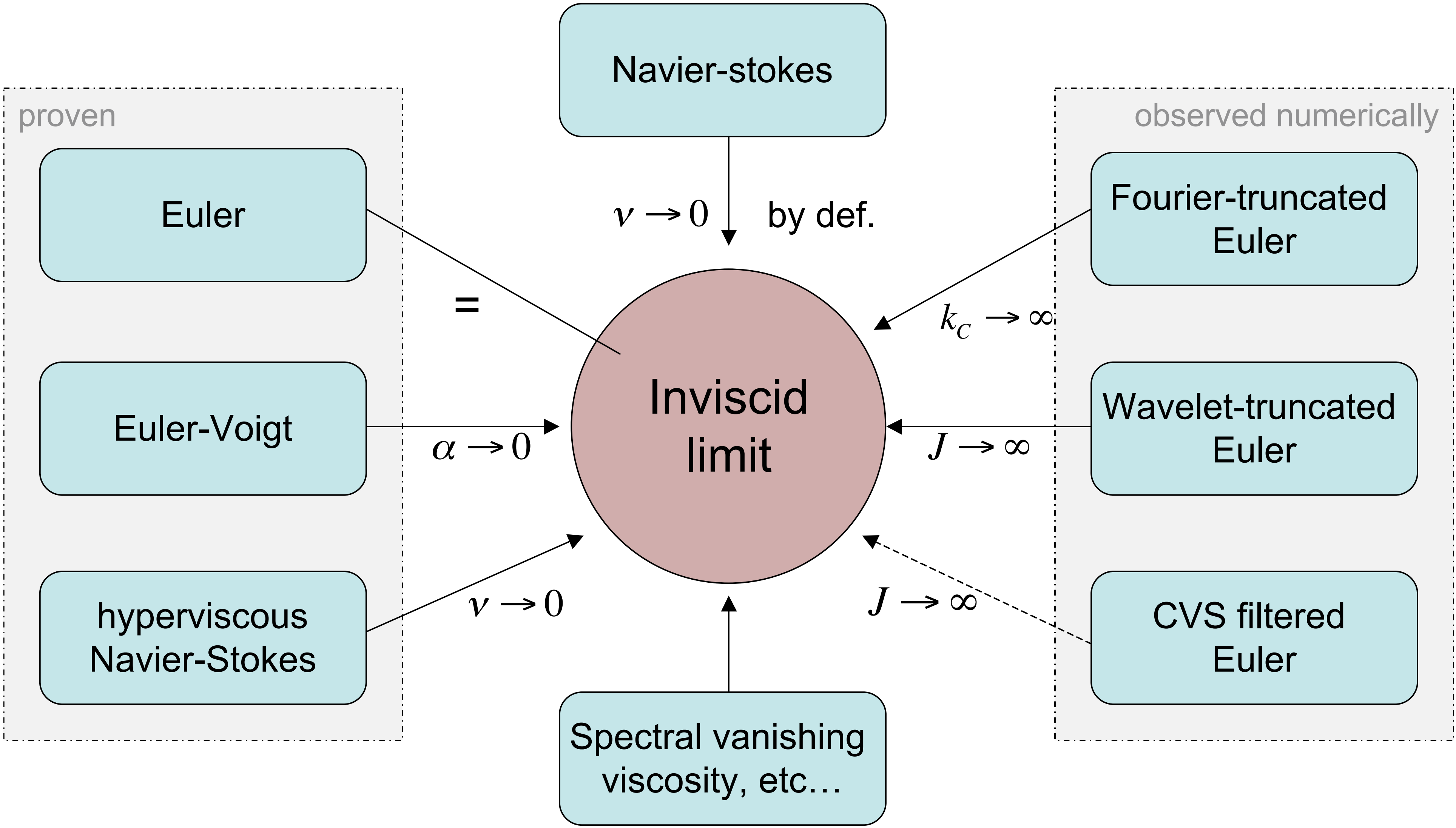
(and as long as the smooth Euler solution exists in 3D).

(the constant in the O may blow up very fast in time)

- <u>With walls</u>, the main questions are still open (see later).

Known convergence results

2D wall-less case, smooth initial data



Remarks on numerical approximation

- There exists exponentially accurate schemes for the **wall-less** Navier-Stokes equations (i.e. the error decreases exponentially with computing time),

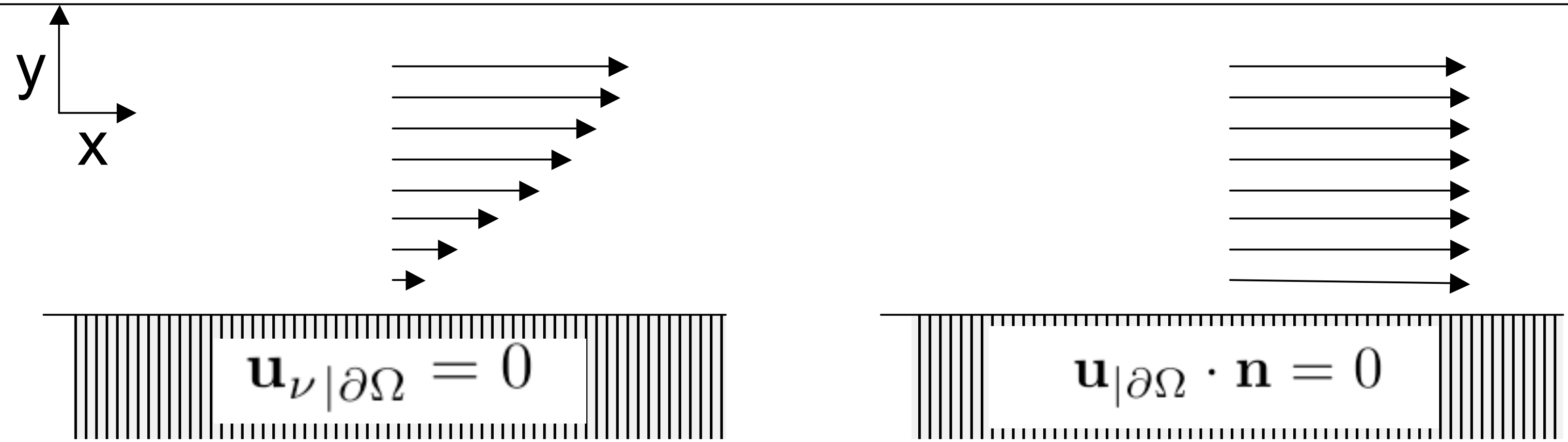
- in the 2D **wall-less** case, the numerical discretization size should satisfy:

$$\boxed{\delta x \propto \mathrm{Re}^{-\frac{1}{2}}}$$

(remark : the proof of that is not yet complete)

- therefore, **in the 2D wall-less case**, solving NS provides an **order 2 scheme to approach the inviscid limit** (i.e. solving Euler), (at least in the energy norm!)

What is the problem with walls ?



Navier-Stokes   Euler

- the wall imposes a strong tangential constraint on viscous flows,
- in contrast, no boundary condition affects the tangential velocity for Euler flows.
- **d'Alembert's paradox**
- Mathematically, the main obstacle to proofs is that **vorticity is not conserved** (same issue in 3D even without walls).

Über Flüssigkeitsbewegung bei sehr kleiner Reibung

- Prandtl (1904) and later authors proposed to use the following hypotheses :

  « The viscosity is assumed to be so small that it can be disregarded *wherever* there are no great velocity differences nor accumulative effects. […] The most important aspect of the problem is the behavior of the fluid on the surface of the solid body. […] In the thin transition layer, the great velocity differences will […] produce noticeable effects in spite of the small viscosity constants. »*

- this leads to

  Inviscid limit = Euler eq. + Prandtl eq.

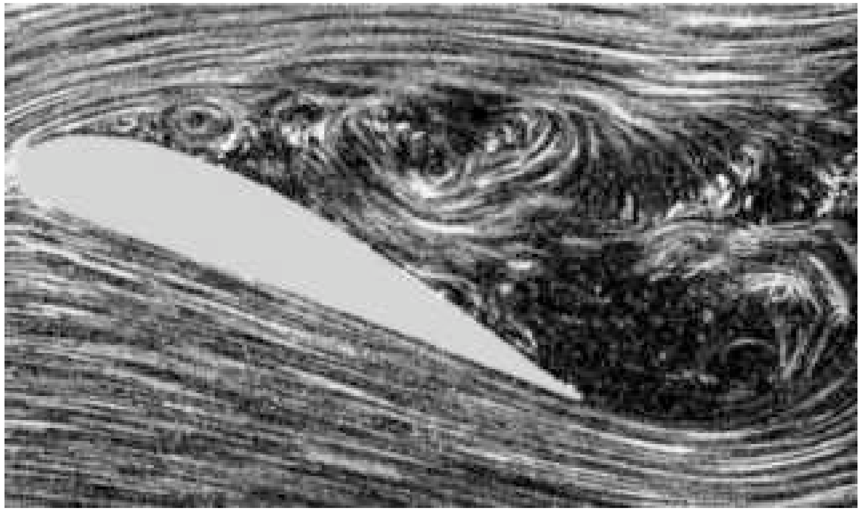
- when this applies, the question of the inviscid limit is solved everywhere except inside the **boundary layer** :

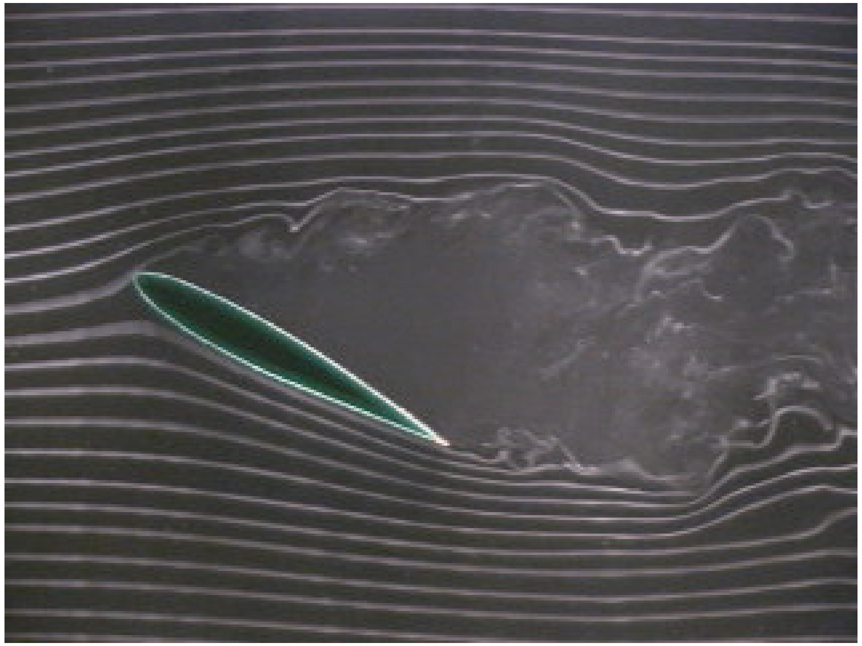
  « It is therefore possible to pass to the limit $\nu = 0$ and still retain the same flow figure. »*

* Prandtl 1927, engl. trans. NACA TM-452 available online

Separation

- Prandtl and others were aware that this approach was valid only away from separation points,
- separated flow regions have to be included « by hand » since the theory doesn't predict their behavior,





Picture from Wikimedia Commons, movie from "Digital fluid mechanics DVD"

Some consequences

- In unseparated regions, all convergence results presented above for the wall-less case should apply,
- the Prandtl boundary layer theory implies the following scaling for **energy dissipation** between two instants t_1 and t_2:

$$\Delta E(t_1, t_2) \underset{\mathrm{Re}\to\infty}{\sim} \mathrm{Re}^{-\frac{1}{2}}$$

- since the boundary layer thickness also scales like $\mathrm{Re}^{-\frac{1}{2}}$, the same scaling should apply for numerical discretization:

$$\boxed{\delta x \propto \mathrm{Re}^{-\frac{1}{2}}}$$

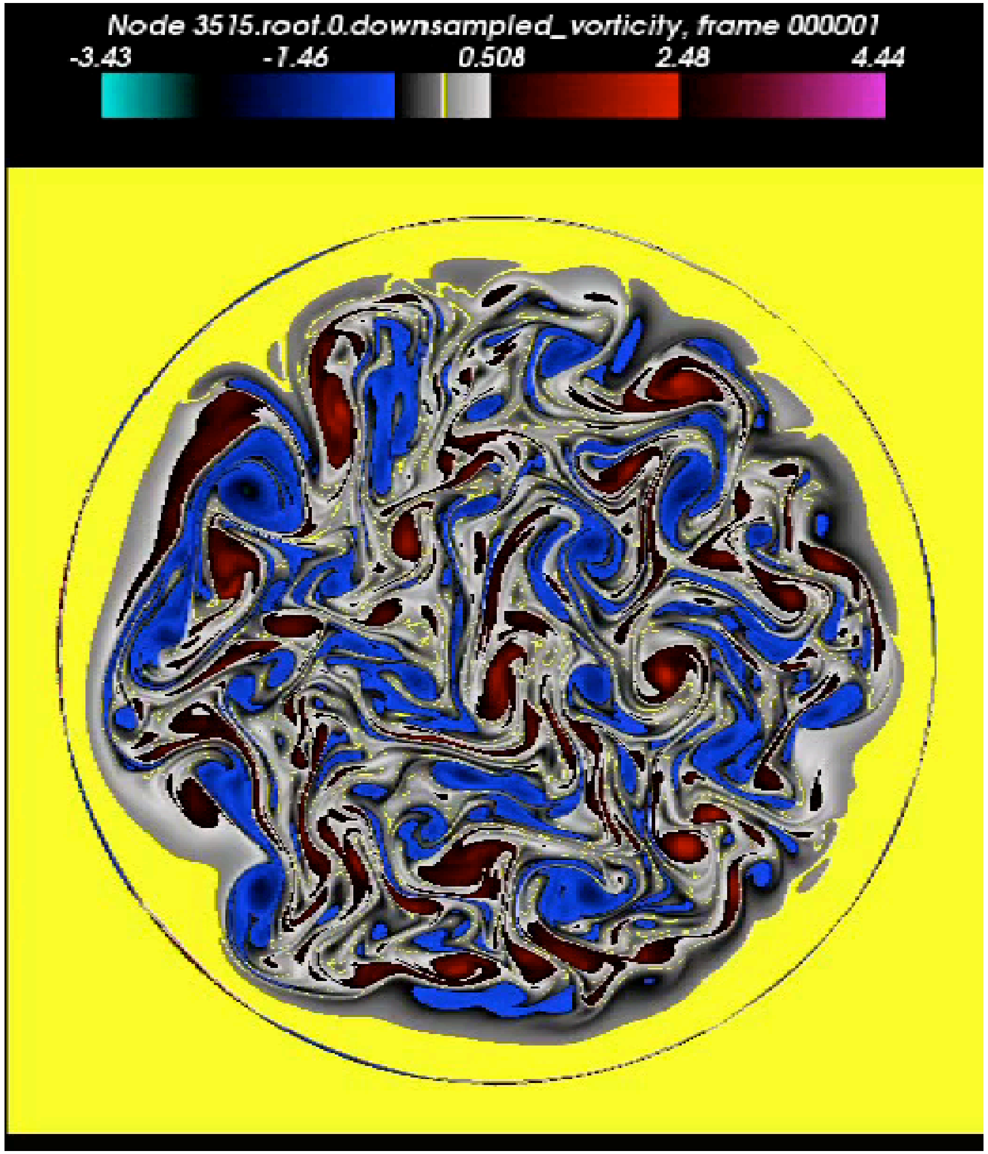
(as long as the solution is well behaved inside the BL)

- all of this phenomenology was observed by Clercx & van Heisjt* by computing flows up to Re=160 000

PRE* **65 (2002)

Introductory movie

Time evolution of vorticity field for 2D wall bounded turbulence.



<u>Qualitative features</u>:

•intense production of vorticity at the walls

•dipole-wall collisions

•**VERY THIN** boundary layer (it will get thinner!)

Design of numerical experiments.

Volume penalization method

- For efficiency and simplicity, we would like to stick to a **spectral solver in periodic, cartesian coordinates.**
- as a counterpart, we need to add an **additional term** in the equations to approximate the effect of the boundaries,
- this method was introduced by Arquis & Caltagirone (1984), and its spectral discretization by Farge & Schneider (2005),
- it has now become classical for solving various PDEs.

$$\text{(PNS)} \begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}} \Delta \mathbf{u} - \frac{1}{\eta} \chi_0 \mathbf{u} \\ \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}(0, \mathbf{x}) = \mathbf{v} \end{cases} \xrightarrow{\text{solution}} \mathbf{u}_{\mathrm{Re},\eta}$$

Convergence with η

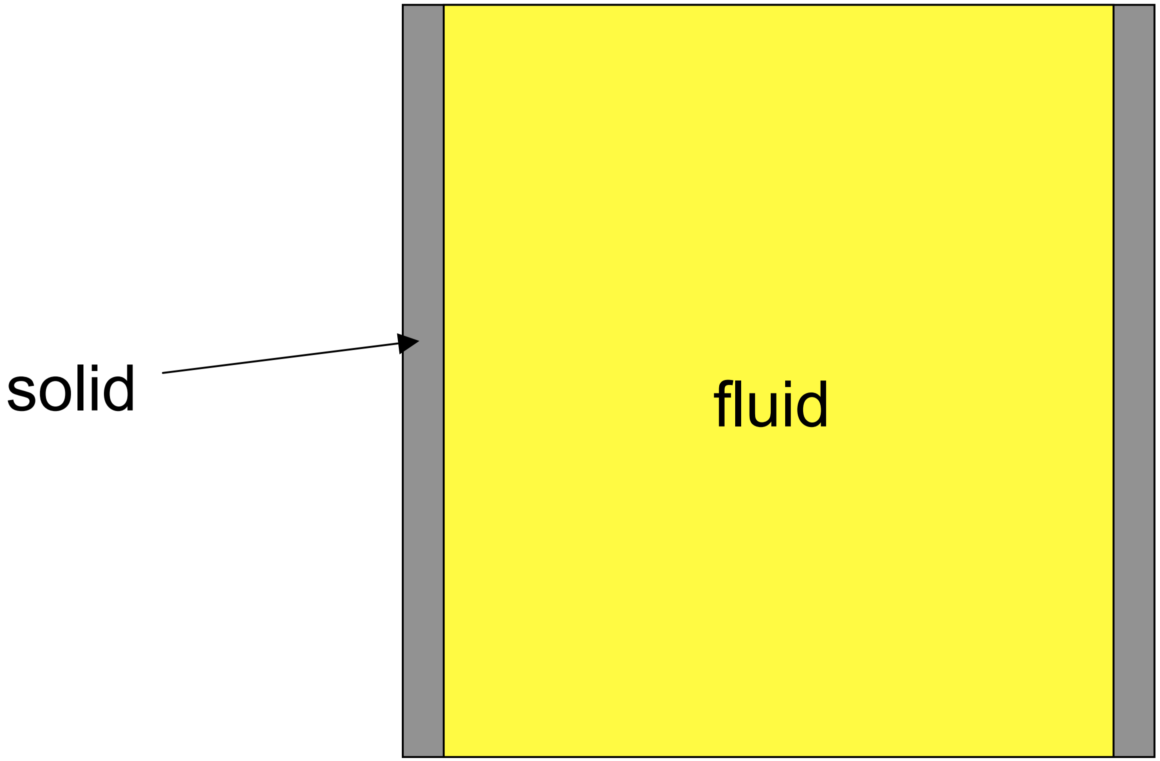
- Convergence in L^2 and H^1 norms for fixed Re was proven by Angot et al. (1999),

$$\left\| \mathbf{u}_{\mathrm{Re},\eta} - \mathbf{u}_{\mathrm{Re}} \right\| \leq C(\mathrm{Re})\eta^{\frac{1}{2}}$$

- **all known bounds diverge exponentially with Re**,
- arbitrarily small η cannot be achieved due to discretization issues,
- hence in practice, we do not have rigorous bounds on the error,
- we need careful validation of the numerical solution (and some faith!)

Choice of geometry

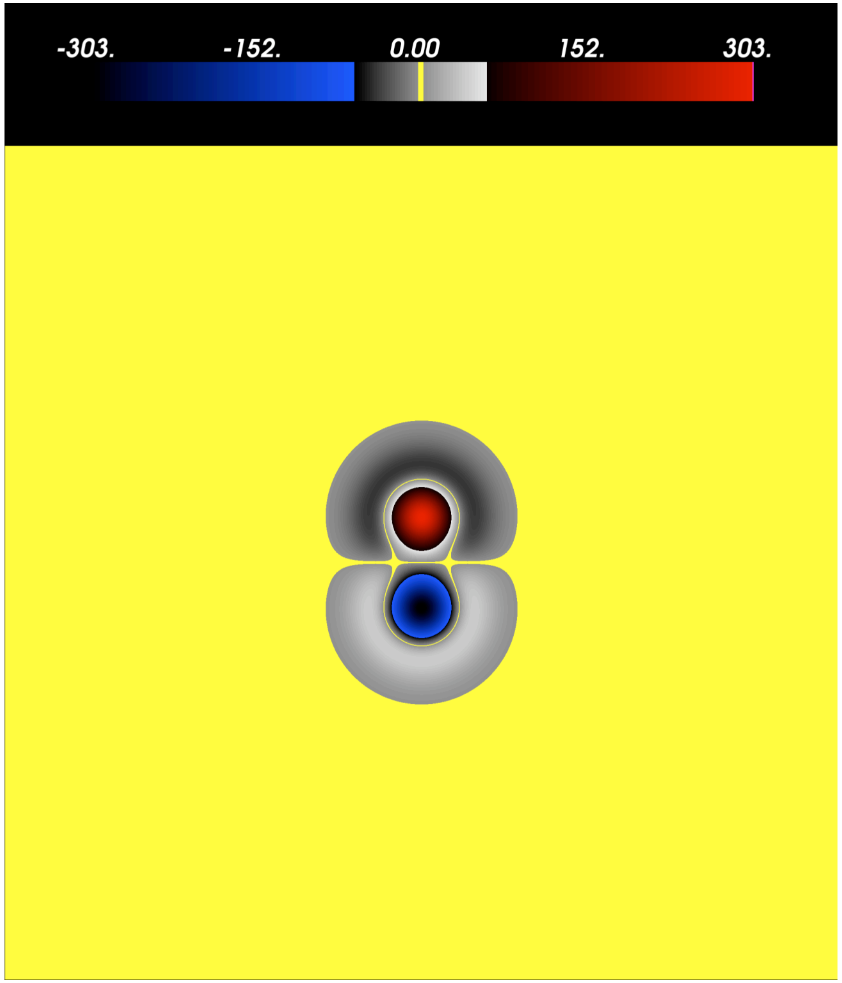
- We consider a channel, periodic in the y direction



$$\mathrm{Re} = \frac{UL}{\nu}$$ where U is the RMS velocity and L is the half-width.

Choice of initial conditions

vorticity dipole



Vorticity **decays exponentially** and the circulation is zero. Velocity also decays exponentially.

Pressure compatibility condition not satisfied exactly but we have checked that it does not creat too much problems at t=0.

Choice of parameters

- To resolve the Kato layer (see later), we impose:

$$N \propto \mathrm{Re}^{-1}$$

- We take for η the minimum value allowed by the CFL condition, which implies:

$$\eta \propto \mathrm{Re}^{-1}$$

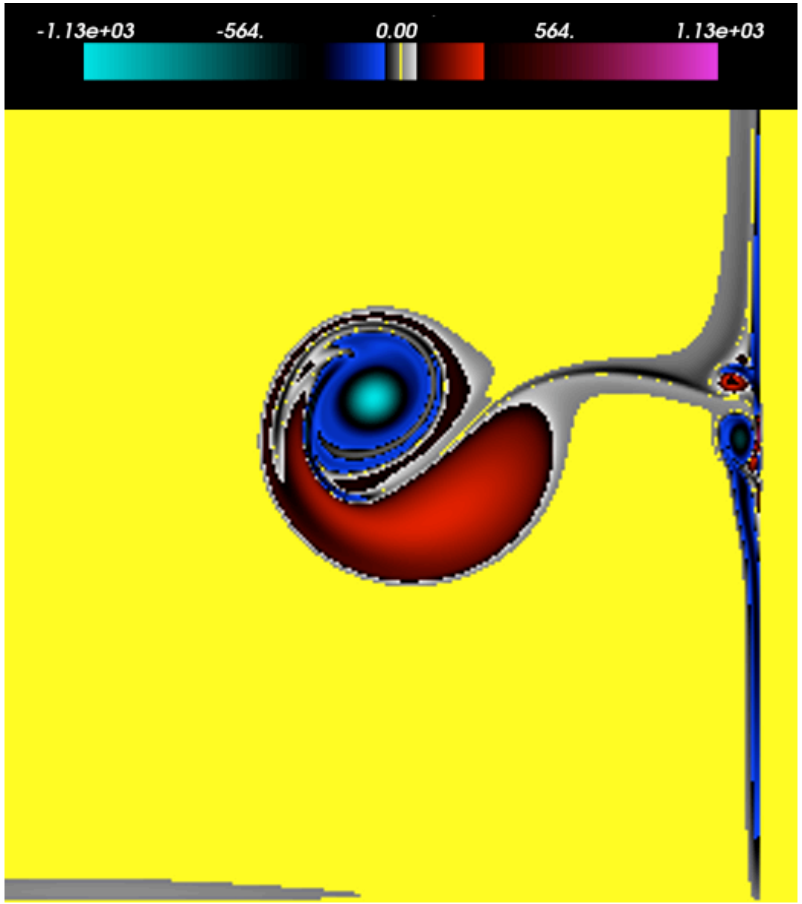
<u>Parameters of all reported numerical experiments</u>

Re	985	1970	3940	7880	*7880*
N	2048	4096	8192	16384	*8192*
η	$4 \cdot 10^{-5}$	$2 \cdot 10^{-5}$	10^{-5}	$0.5 \cdot 10^{-5}$	$\mathit{0.5 \cdot 10^{-5}}$

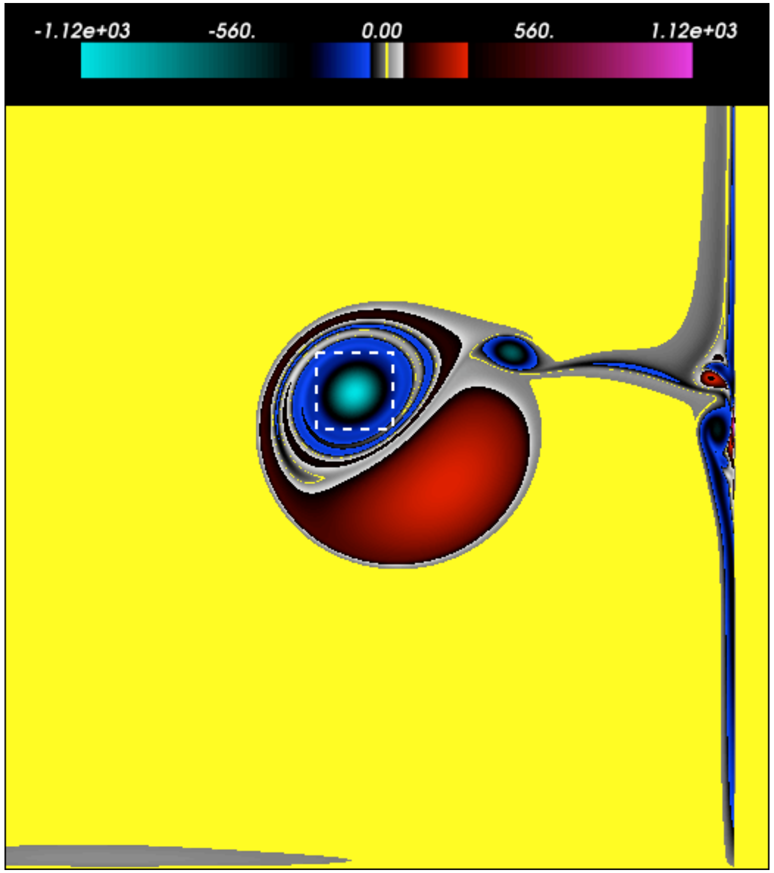
Convergence test

for Re > 1000, we did not have access to a reference solution, => auto-comparison for Re = 8000. RMS velocity difference 20%.

N = 8192



N = 16384



Dissipation of energy ?

Why is dissipation of energy so essential ?

- Kato (1984) proved (roughly stated):

The NS solution converges towards the Euler solution in L²

$$\forall t \in [0,T], \left\| u_{\mathrm{Re}}(t) - u(t) \right\|_{L^2(\Omega)} \underset{\mathrm{Re}\to\infty}{\longrightarrow} 0$$

if and and only if

the energy dissipation during this interval vanishes,

$$\Delta E_{\mathrm{Re}}(0,T) = \mathrm{Re}^{-1} \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}\mathbf{x} \left| \nabla \mathbf{u}(t,\mathbf{x}) \right|^2 \underset{\mathrm{Re}\to\infty}{\longrightarrow} 0$$

and even if and only if

it vanishes in a strip of width prop to **Re⁻¹** around the solid.

$$\mathrm{Re}^{-1} \int_0^T \mathrm{d}t \int_{\Gamma_{c\,\mathrm{Re}^{-1}}} \mathrm{d}\mathbf{x} \left| \nabla \mathbf{u}(t,\mathbf{x}) \right|^2 \underset{\mathrm{Re}\to\infty}{\longrightarrow} 0 \qquad \Gamma_{c\,\mathrm{Re}^{-1}} = \left\{ \mathbf{x} \middle| d(\mathbf{x},\partial\Omega) < c\,\mathrm{Re}^{-1} \right\}$$

An important practical consequence

- To have any chance of observing energy dissipation (i.e. default of convergence towards the Euler solution), we need a smaller grid than the one traditionally used !

$$\cancel{\delta x \propto \mathrm{Re}^{-\frac{1}{2}}}$$

$$\Downarrow$$

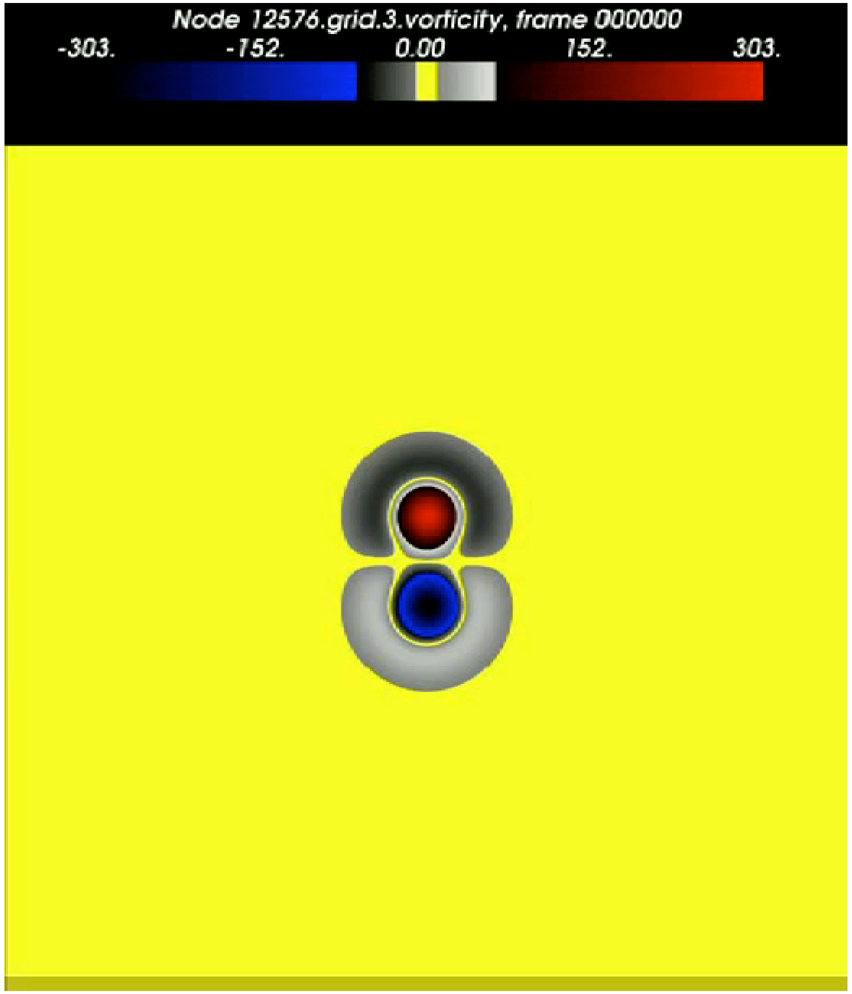
$$\boxed{\delta x \propto \mathrm{Re}^{-1}}$$

- **This is the essential message in this talk !!**

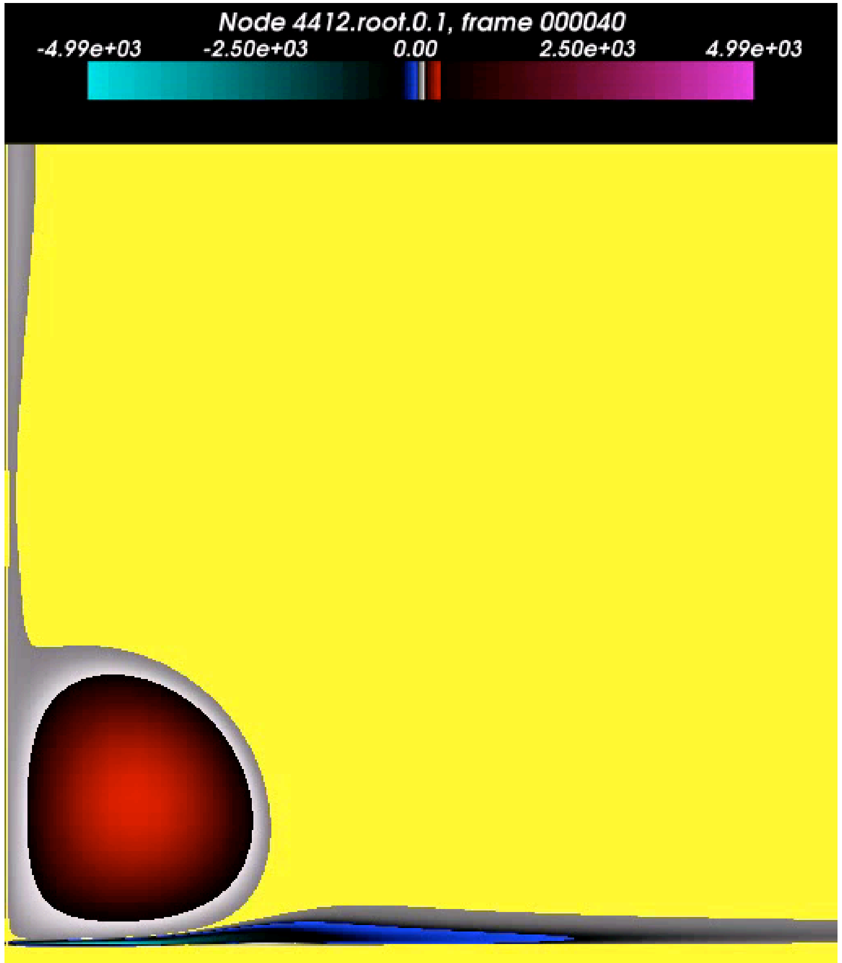
Results

Results

- We focus on the dipole-wall collision.



vorticity movie for Re = 3940



$\frac{L}{\sqrt{\mathrm{Re}}}$

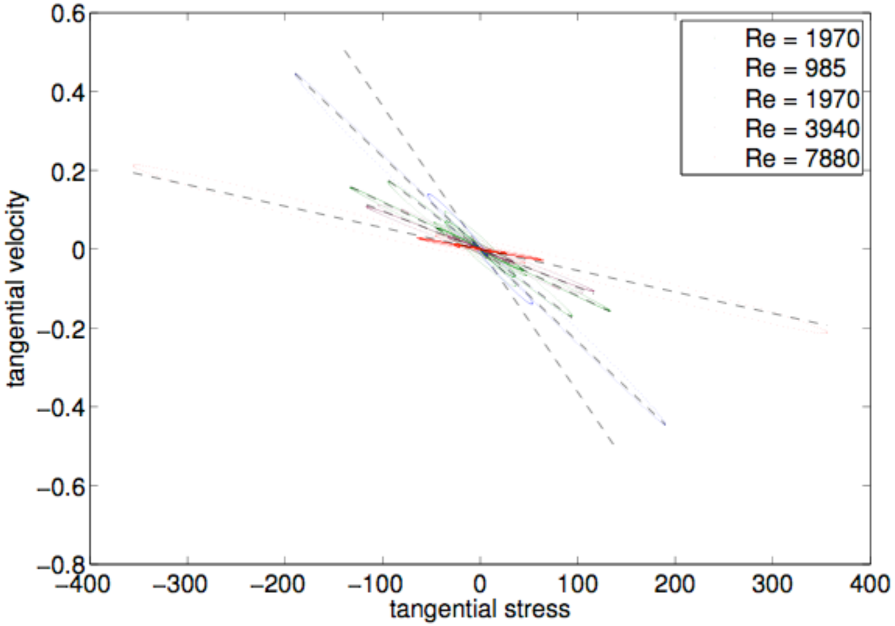
zoom on collision for Re = 7980

Boundary conditions

- A posteriori, we want to check what were the boundary conditions seen by the flow.
- We define the boundary as the isoline $\chi = 0.02$, where the viscous term approximately balances the penalization term in the PNSE.
- To avoid grid effects we interpolate the fields along this isoline.
- The **normal velocity** is smaller than 10^{-3} (to be compared with the initial RMS velocity 0.443),
- but the **tangential velocity** reaches values of order 0.1 !!

Boundary conditions

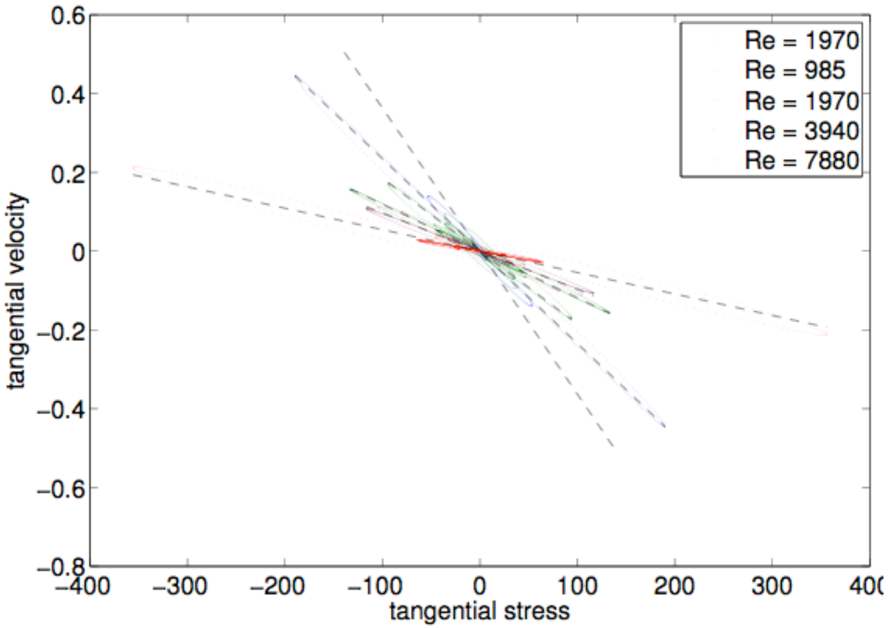
- We plot the tangential velocity as a function of the tangential stress:

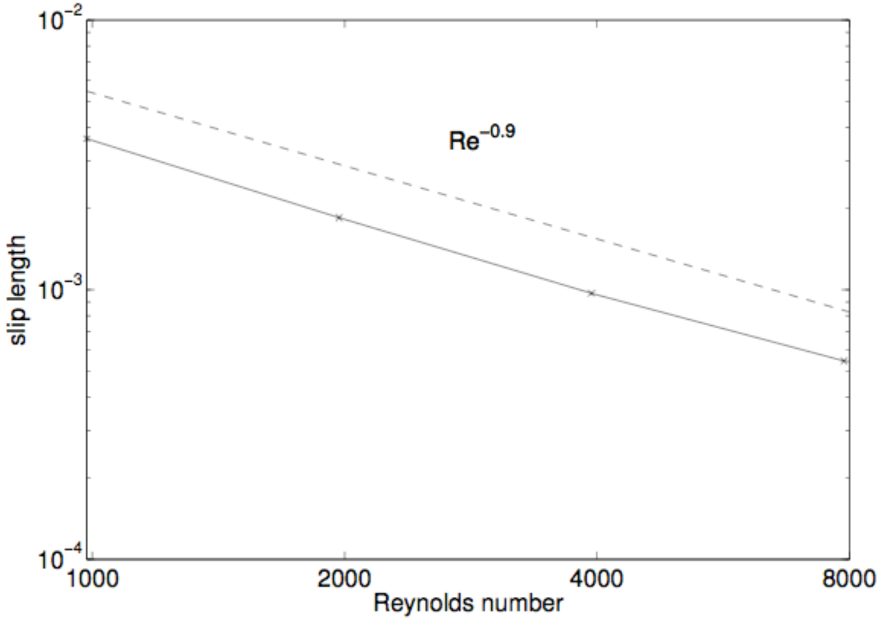


Boundary conditions

- A linear relationship with correlation coefficient above 0.98 appears:

$$u_y + \alpha(\mathrm{Re},\eta,N)\partial_x u_y = 0$$



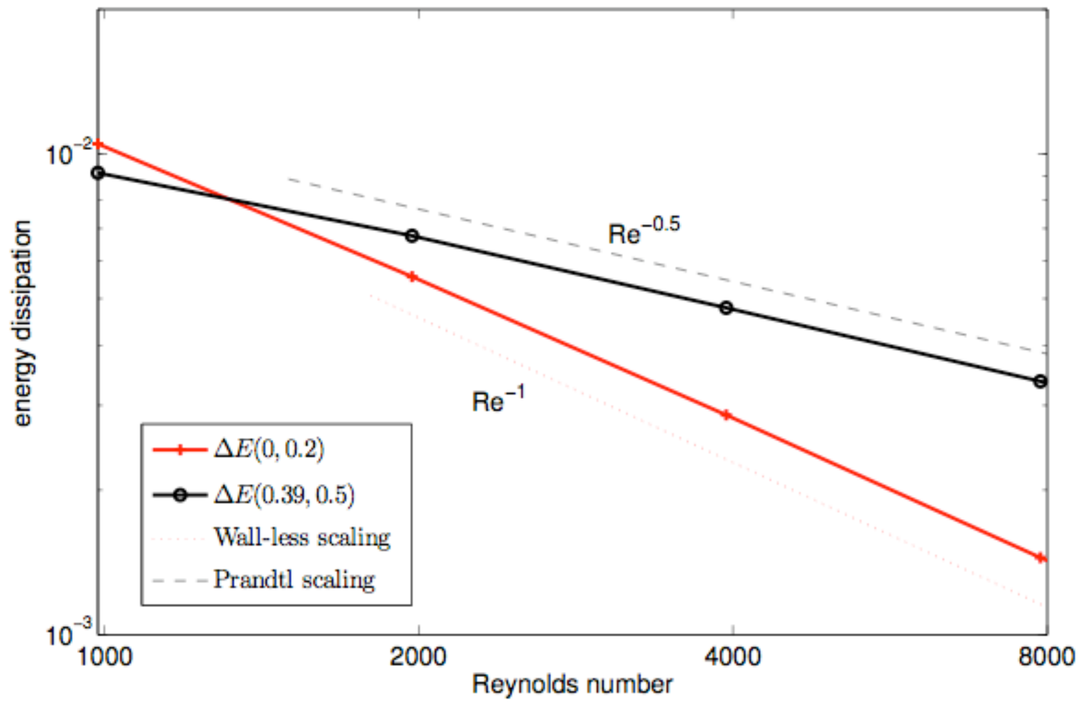


- The flow hence sees **Navier boundary conditions** with a slip length α satisfying: $\alpha \propto \mathrm{Re}^{-0.9}$

Energy dissipation

- We now look at the energy dissipated during the collision for increasing Reynolds numbers.

<u>energy dissipation as a function of Reynolds</u>

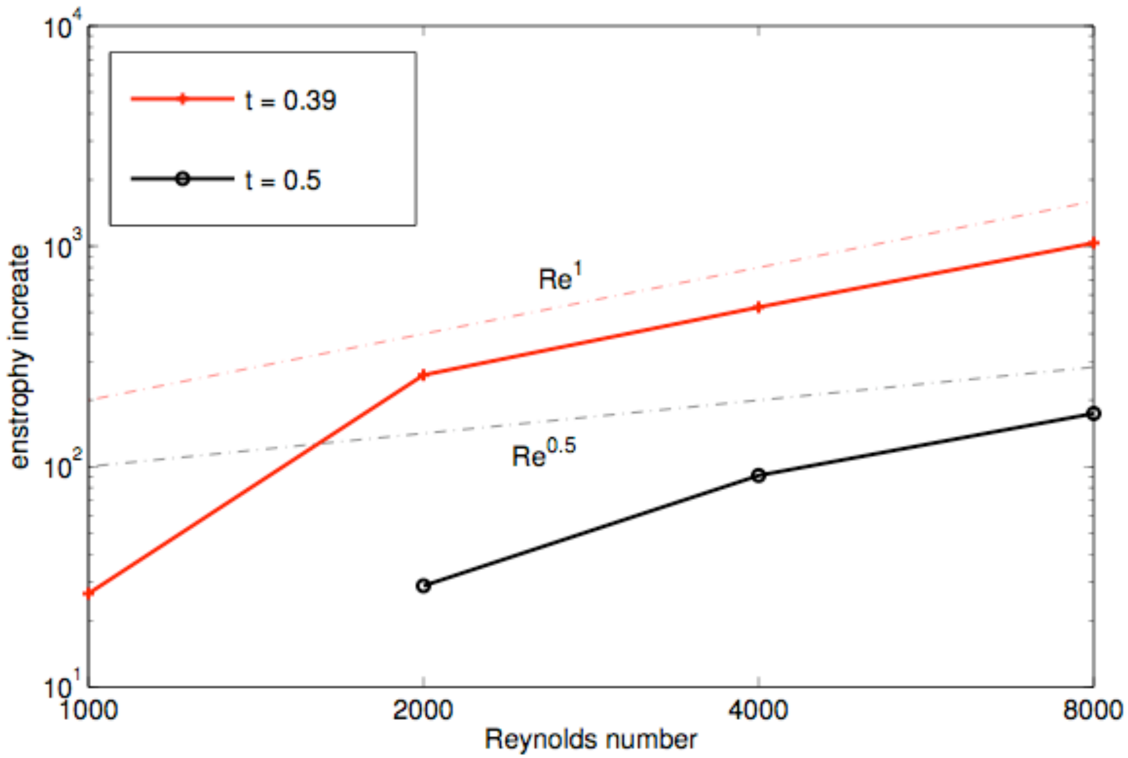


Prandtl scaling ??

Enstrophy production

- To isolate the effect of the enstrophy produced at the boundary, we consider Z(t)-Z(0):

enstrophy increase as a function of Reynolds



Dissipative scaling !

Dissipative structures ?

Subregions

We define two subregions of interest in the flow :

- **region A** : a vertical slab of width $10N^{-1}$ along the wall,
- **region B** : a square box of side 0.025 around the center of the main structure.

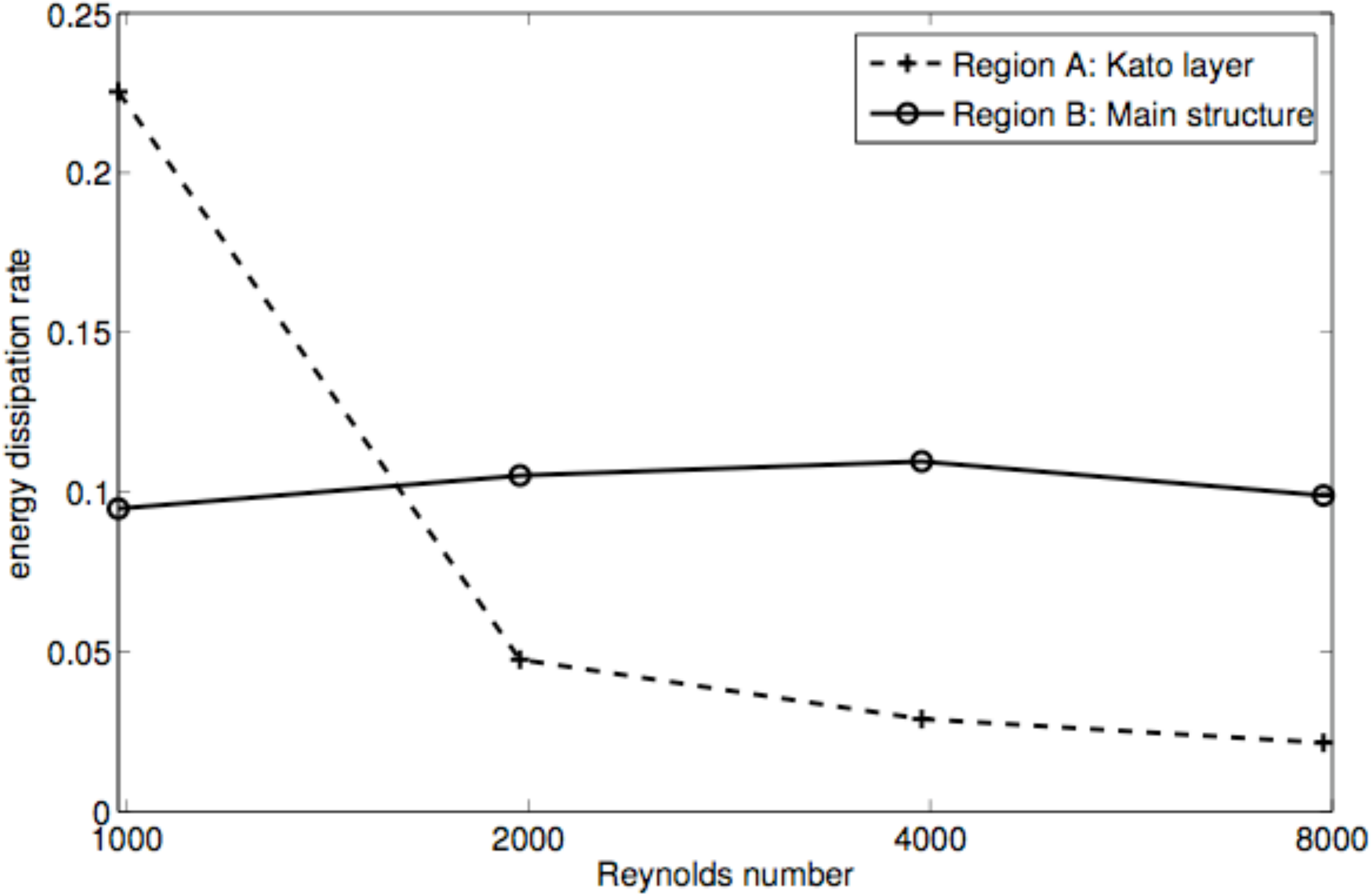
Now we consider the **local** energy disspation rate:

$$\varepsilon = \nu |\nabla u|^2$$

(sometimes called pseudo-dissipation rate)

and we integrate it over A and B respectively.

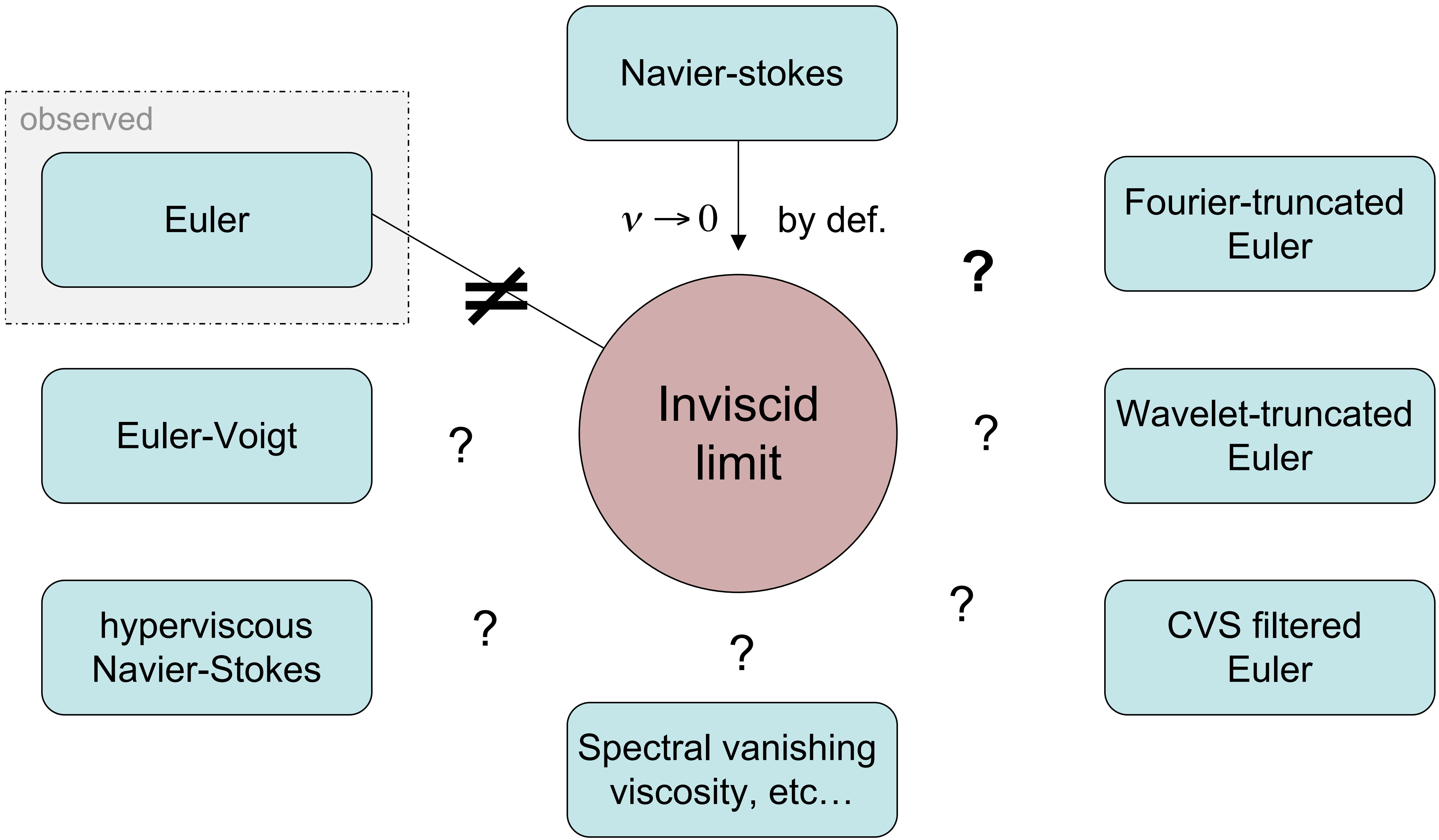
Subregions



0.25
0.2
0.15
0.1
0.05
0
1000
2000
4000
8000
Reynolds number
energy dissipation rate
Region A: Kato layer
Region B: Main structure

Convergence (for $t > t_C$) ?

2D wall-bounded case, smooth initial data



Summary

- We have studied the behavior as a function of Reynolds number of a flow modeling a dipole-wall collision in a 2D channel,
- we have shown that the flow approximately satisfies Navier boundary conditions with a slip length proportional to $Re^{-0.9}$,
- we have then shown that the enstrophy production during the collision scales like Re, implying **nonzero energy dissipation** in the vanishing viscosity limit,
- we have outlined two regions where the energy dissipation seems not to go to zero: a “**Kato layer**”, of thickness proportional to Re^{-1} along the wall, and an intense vortex. These are two examples of **energy dissipating structures** (they may play the same role as shocks in compressible flows).

Perspectives

- Energy dissipating structures could be observed **experimentally**, for example in soap films or in oceanic flows (??).
- Current **statistical theories** of 2D turbulent flows cannot account for energy dissipation. Understanding the statistical properties of 2D flows containing energy dissipating structures is an open question.
- The structure should be studied in more detail, and the link with the **Kelvin-Helmholz** instability should be clarified.
- The limiting flow has unbounded vorticity. Is it a weak Euler solution? Or do we need an additional term to describe it? Nobody knows (cf Onsager conjecture…).
- **3D computations** with the resolution required to resolve the Kato layer would be very costly, but highly relevant.

Thank you !

many thanks to Claude Bardos for pointing us to the paper by Kato,

and also to Gregory Eyink, Dmitry Kolomenskiy, Anna Mazzucato, Helena Nussenzveig-Lopes and Zhouping Xin for fruitful discussions.

Most of the results were obtained using the Kicksey-Winsey C++ code, which is available online under a GPL license:

http://justpmf.com/kicksey_winsey

Publications are available on:

http://wavelets.ens.fr

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Thanks to CEMRACS and SMAI for hospitality.

Discretization

<u>Space discretization</u>

- Galerkin method, Fourier modes with wavenumber $|\mathbf{k}| \leq K$
- pseudo-spectral evaluation of products, using a N x N grid,

with $N = 3K$

to ensure full **dealiasing**.

<u>Time discretization</u>

- 3rd order, low-storage, fully **explicit Runge-Kutta** scheme for the nonlinear and penalization terms,
- integrating factor method for the viscous term.

Regularization

- One of our main goals is to diagnose energy dissipation,
- hence we have introduced a regularized problem

$$\text{(RPNS)} \quad \begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \dfrac{1}{\mathrm{Re}}\Delta\mathbf{u} - \dfrac{1}{\eta}\chi\mathbf{u} \\ \nabla\cdot\mathbf{u} = 0, \quad \mathbf{u}(0,\mathbf{x}) = \mathbf{v} \end{cases} \xrightarrow{\text{solution}} \mathbf{u}_{\mathrm{Re},\eta,\chi}$$

mollified mask function

- the Galerkin truncation of (RPNS) with K modes admits the following energy equation :

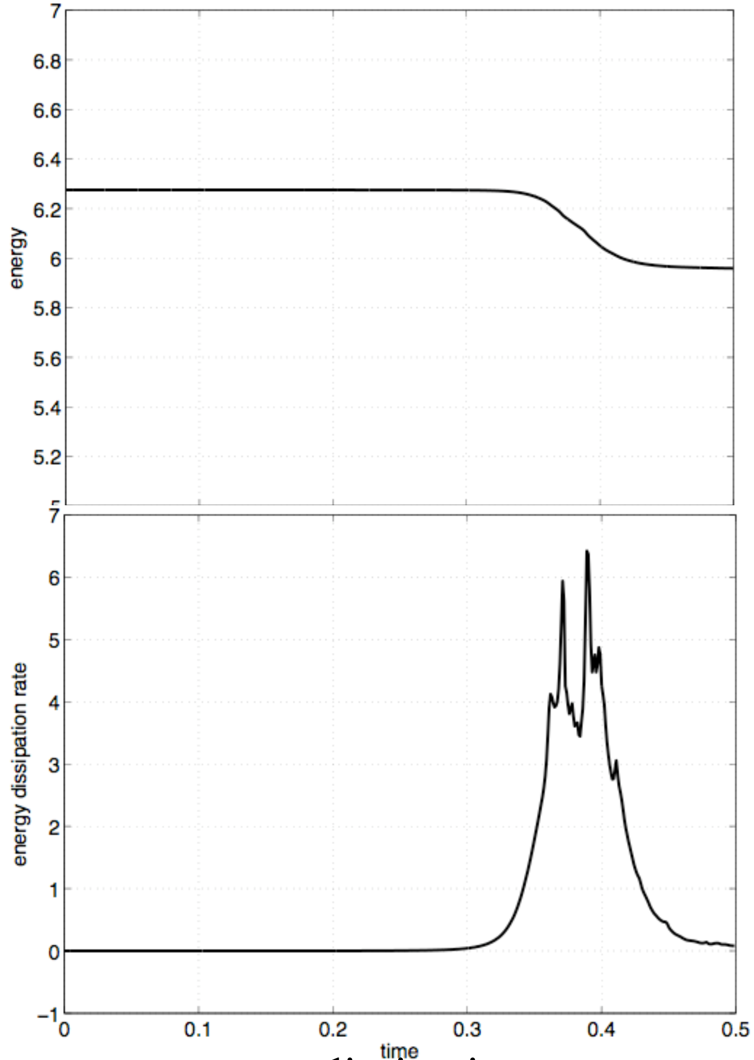
$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{u}_{\mathrm{Re},\eta,\chi,K}\right\|^2 = -2\nu\left\|\nabla\mathbf{u}_{\mathrm{Re},\eta,\chi,K}\right\|^2 - \frac{1}{\eta}\int\chi\left|\mathbf{u}_{\mathrm{Re},\eta,\chi,K}\right|^2$$

spurious dissipation can be monitored easily.

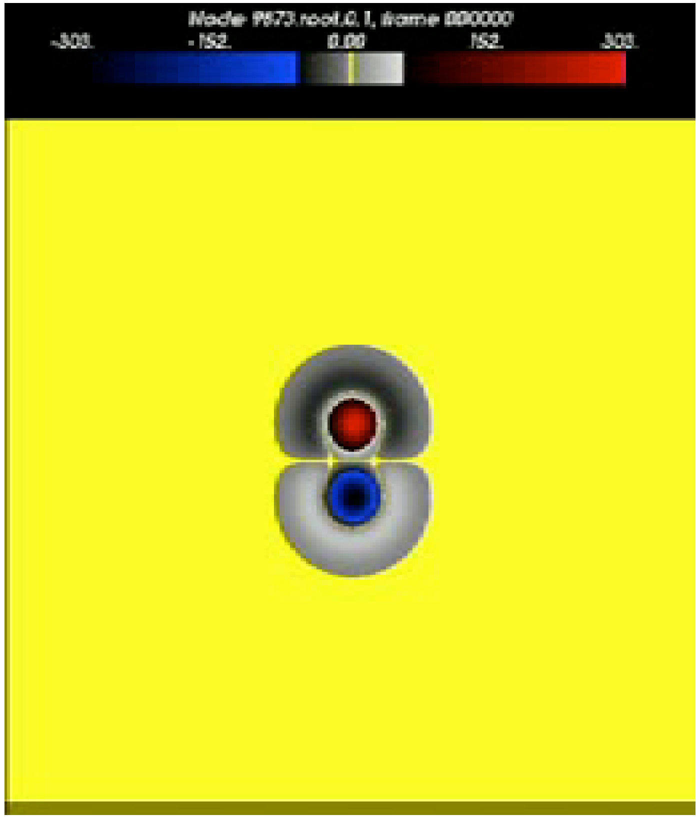
Illustration : Fourier-truncated inviscid RPNS

- to check conservation properties we perform some runs with ν **= 0**,
- this is an example with a dipolar initial condition.

time evolution of energy



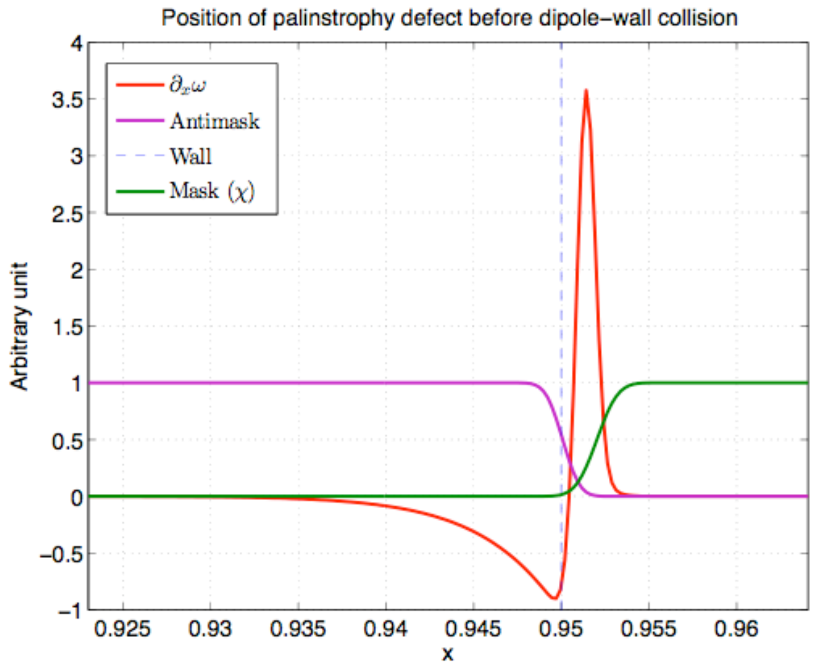
energy dissipation rate



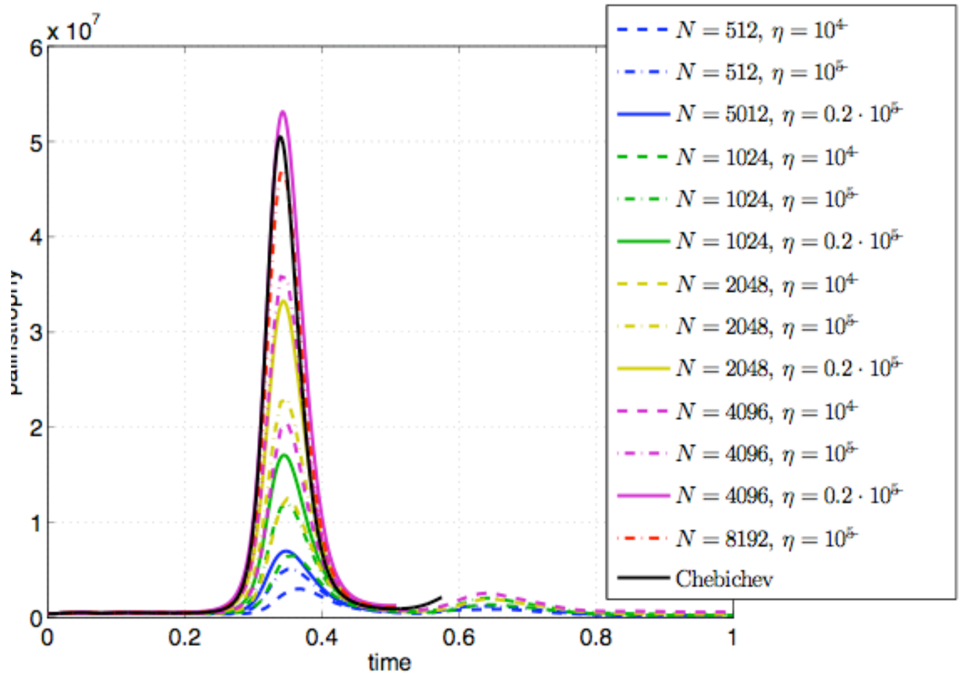
vorticity field

Convergence tests

- **Test 1** : For Re = 1000 we reproduce the palinstrophy $P = \frac{1}{2}\int_{\Omega} |\nabla \omega|^2$ obtained by H. Clercx using a Chebichev method,
- our method allows a clean elimination of the palinstrophy defect due to the discontinuity in the penalization term,
- fully capturing the palinstrophy requires **very high resolutions**.



palinstrophy defect



time evolution of palinstrophy

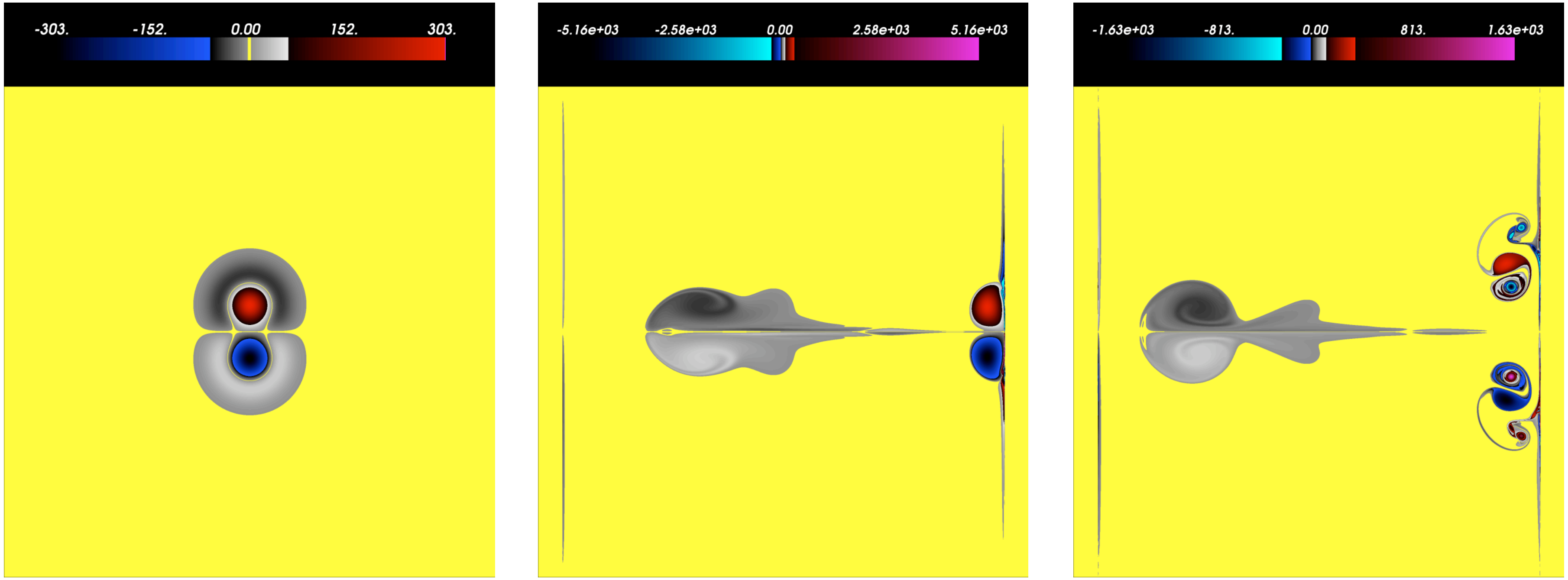
Lagrangian viewpoint

- What does the trajectory of a particle initially sitting very close to the boundary looks like when Re >> 1 ?

$$\mathbf{a} = -\nabla p + \nu \Delta \mathbf{u}$$

- to detach from the wall, a particle must jump from **u** = **0** to a finite **u** => **infinite acceleration**,
- we conjecture that energy will then **continue to be dissipated** along those trajectories starting from the wall,
- we should check numerically this using Lagrangian tracers.

Dipole wall collision at Re = 8000



When should we expect the flow to dissipate ?

- Sammartino & Caflisch (1998) proved :

For analytic initial data, and when Ω is a half-plane, there is a time $t_0 > 0$ such that in $[0,t_0[$

– the NS solution converges to the Euler solution in L^2,

– the Prandtl equation has a unique solution which describes the boundary layer to first order in Re^{-1}.

In other words, flow separation can occur only after a positive time, and not at $t = 0$.

Note : all our initial conditions are analytic.

Suitable initial conditions

- the incompressible Navier-Stokes equations are **non-local** due to the divergence free condition,
- boundary conditions impose that the following **force balance** holds for all time at the boundary:

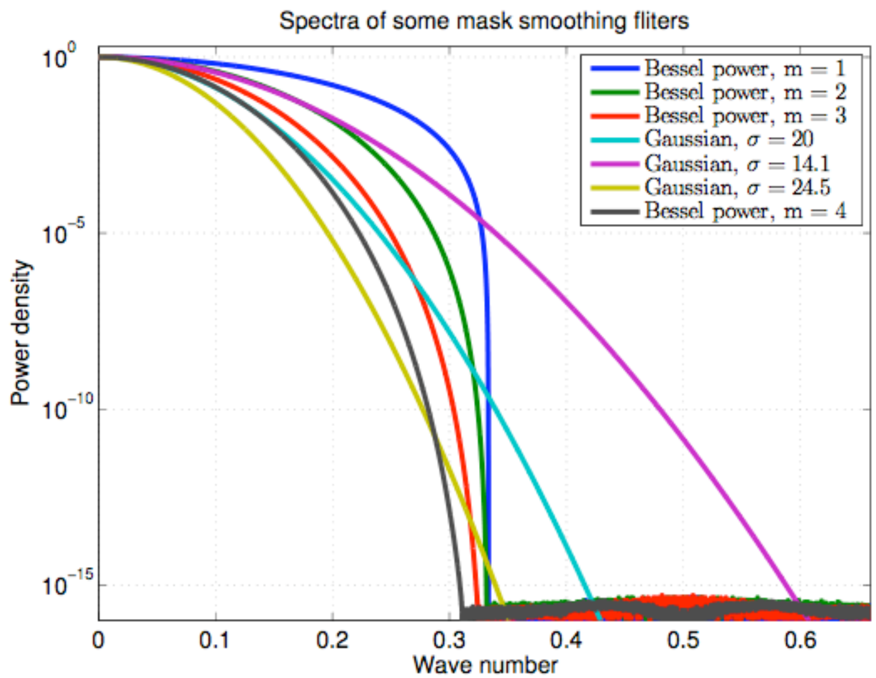
$$\text{(C)} \qquad (-\nabla p + \nu \Delta \mathbf{u})_{|\partial\Omega} = 0$$

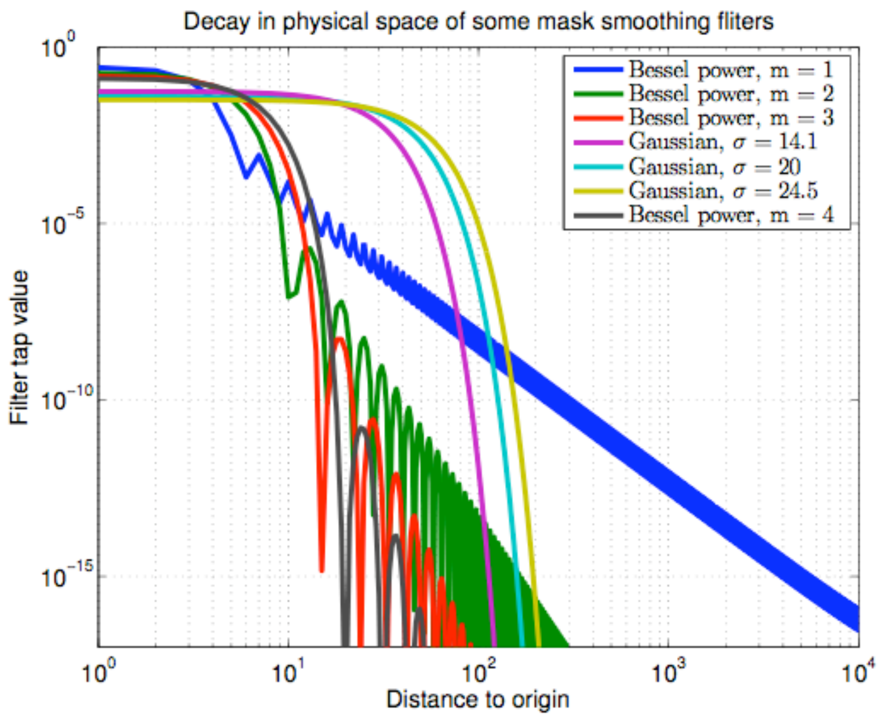
- this translates into a compatibility condition at t = 0,
- in practice this is very hard to satisfy exactly since the initial pressure is usually not localized even if the initial velocity is,
- failure to satisfy (C) at t=0 makes the vorticity discontinuous in time, and creates an artificial boundary layer depending on ν,
- care has to be taken to approximately match (C).

see R. Temam, *JCP* **218**, 443-450 (2006)

Choice of mask function

- For stability reasons we impose that : $0 \leq \chi \leq 1$
- such a χ can be obtained by convolving χ_0 with a smooth positive kernel,
- we use some well localized kernels based on Bessel functions*,





*Ehm et al, *Trans. Am. Math. Soc.* **356** (2004)